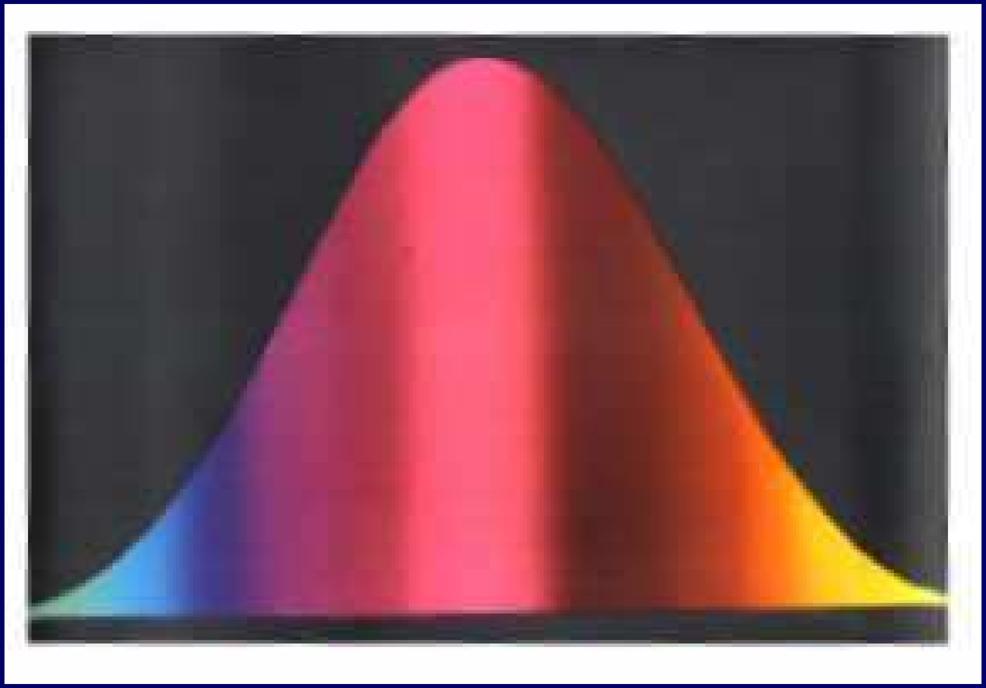
Great Theoretical Ideas In Computer Science

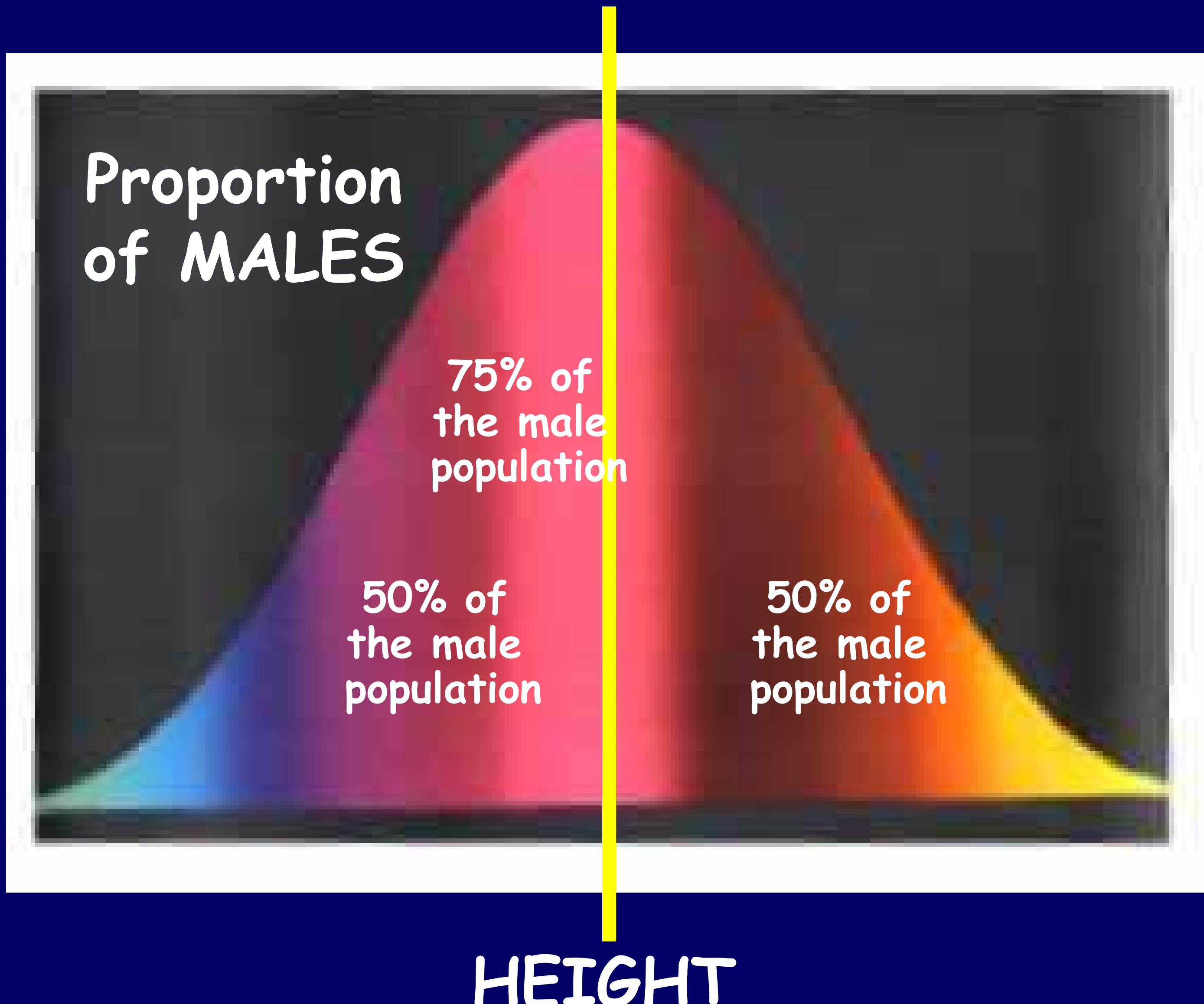
Steven Rudich, Anupam Gupta CS 15-251 Spring 2005

Lecture 20 March 24, 2005 Carnegie Mellon University

Probability Theory: Counting in Terms of Proportions



A Probability Distribution



The Descendants Of Adam

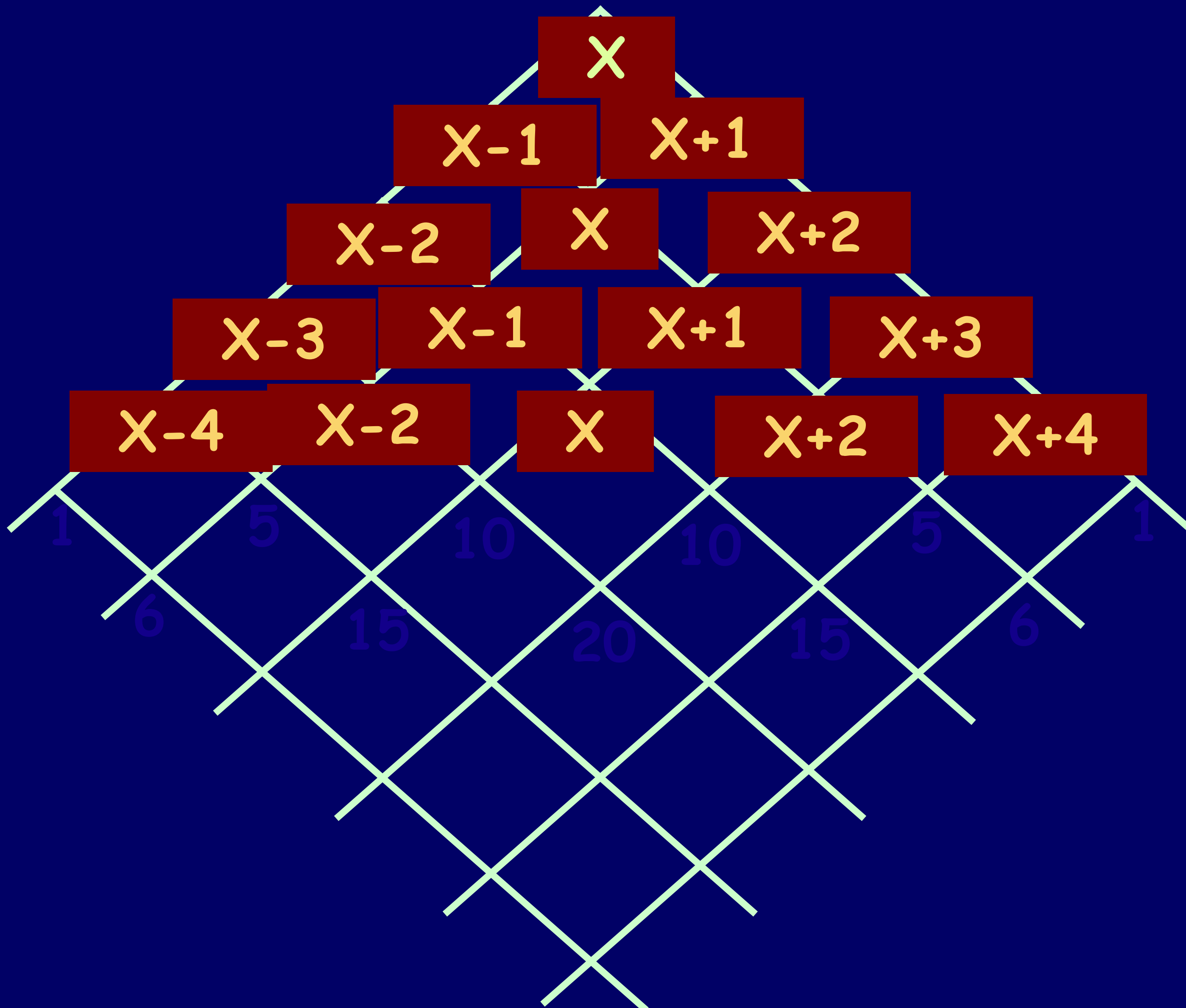
Adam was X inches tall.

He had two sons

One was X+1 inches tall

One was X-1 inches tall

Each of his sons had two sons …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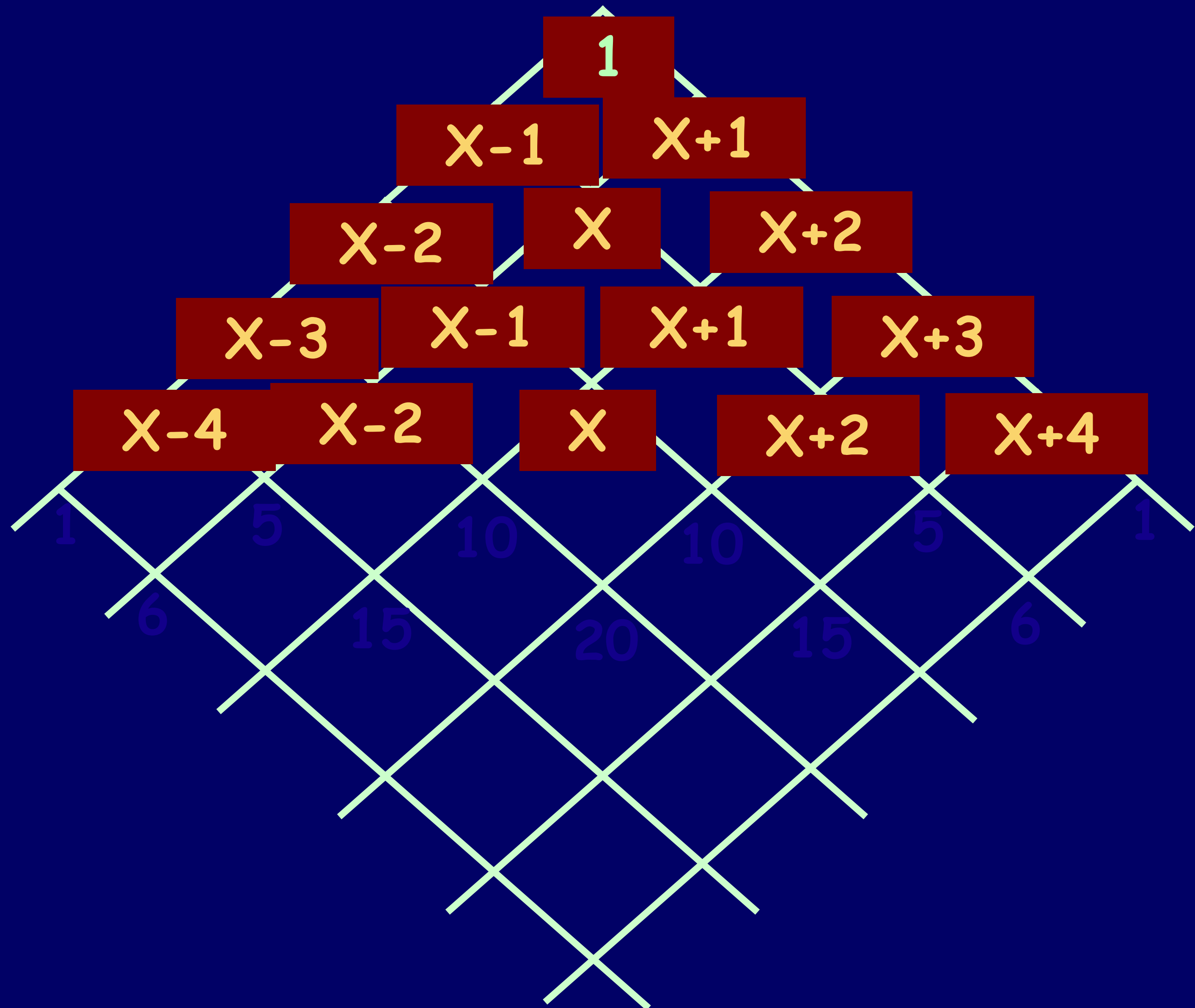
6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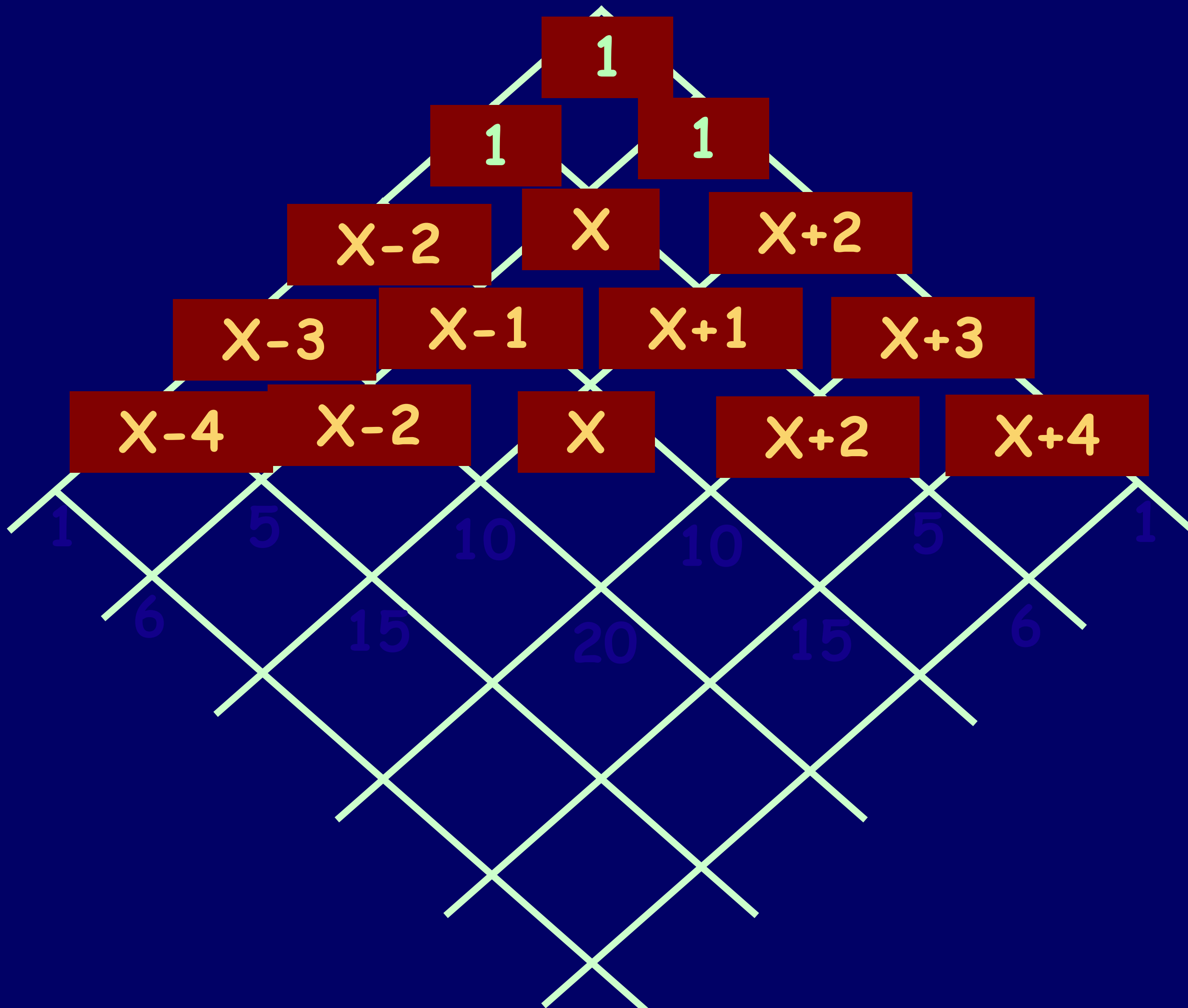
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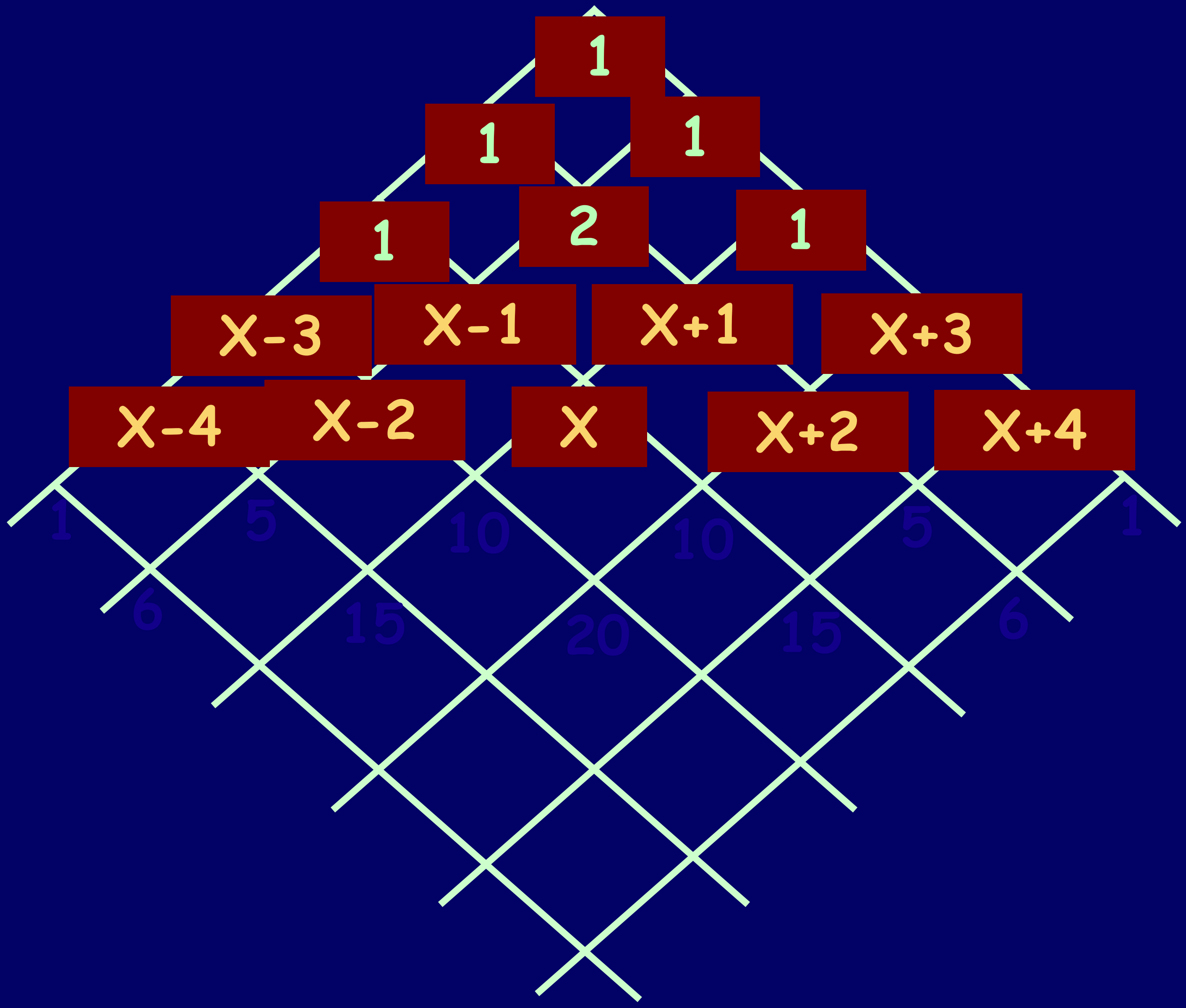
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10
10
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1
6
15
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15
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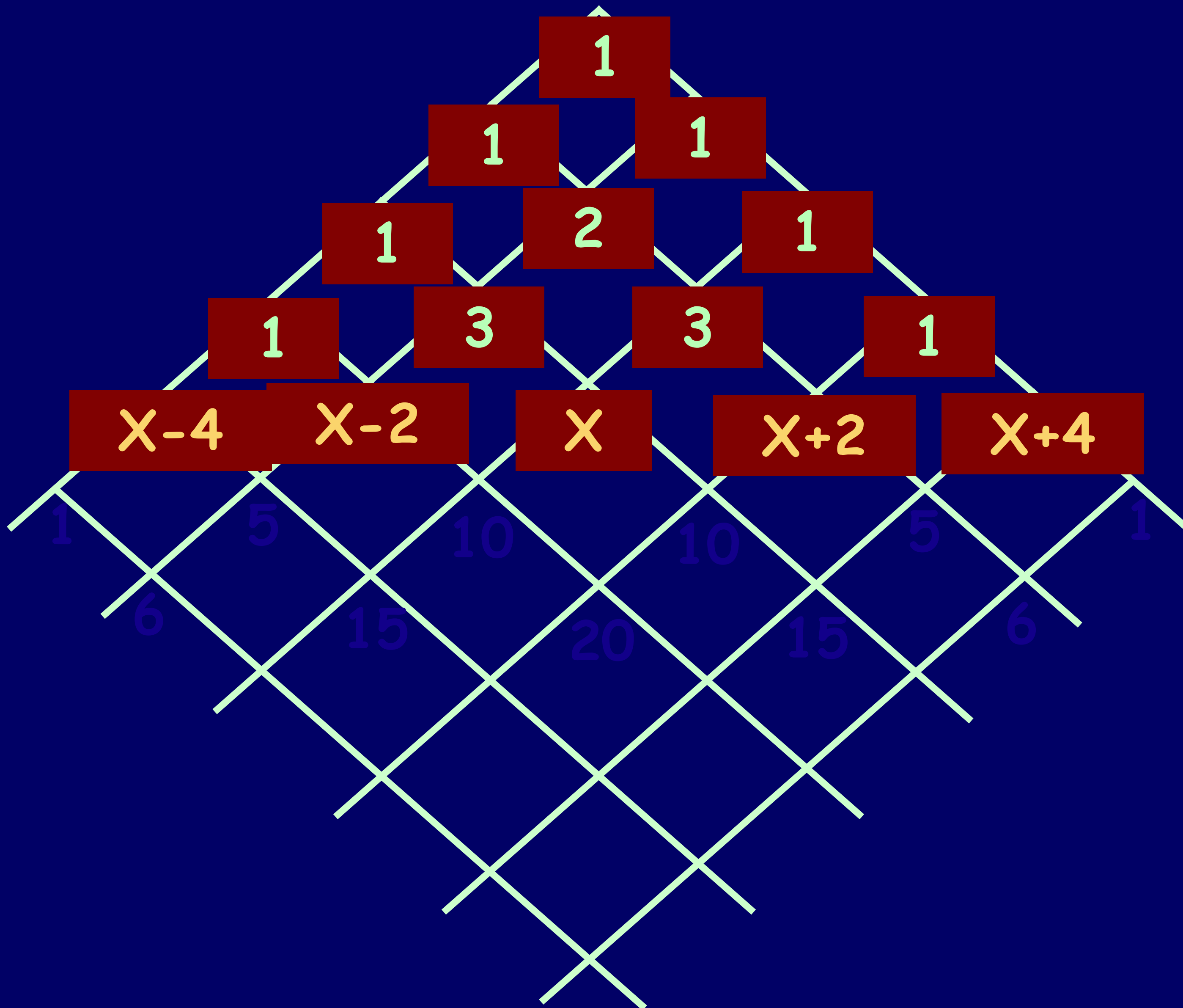
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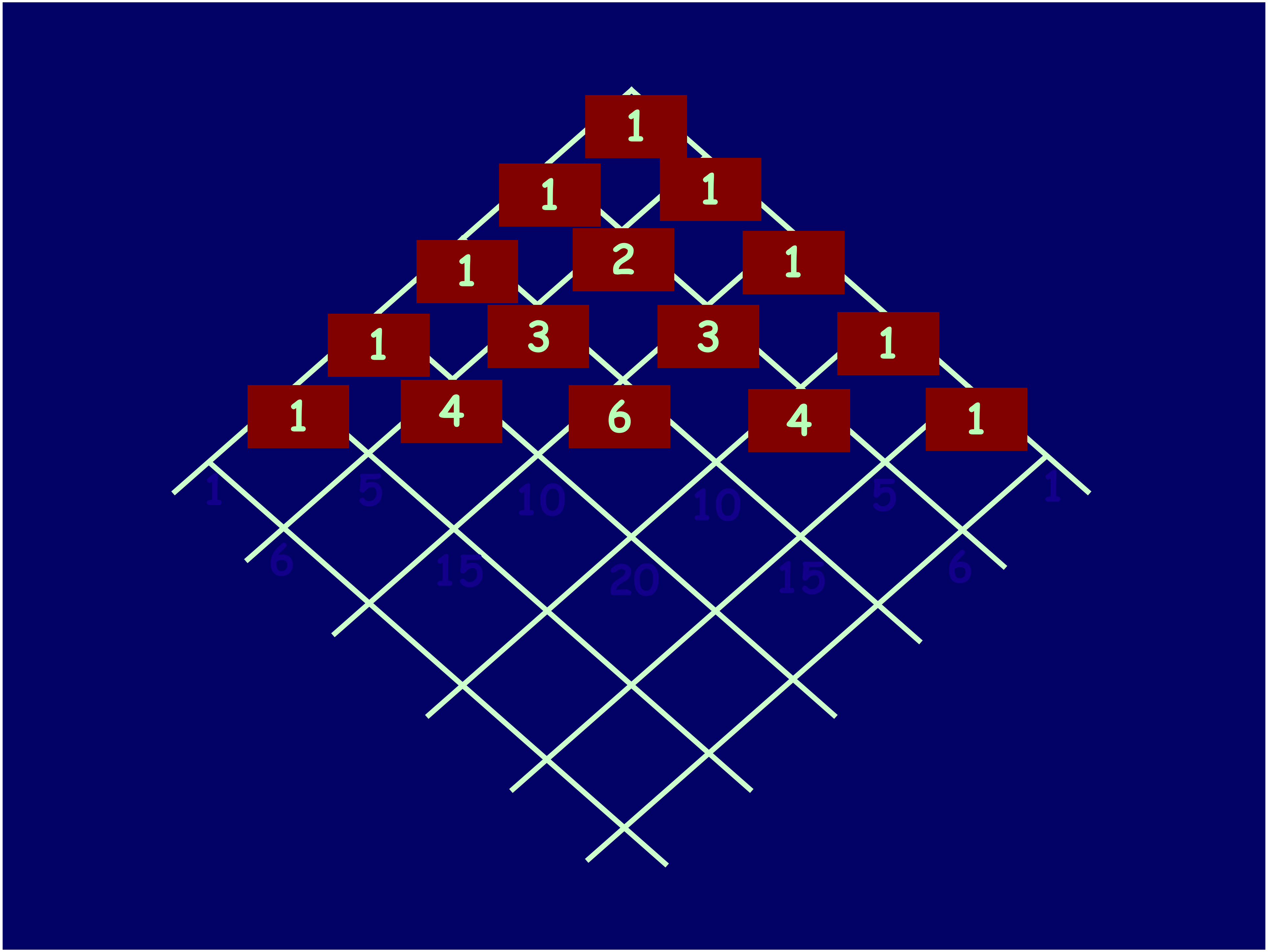
1 2 1
1 3 3 1
X-4 X-2 X X+2 X+4
1 5 10 10 5 1
6 15 20 15 6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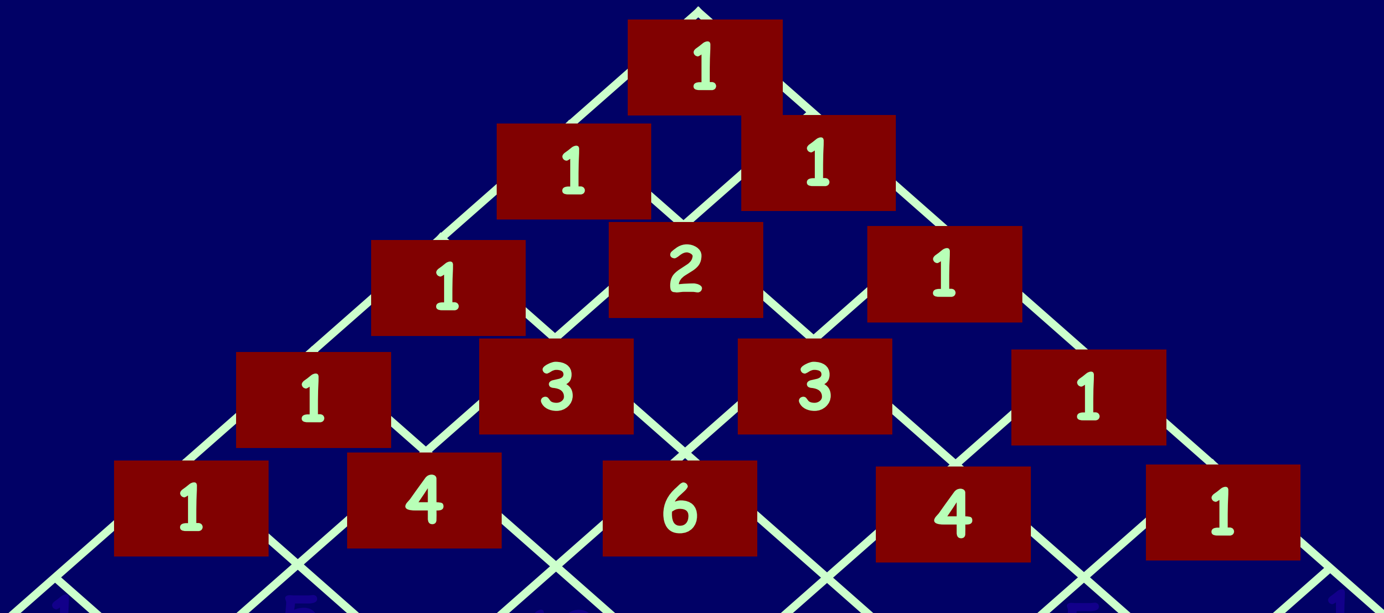
1 5 10 10 5 1
6 15 20 15 6



In n^{th} generation, there will be 2^n males, each with one of $n+1$ different heights: $h_0 < h_1 < \ldots < h_n$.

$h_i = (X - n + 2i)$ occurs with proportion

$$\frac{\binom{n}{i}}{2^n}$$

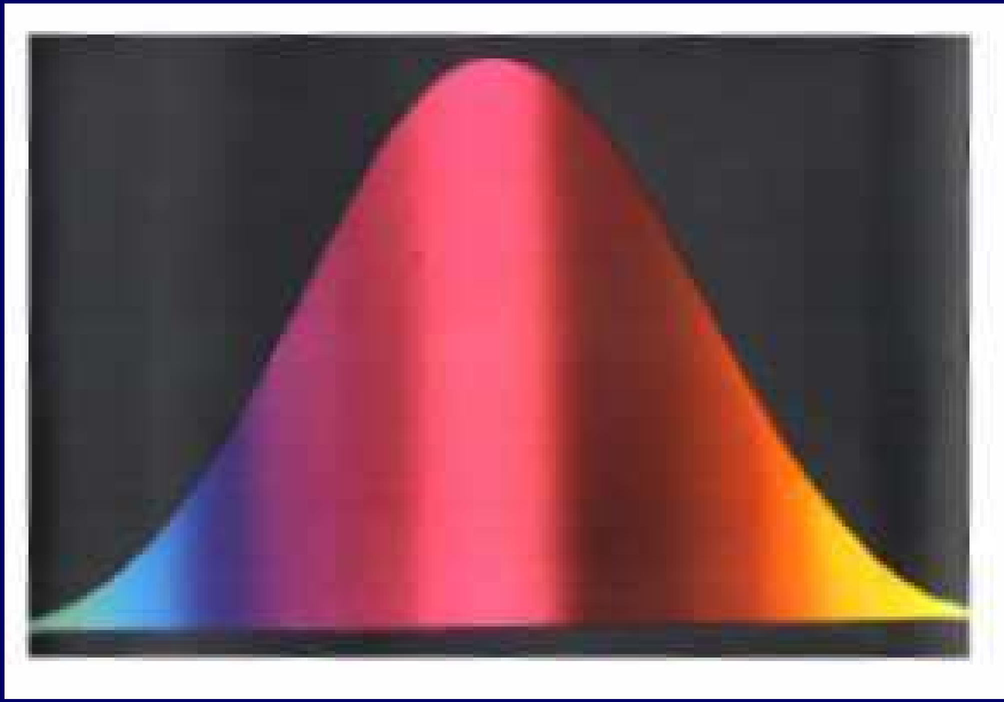
Unbiased Binomial Distribution On n+1 Elements.

Let S be any set $\{h_0, h_1, \ldots, h_n\}$ where each element h_i has an associated probability

$$\frac{\binom{n}{i}}{2^n}$$

Any such distribution is called an <u>Unbiased Binomial Distribution</u> or an <u>Unbiased Bernoulli Distribution</u>.

As the number of elements gets larger, the shape of the unbiased binomial distribution converges to a Normal (or Gaussian) distribution.

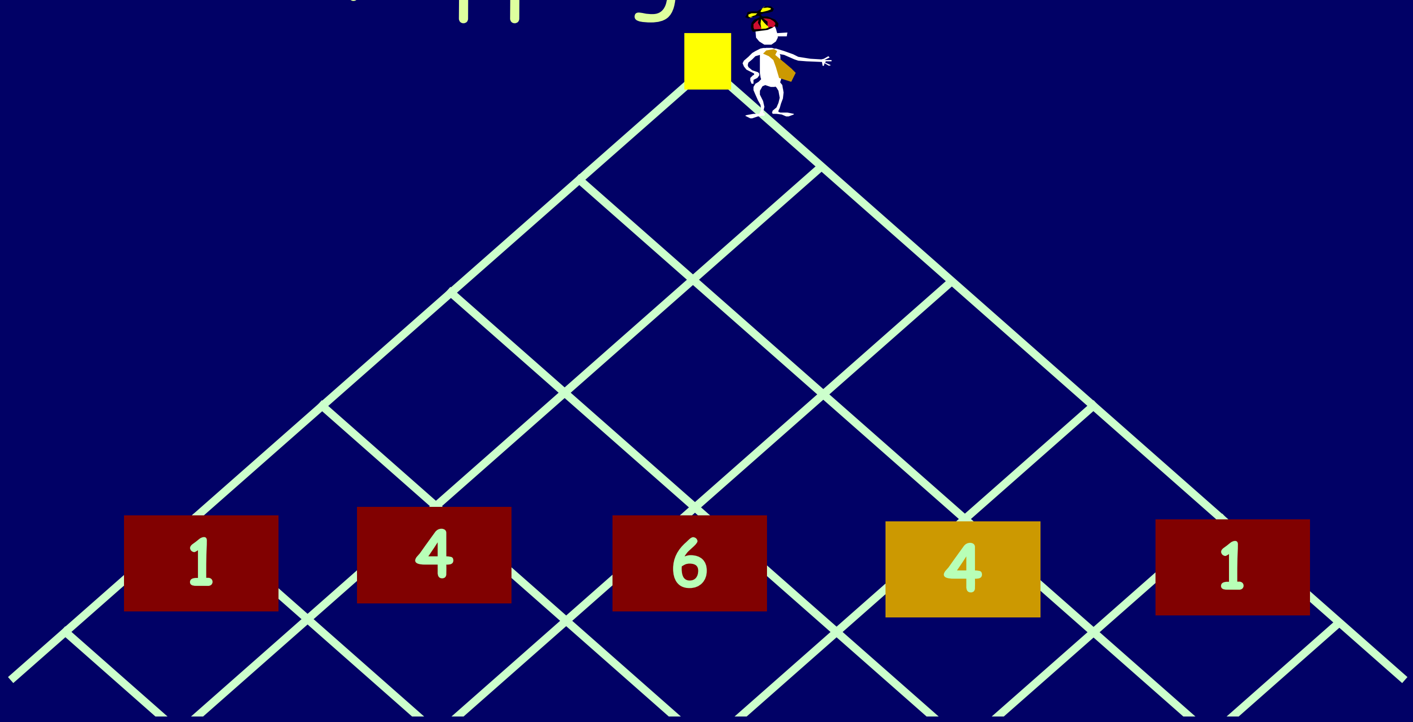


Mean

As the number of elements gets larger, the shape of the unbiased binomial distribution converges to a Normal (or Gaussian) distribution.

$$\frac{\binom{2m}{m-t}}{\binom{2m}{m}} \approx e^{-(t^2/m)}$$

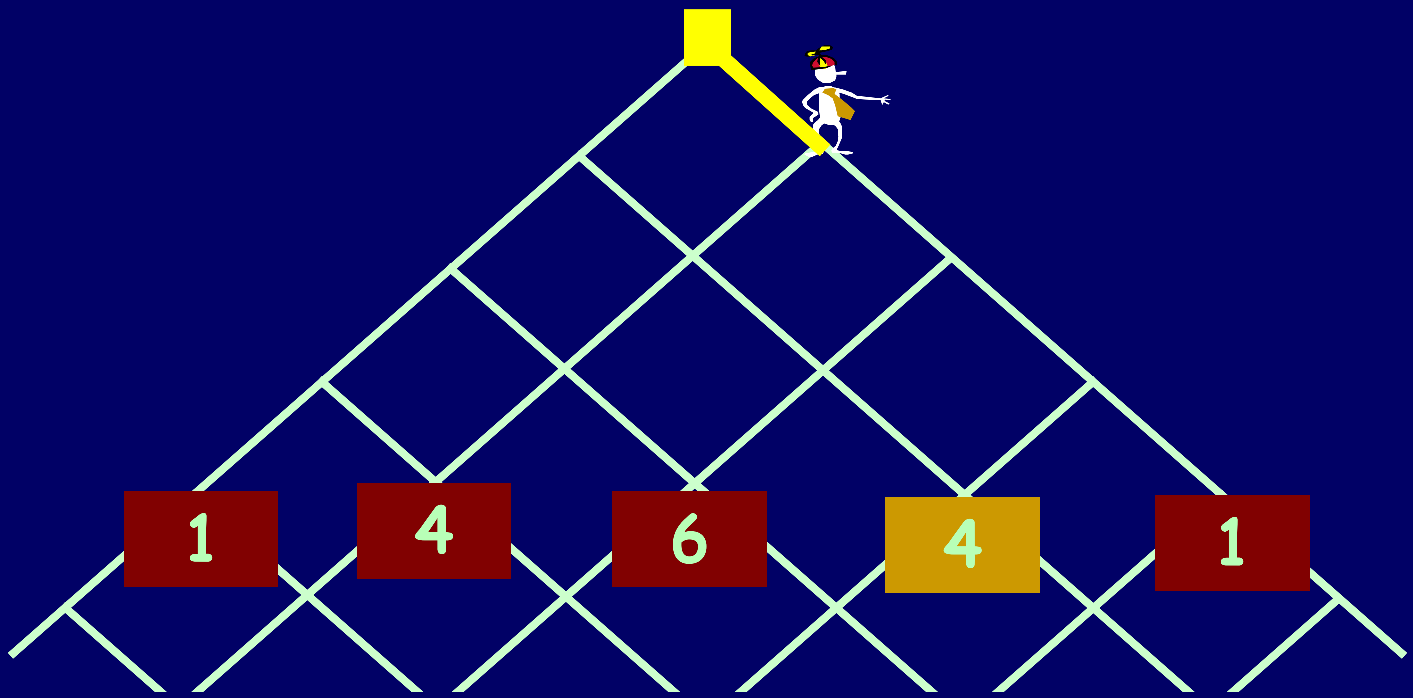
Coin Flipping in Manhattan



At each step, we flip a coin to decide which way to go.

What is the probability of ending at the intersection of Avenue i and Street (n-i) after n steps?

Coin Flipping in Manhattan



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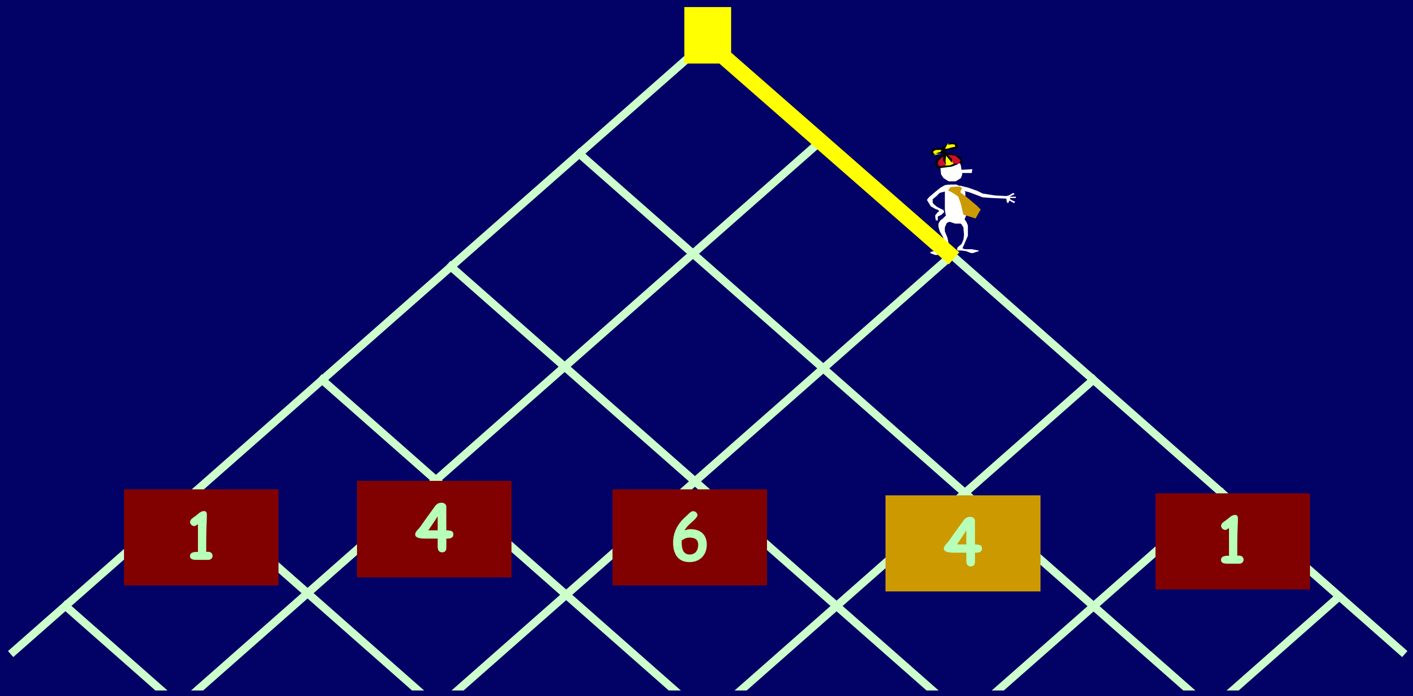

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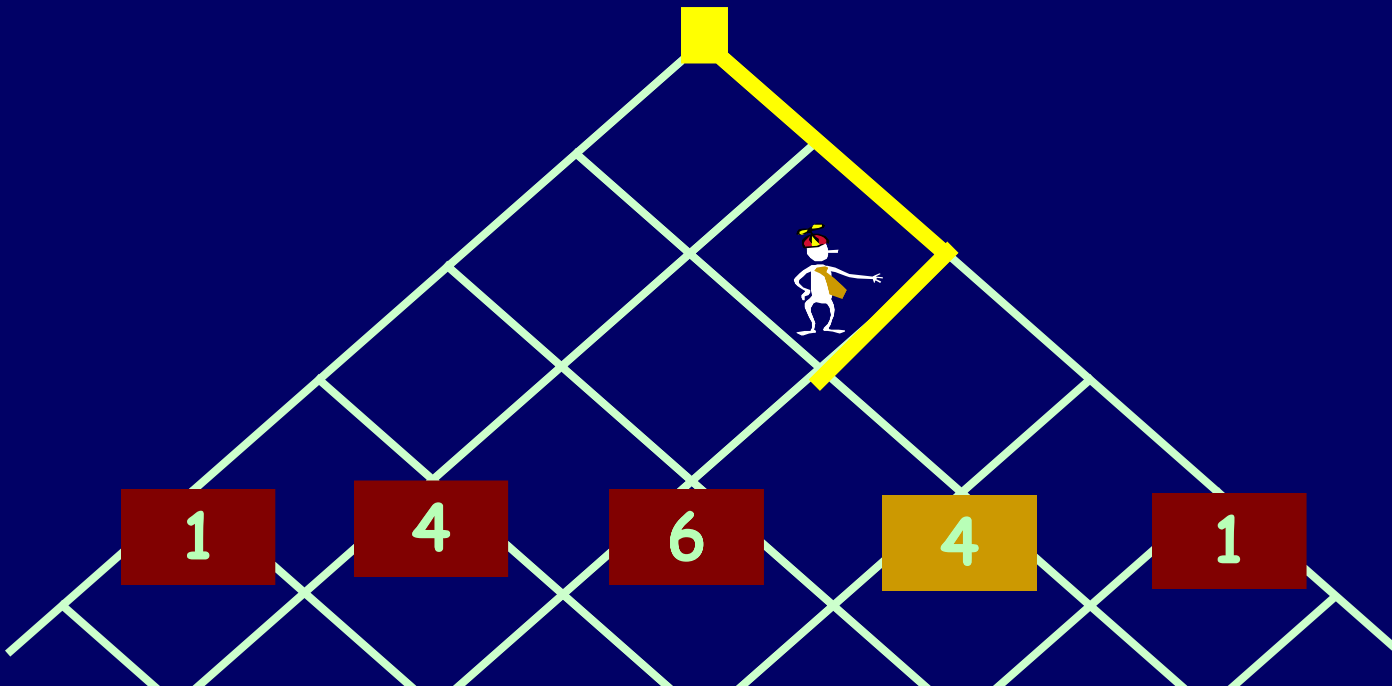

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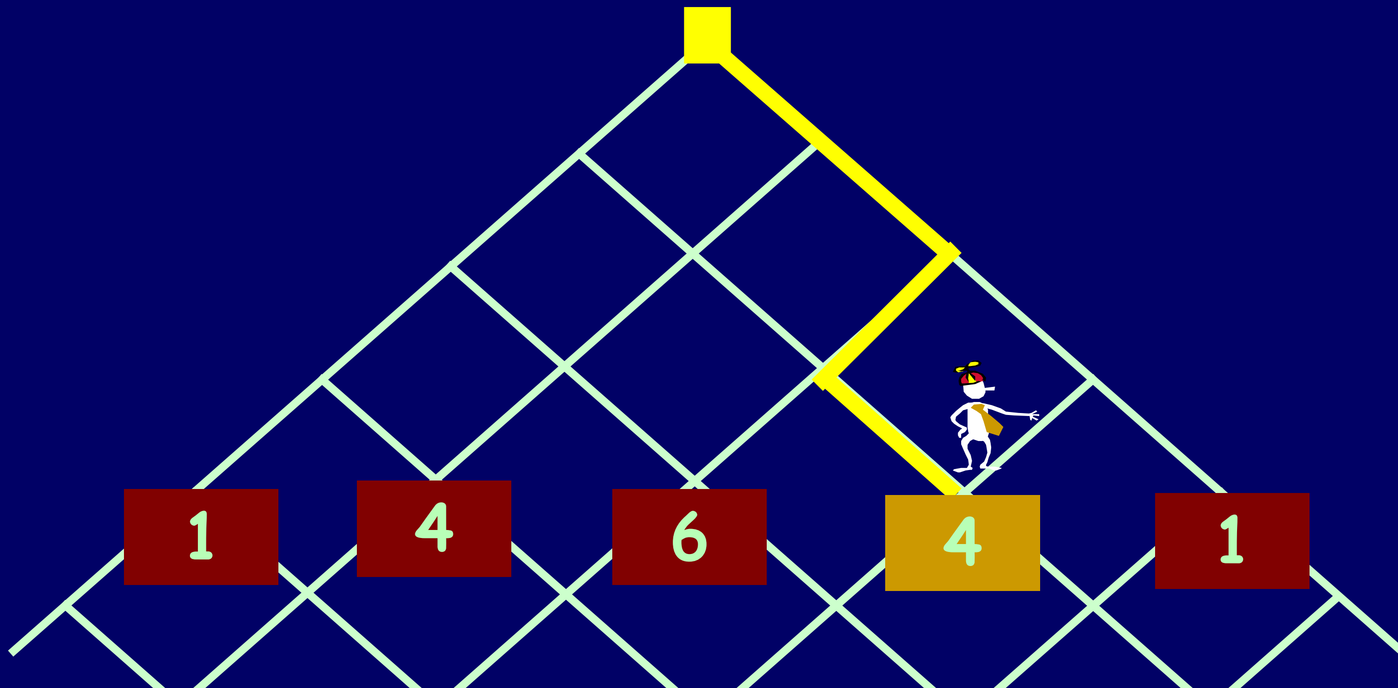

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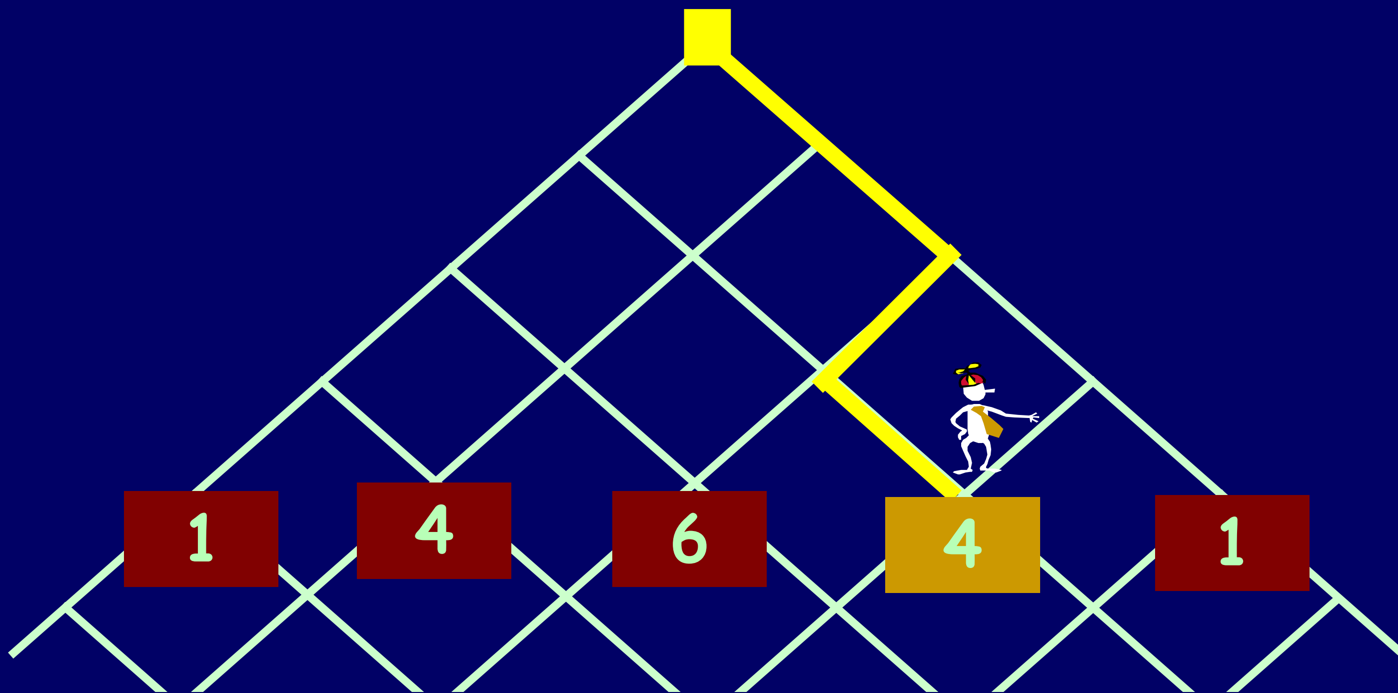
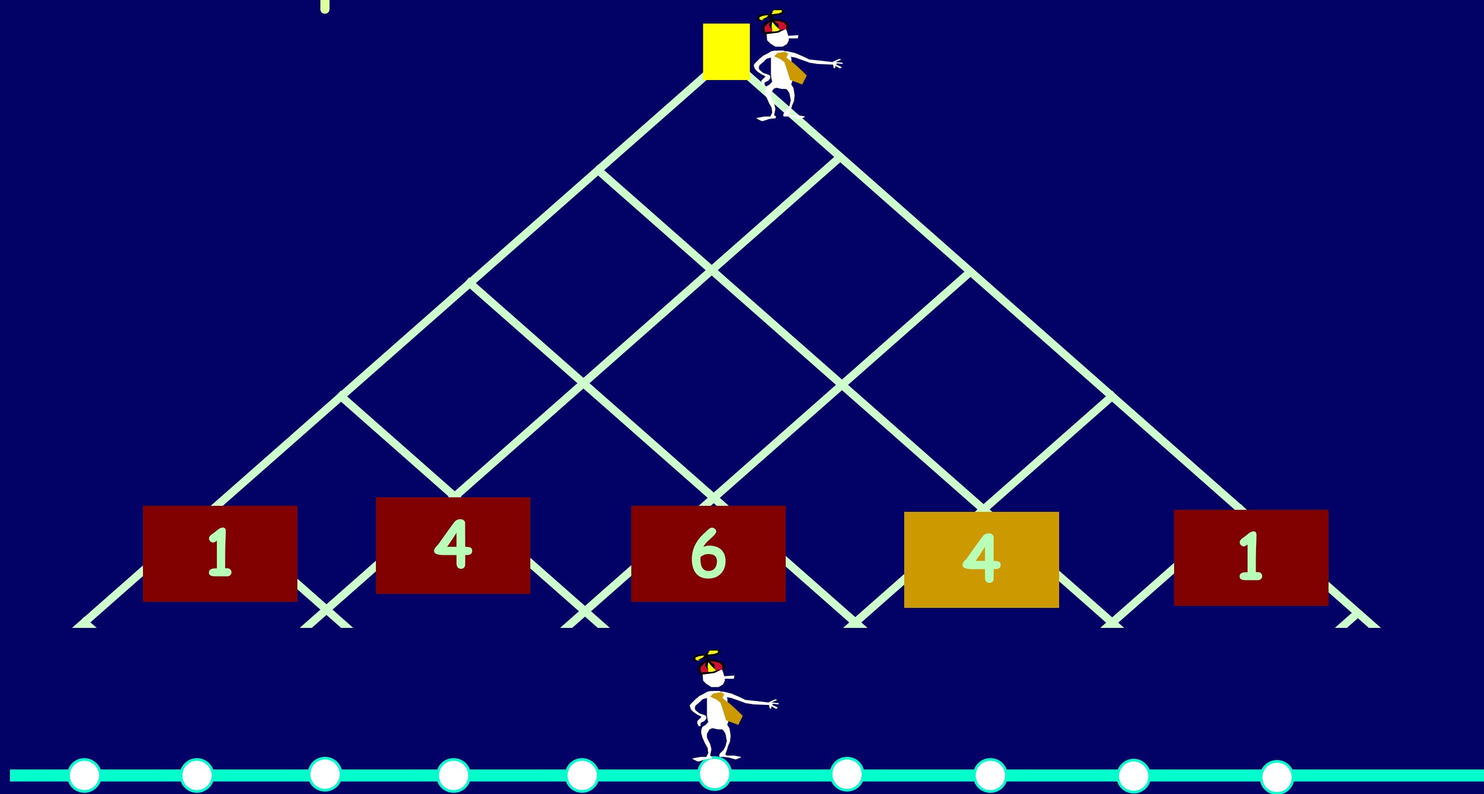


2^n different paths to level n, each equally likely.

The probability of i heads occurring on the path we generate is: $\dfrac{\binom{n}{i}}{2^n}$

n-step Random Walk on a line

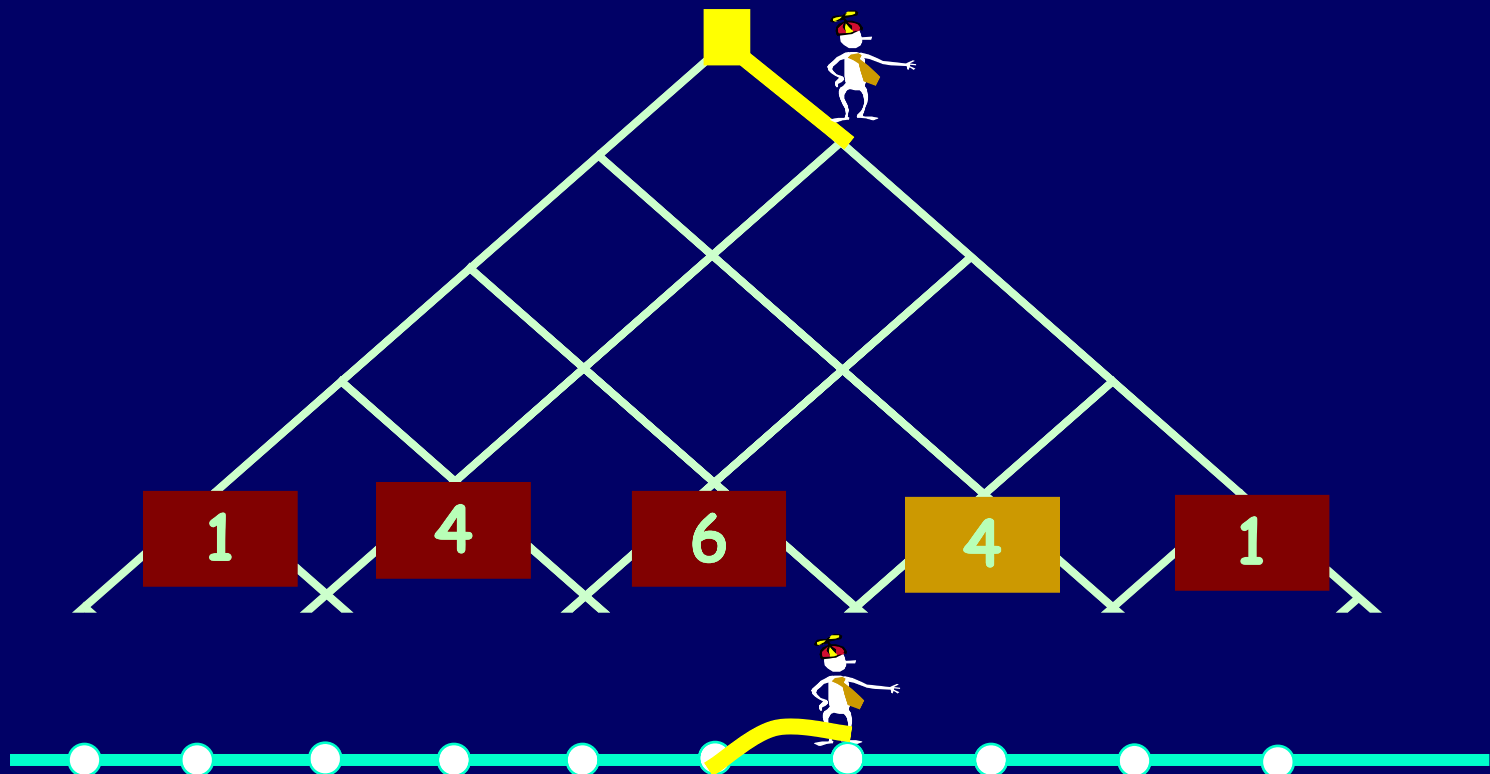


Start at the origin: at each point, flip an unbiased coin to decide whether to go right or left.

The probability that, in n steps, we take i steps to the right and n-i to the left (so we are at position 2i-n) is:

$$\frac{\binom{n}{i}}{2^n}$$

n-step Random Walk on a line



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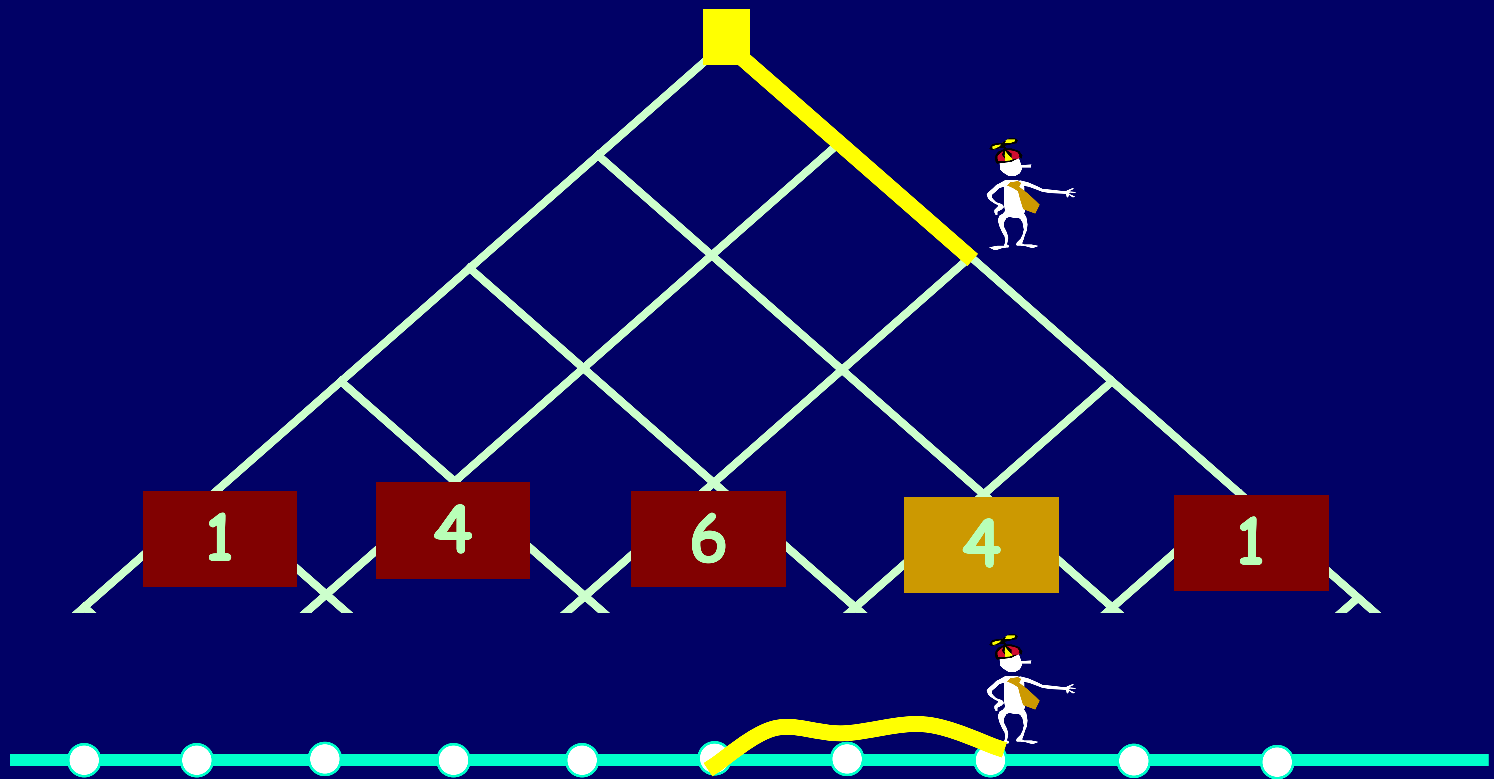

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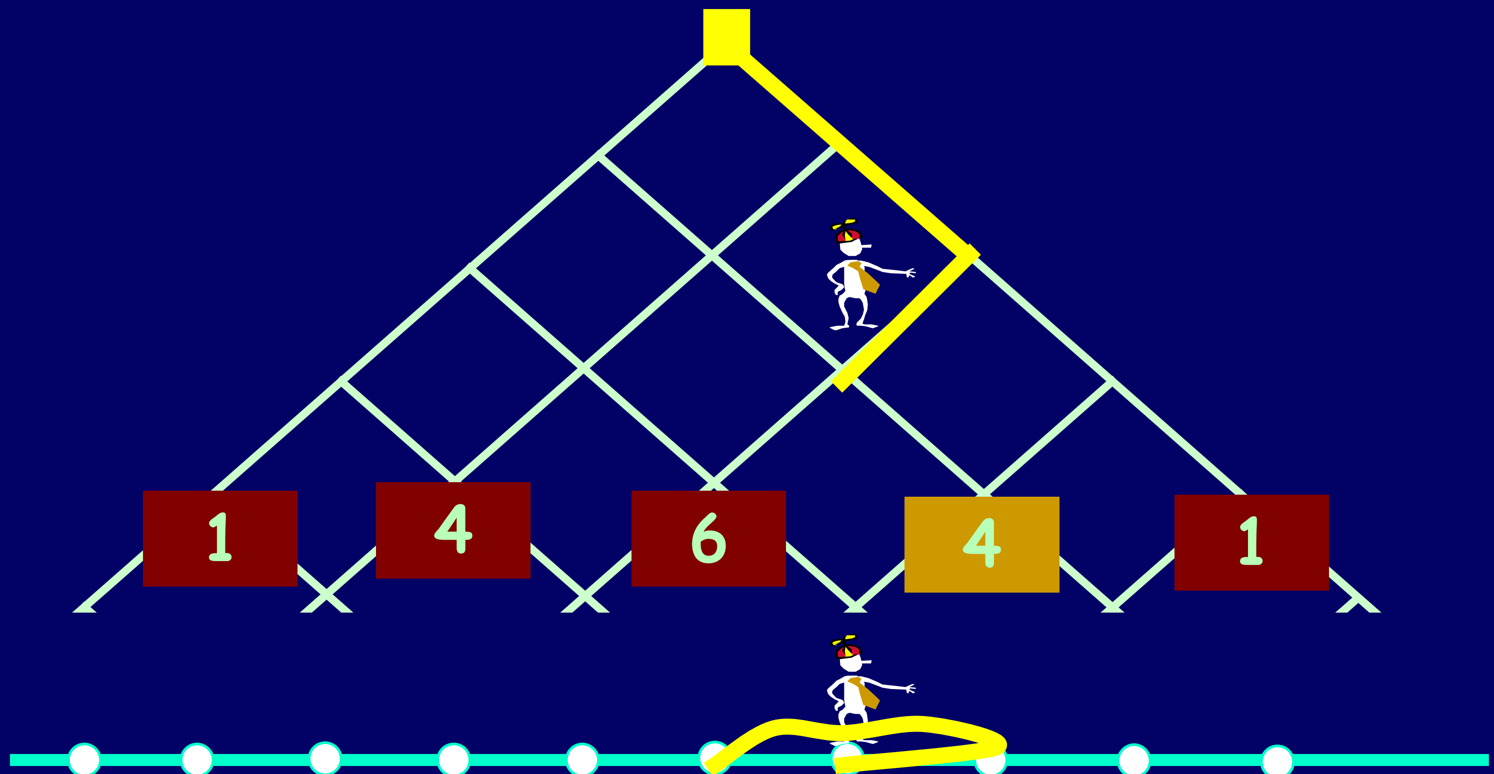

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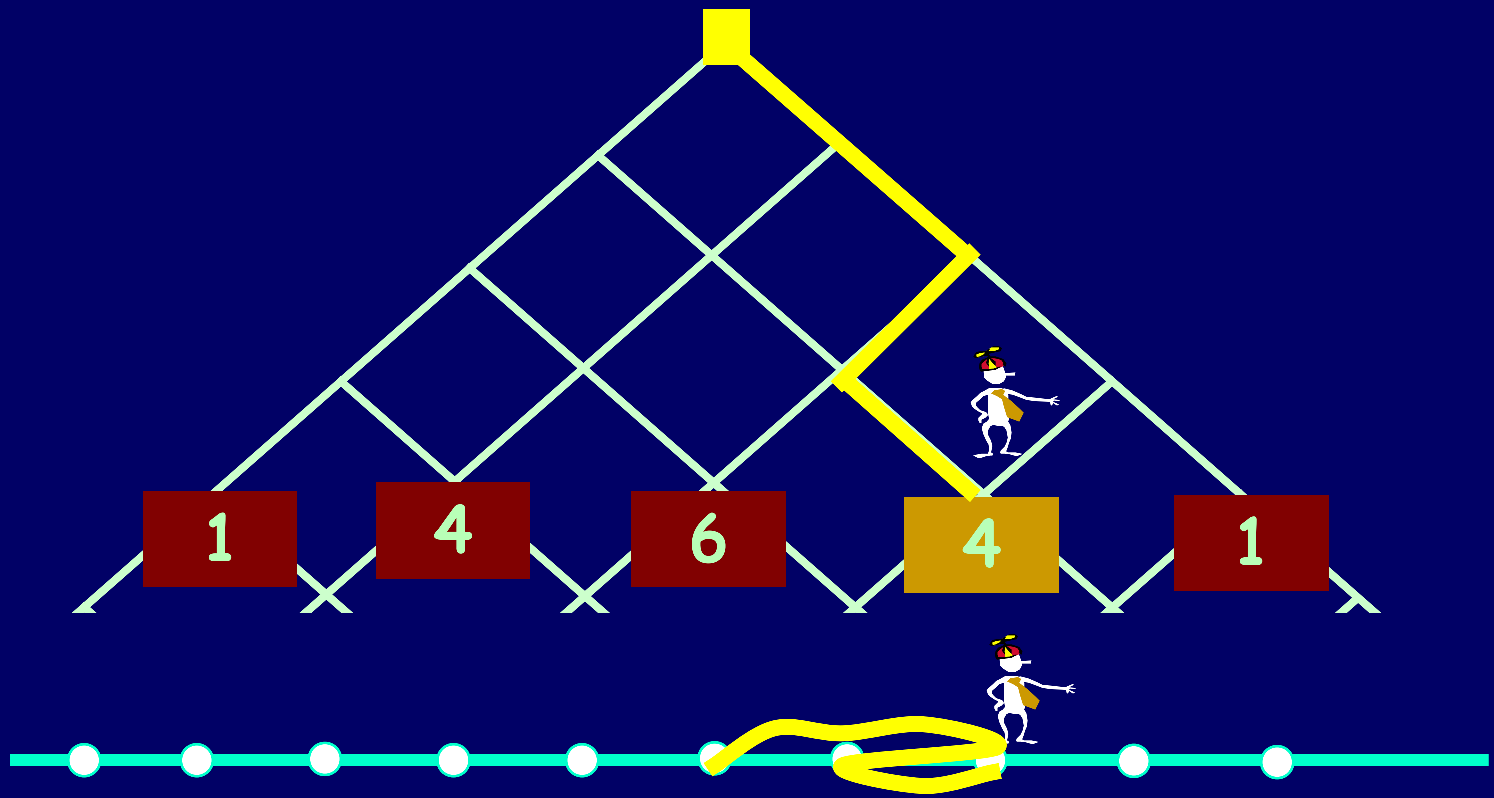

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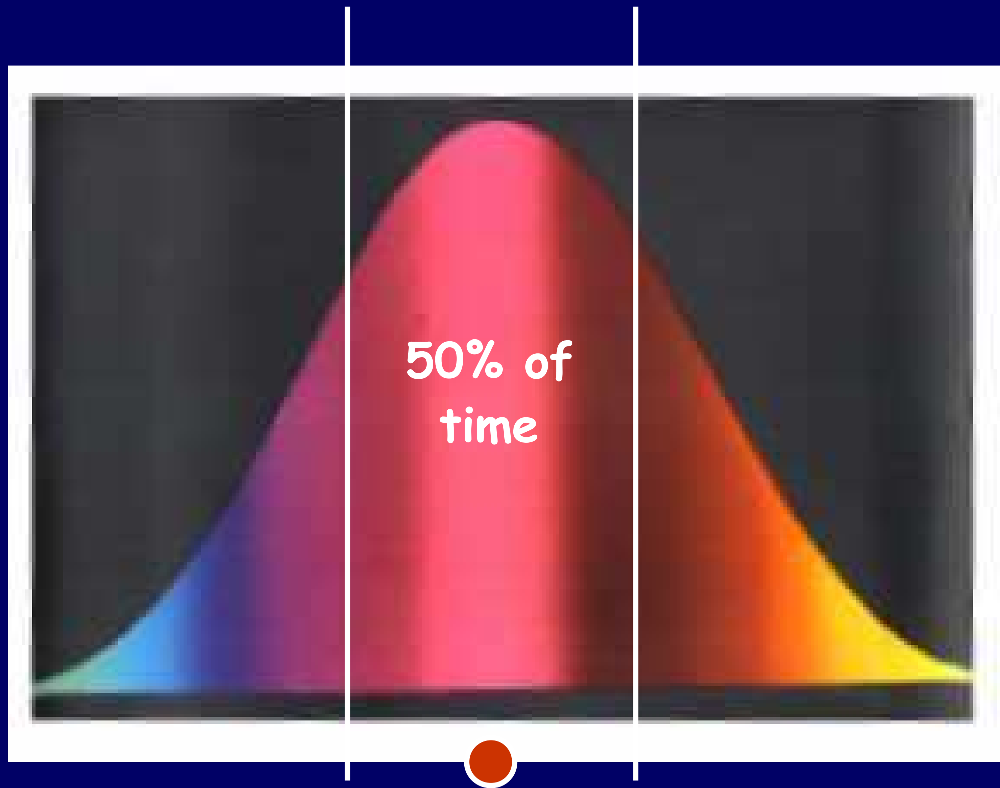
$$\frac{\binom{n}{i}}{2^n}$$

Again, a Normal or Gaussian distribution!

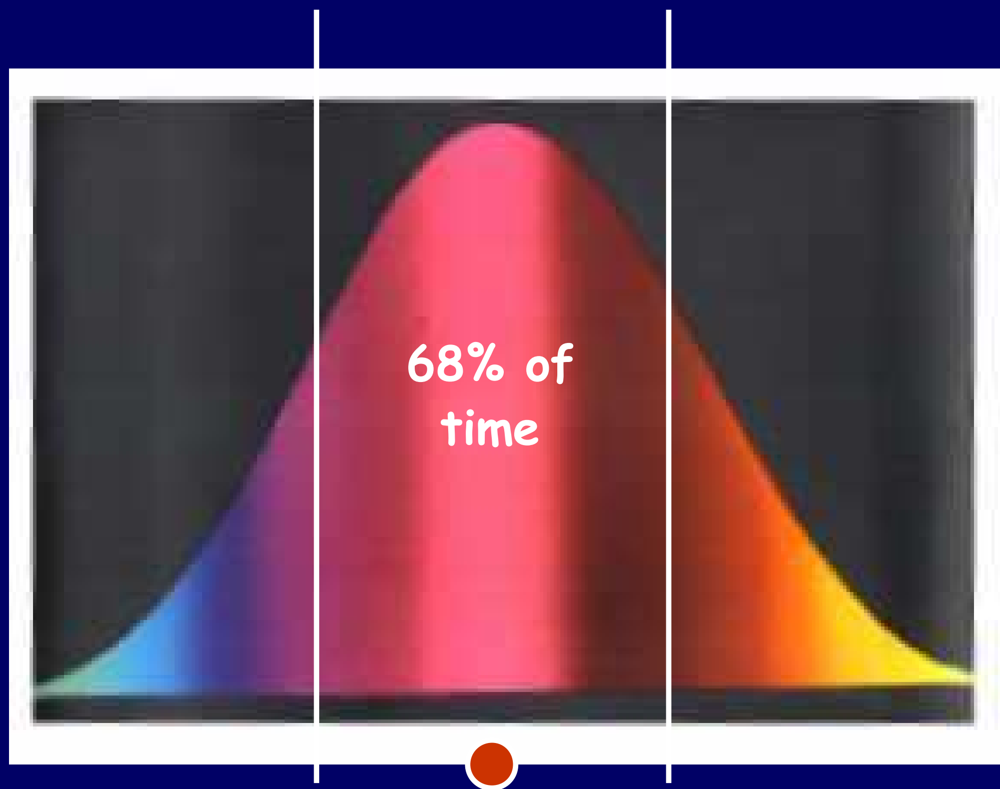
Let's look after n steps



$$\frac{2}{3}\sqrt{n}$$

Again, a Normal or Gaussian distribution!

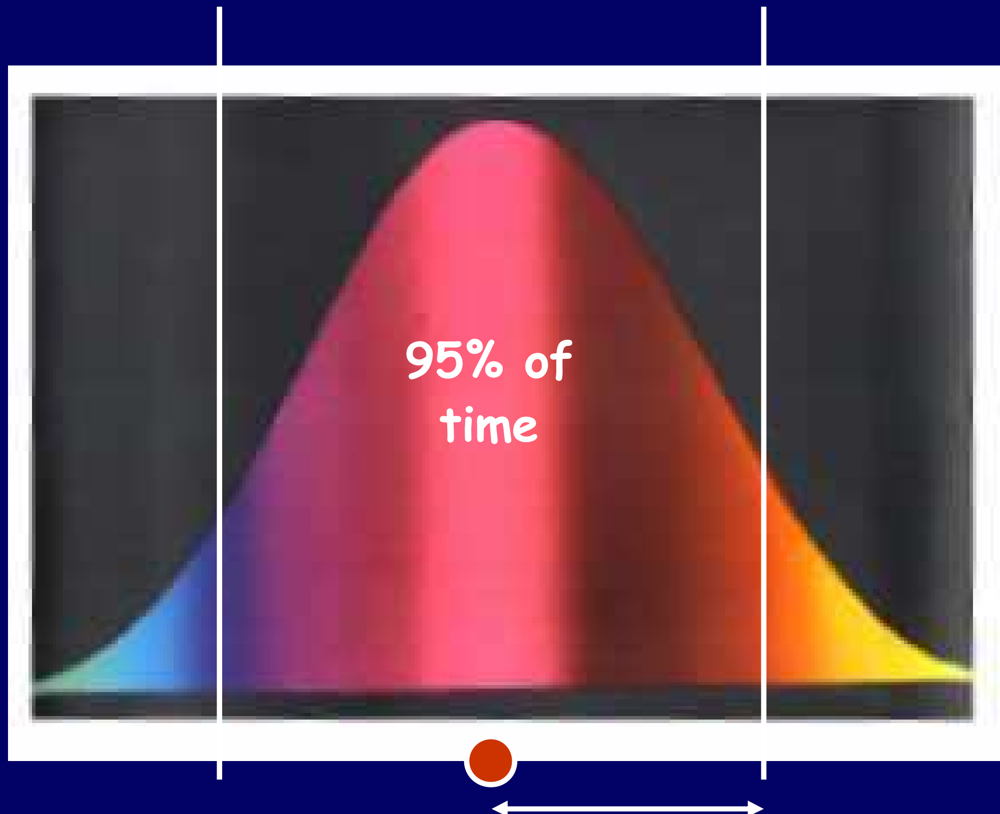
Let's look after n steps



$$\sqrt{n}$$

Again, a Normal or Gaussian distribution!

Let's look after n steps



$$2\sqrt{n}$$

Again, a Normal or Gaussian distribution!

Let's look after n steps

$3\sqrt{n}$

Probabilities and Counting
are intimately related ideas…

Probabilities and counting

Say we want to count

the number of X's with property P

One way to do it is to ask

"if we pick an X at random,
what is the probability it has property P?"

and then multiply by the number of X's.

$$\begin{bmatrix} \text{Probability of X} \\ \text{with prop. P} \end{bmatrix} = \frac{\text{(\# of X with prop. P)}}{\text{(total \# of X)}}$$

Probabilities and counting

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and then multiply by the number of X's.

$$\times \begin{bmatrix} \text{Probability of X} \\ \text{with prop. P} \end{bmatrix} = \frac{(\text{\# of X with prop. P})}{(\text{total \# of X})}$$

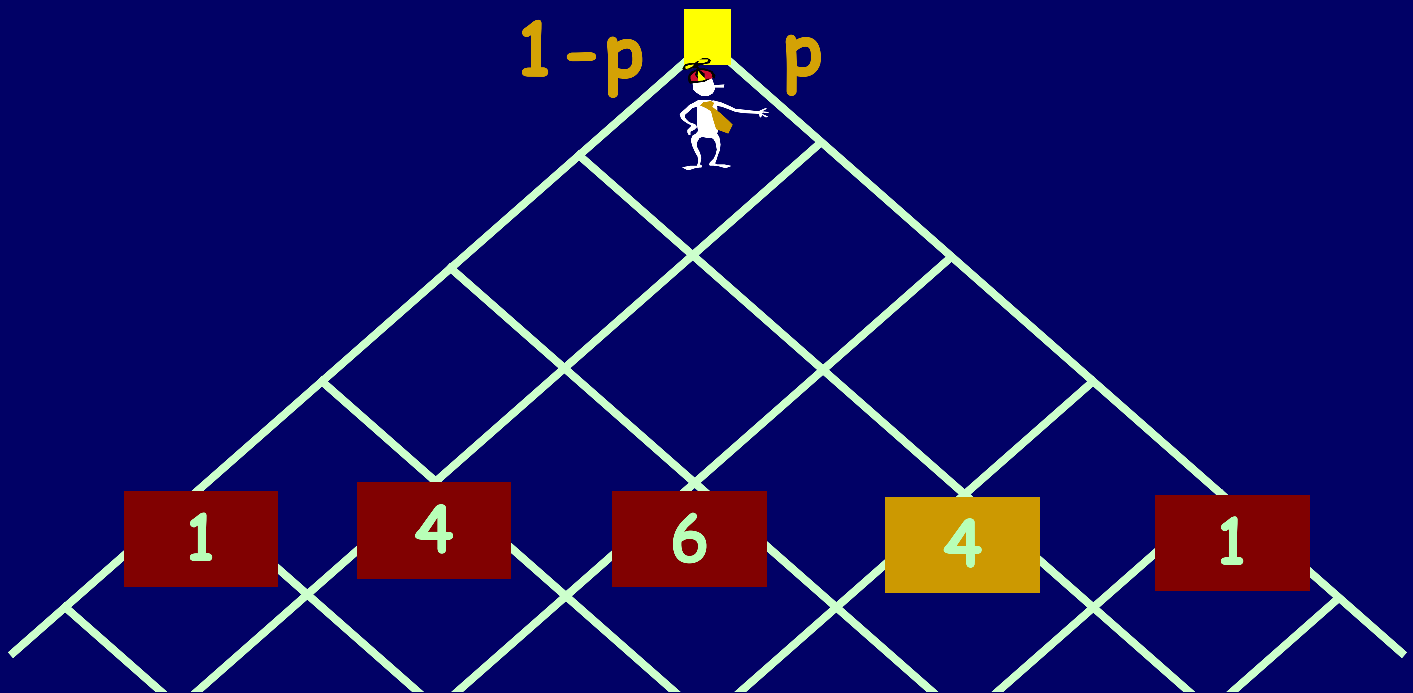
How many n-bit strings have an even number of 1's?

If you flip a coin n times, what is the probability you get an even number of heads? Then multiply by 2^n.

Say prob was q after $n-1$ flips.

$$(\text{total \# of X}) \times \left[\begin{array}{c}\text{Probability of X}\\ \text{with prop. P}\end{array}\right] = (\text{\# of X with prop. P})$$

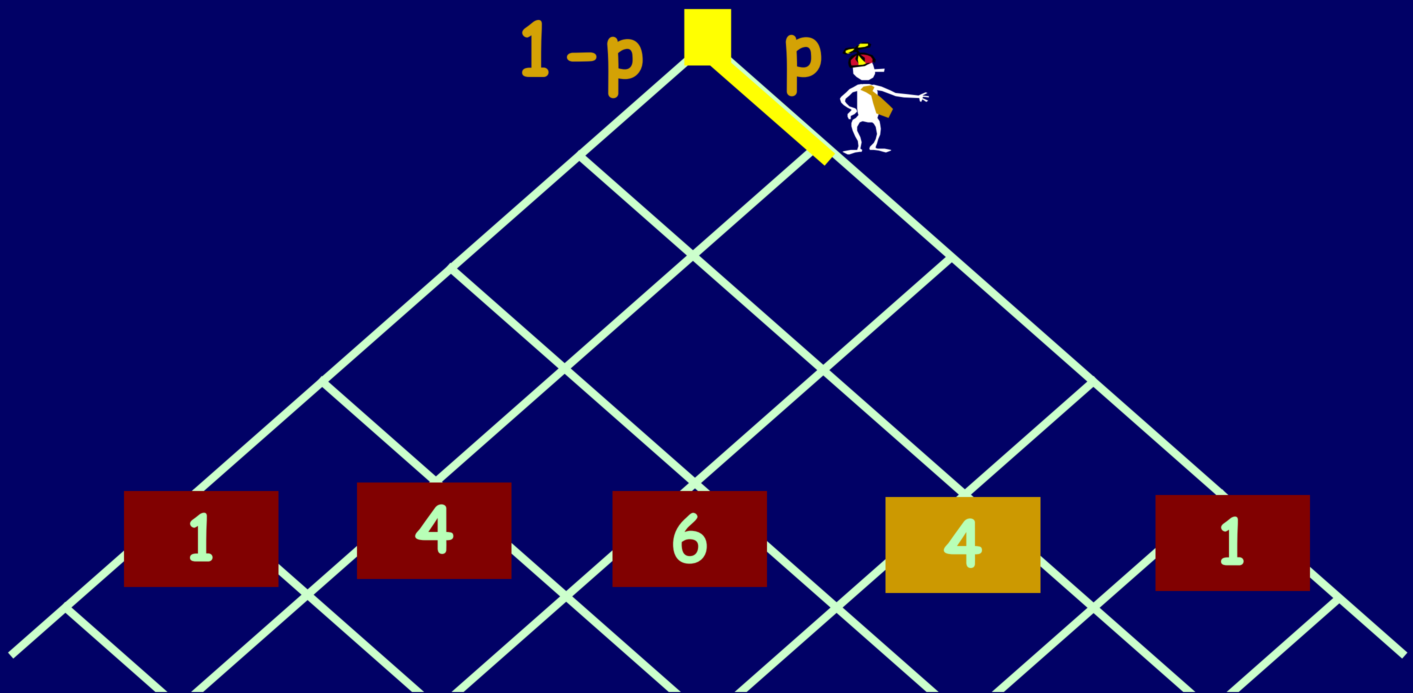
Binomial distribution with bias p



Start at the top. At each step, flip a coin with a bias p of heads to decide which way to go.

What is the probability of ending at the intersection of Avenue i and Street (n-i) after n steps?

Binomial distribution with bias p



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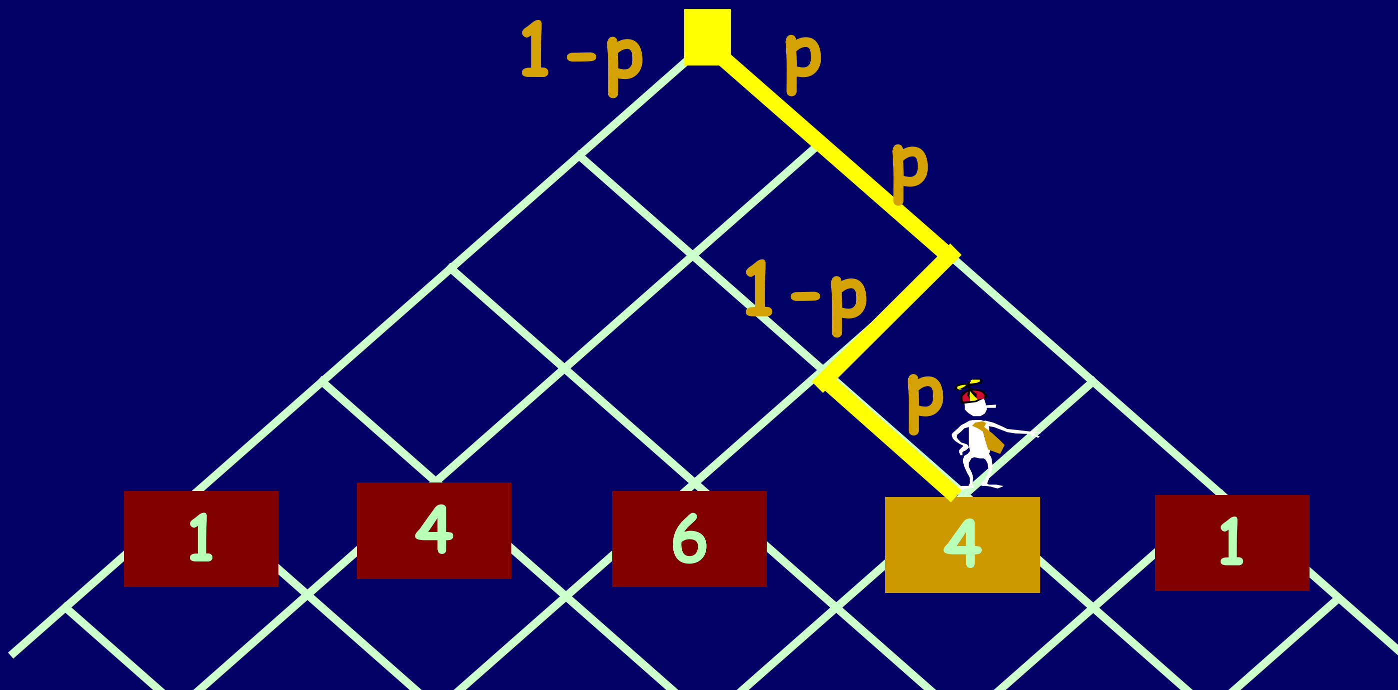

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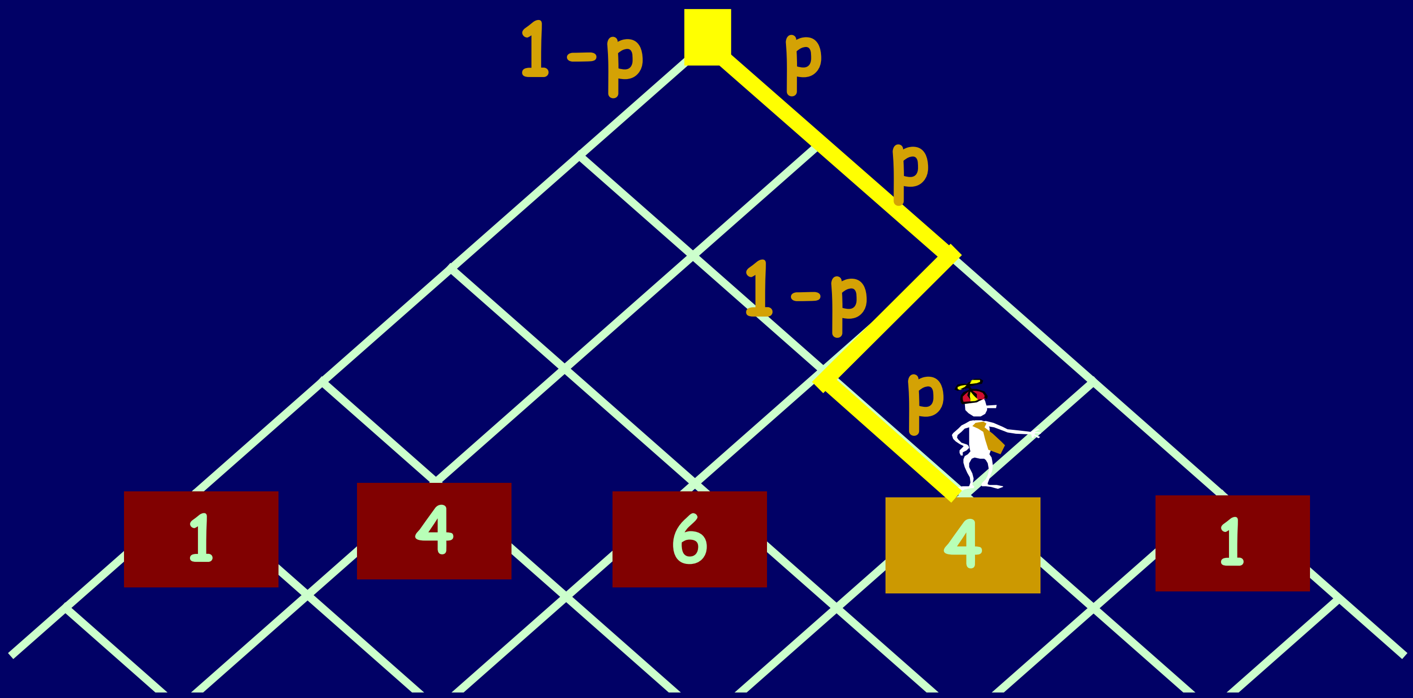
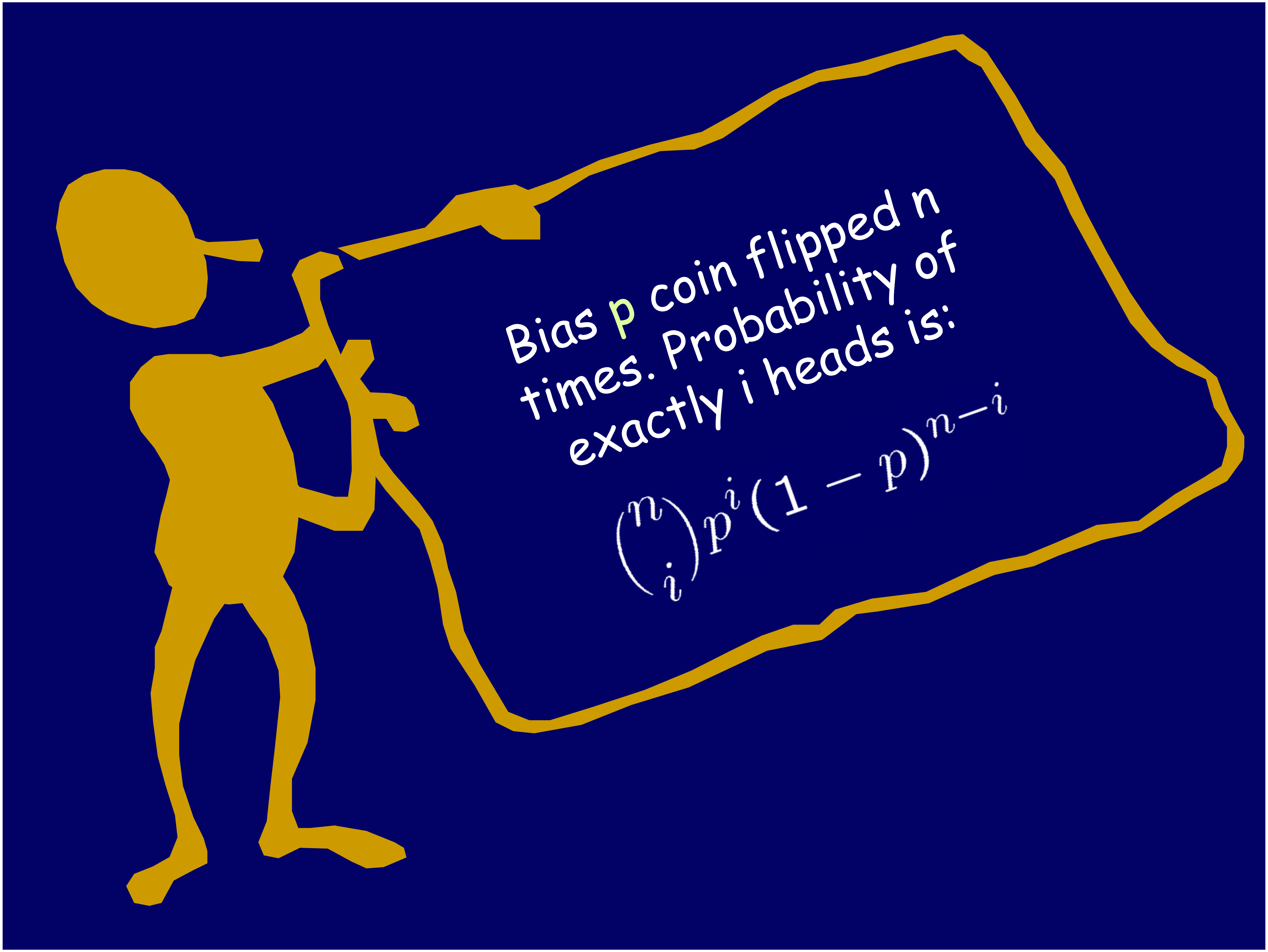
Binomial distribution with bias p



Start at the top. At each step, flip a coin with a bias p of heads to decide which way to go.

The probability of any fixed path with i heads (n-i tails) being chosen is: $p^i (1-p)^{n-i}$

Overall probability we get i heads is: $\binom{n}{i} p^i (1-p)^{n-i}$


Bias p coin flipped n times. Probability of exactly i heads is:
$\binom{n}{i} p^i (1-p)^{n-i}$

How many n-trit strings have even number of 0's?

If you flip a bias 1/3 coin n times, what is the probability q_n you get an even number of heads? Then multiply by 3^n. [Why is this right?]

Then $q_0=1$.

Say probability was q_{n-1} after $n-1$ flips.

Then, $q_n = \boxed{(2/3)q_{n-1}} + \boxed{(1/3)(1-q_{n-1})}$.

$p_n = q_n - \frac{1}{2}$

$\Rightarrow p_n = 1/3\ p_{n-1}$
and $p_0 = \frac{1}{2}$.

Rewrite as: $q_n - \frac{1}{2} = 1/3(q_{n-1} - \frac{1}{2})$

So, $q_n - \frac{1}{2} = (1/3)^n \frac{1}{2}$. Final count = $\frac{1}{2} + \frac{1}{2}3^n$

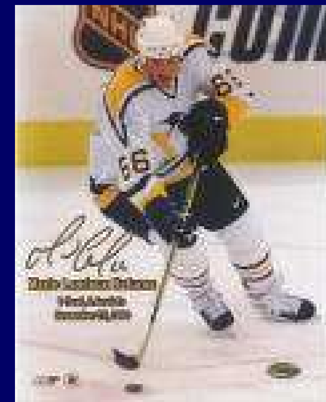
Poisson process (buses in the snow)

Limit of:

- Bus arrives every 20 minutes.
- Every 10 min there is a ½ chance of a bus arriving.
- Every minute there is a 1/20 chance of bus arriving.
- Every second there is a 1/1200 chance of bus arriving.
- …

Might then look at distribution of # buses in given time period, or waiting time for next bus, etc.

Some puzzles



Teams A and B are equally good.

In any one game, each is equally likely to win.

What is most likely length of a "best of 7" series?

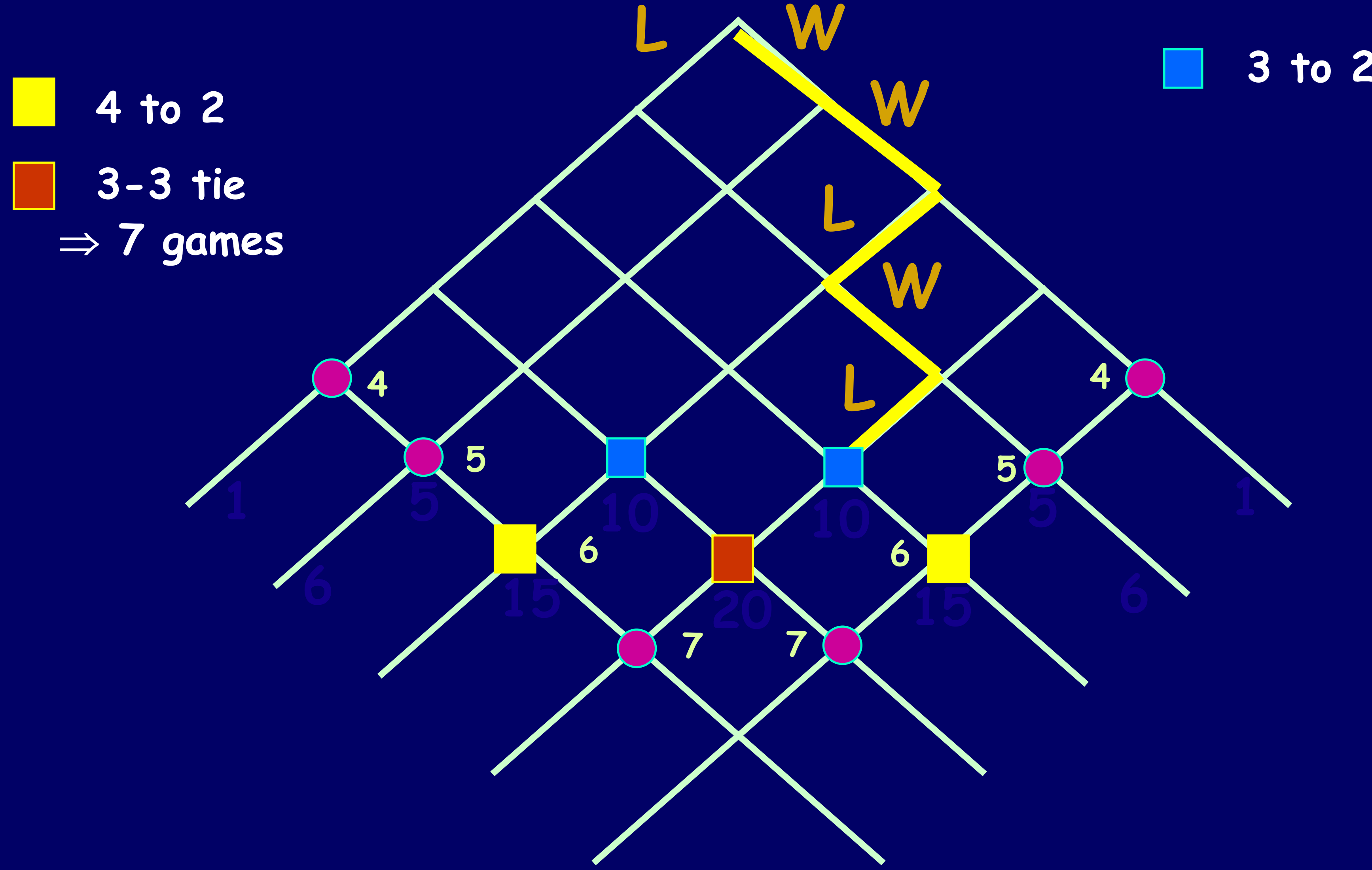
Flip coins until either 4 heads or 4 tails. Is this more likely to take 6 or 7 flips?

Actually, 6 and 7 are equally likely

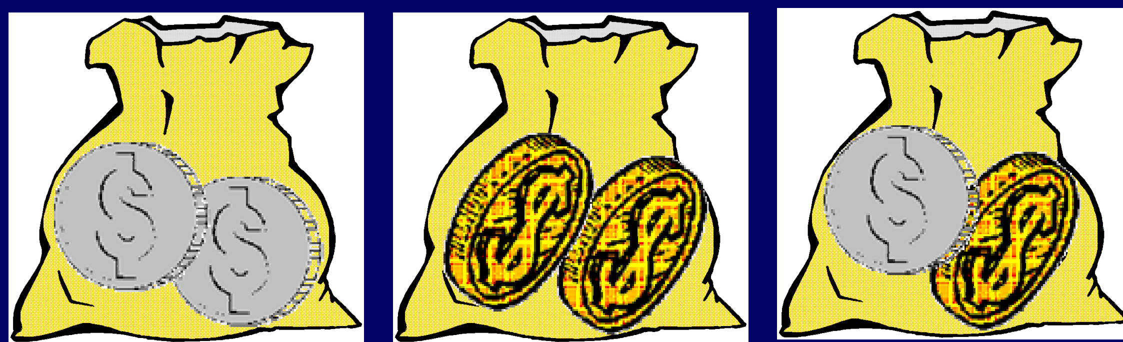
To reach either one, after 5 games, it must be 3 to 2.

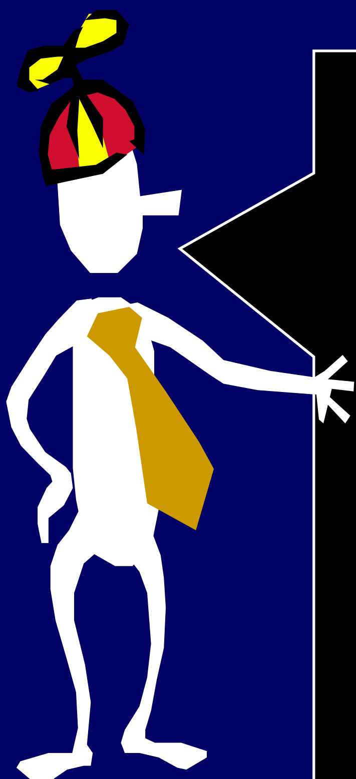
½ chance it ends 4 to 2. ½ chance it doesn't.

Another view



Silver and Gold

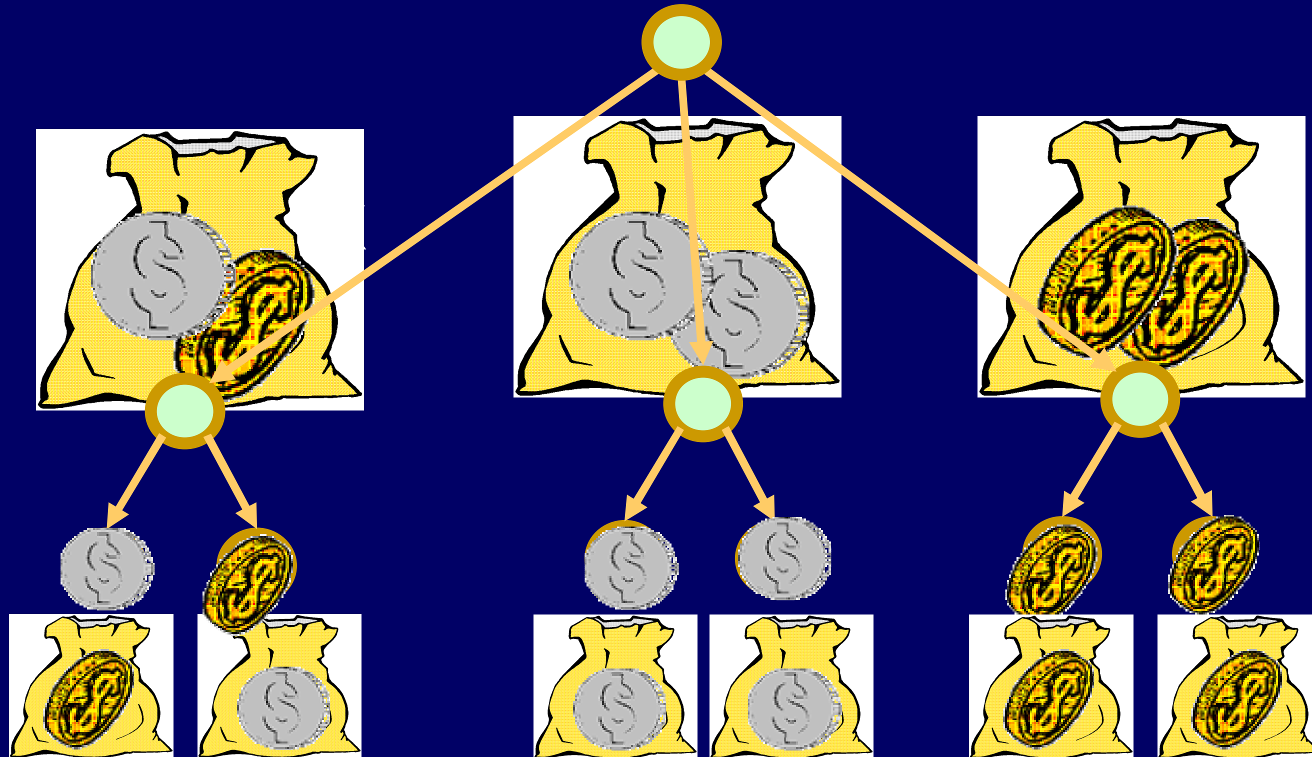


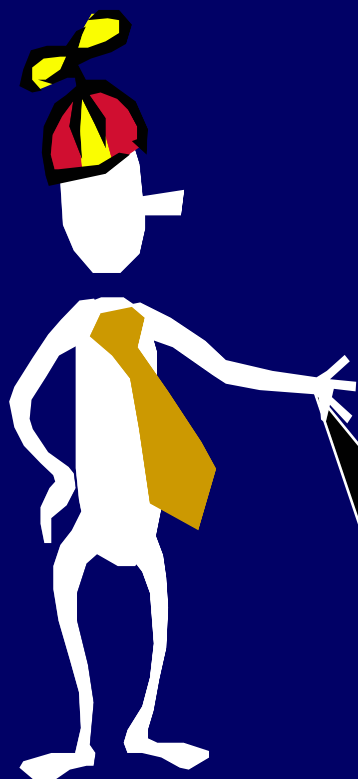


One bag has two silver coins, another has two gold coins, and the third has one of each.

One of the three bags is selected at random. Then one coin is selected at random from the two in the bag. It turns out to be gold.

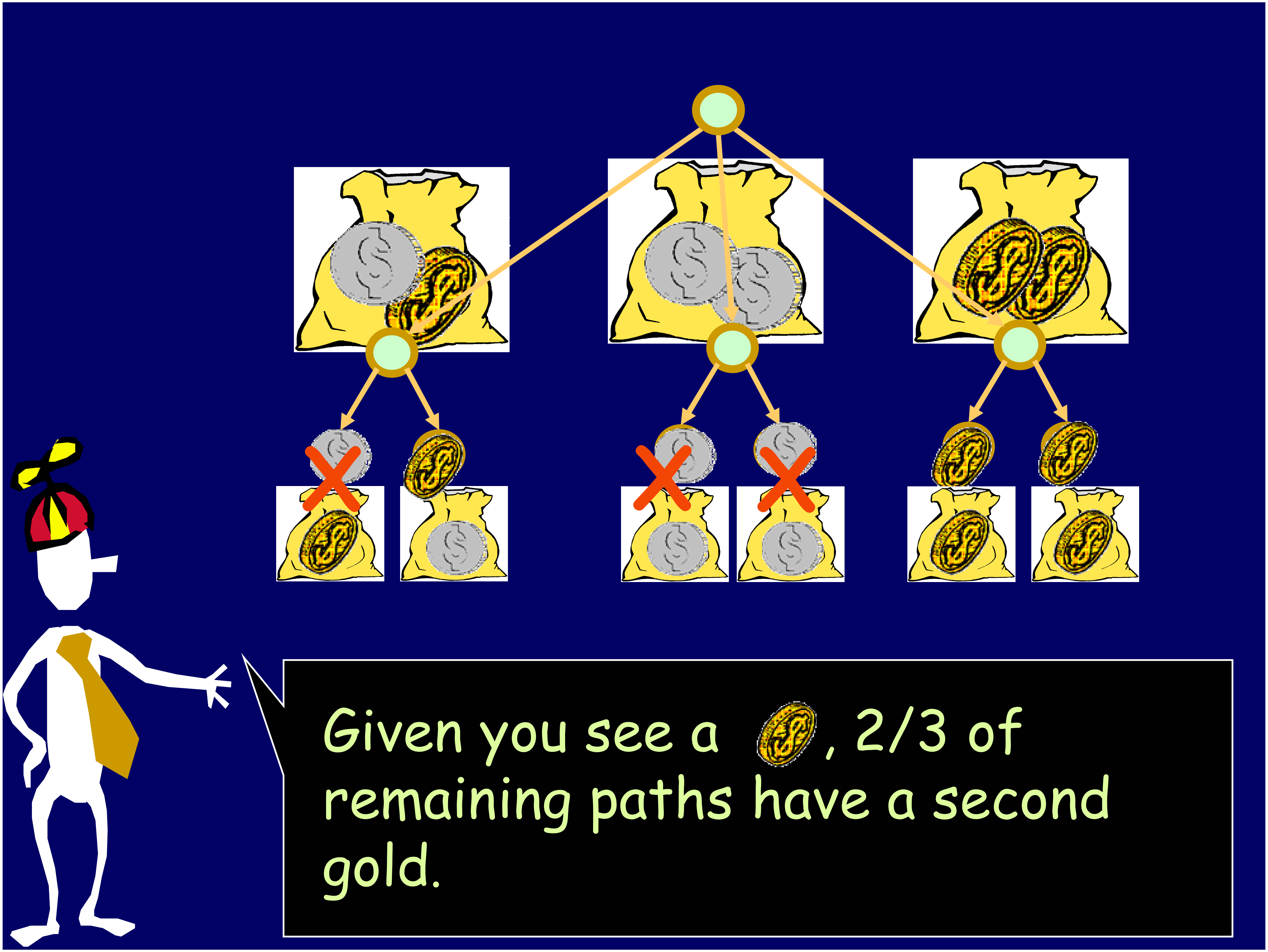
What is the probability that the other coin is gold?





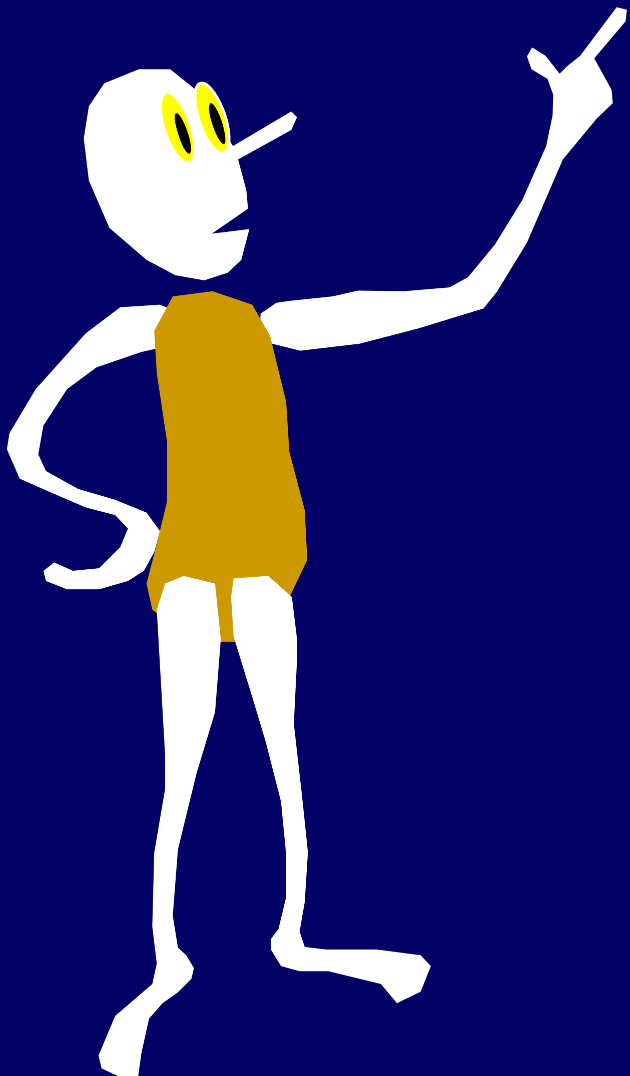
3 choices of bag
2 ways to order bag contents

6 equally likely paths.


Given you see a , 2/3 of remaining paths have a second gold.

So, sometimes, probabilities can be counter-intuitive

Language Of Probability



The formal language of probability is a very important tool in describing and analyzing probability distributions.

Finite Probability Distribution

A (finite) <u>probability distribution</u> D is a finite set S of elements, where each element x in S has a positive real <u>weight</u>, p<u>roportion</u>, or <u>probability</u> p(x).

The weights must satisfy:

$$\sum_{x \in S} p(x) = 1$$

Finite Probability Distribution

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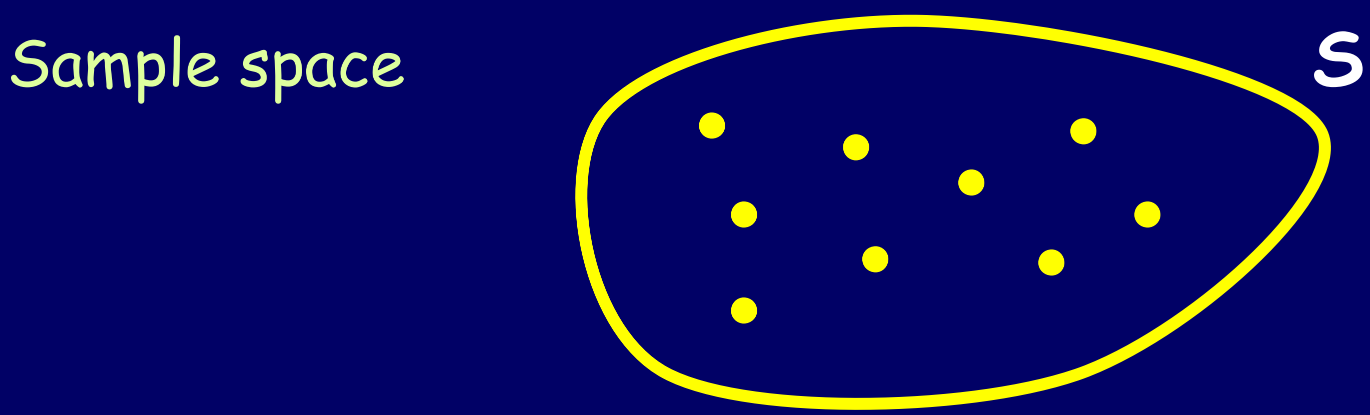
For notational convenience we will define D(x) = p(x).

S is often called the <u>sample space</u> and elements x in S are called <u>samples</u>.

Sample space

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).



Probability

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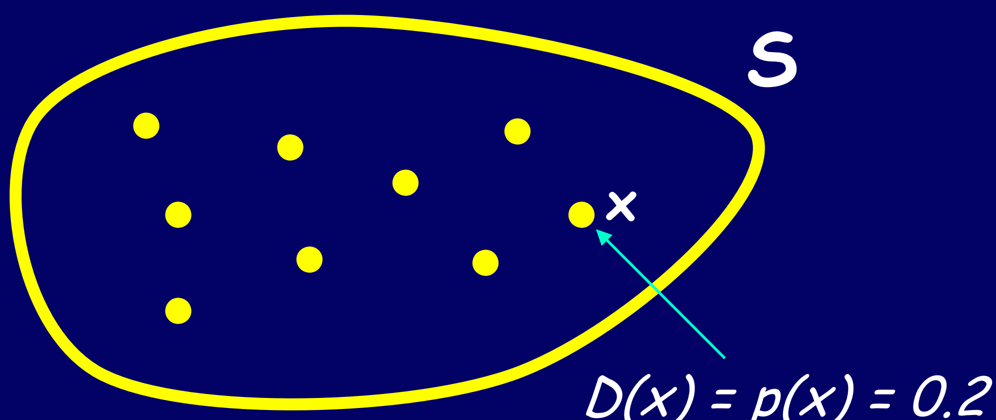
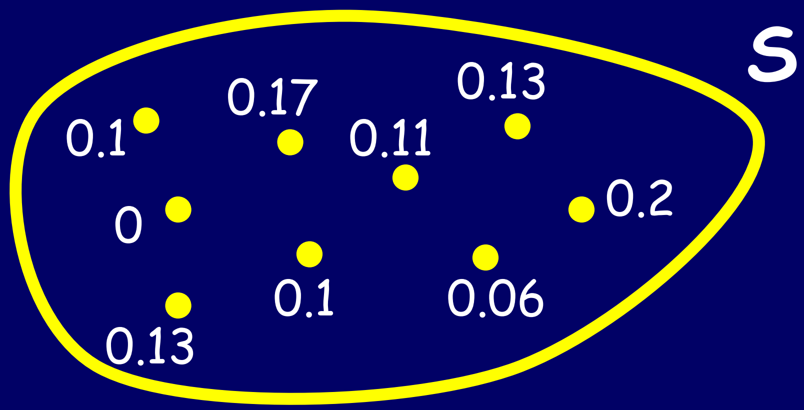


<u>weight</u> or <u>probability</u> of x

Probability Distribution

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

weights must sum to 1



Events

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

Any set $E \subset S$ is called an event.

The probability of event E is

$$Pr_D[E] = \sum_{x \in E} p(x)$$

Events

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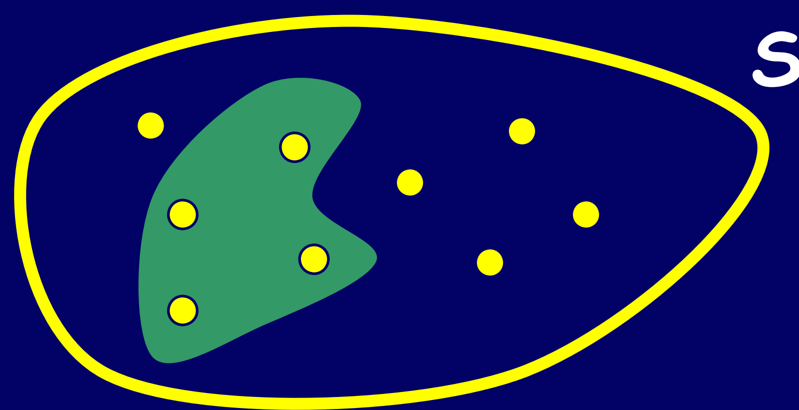

Event E

Events

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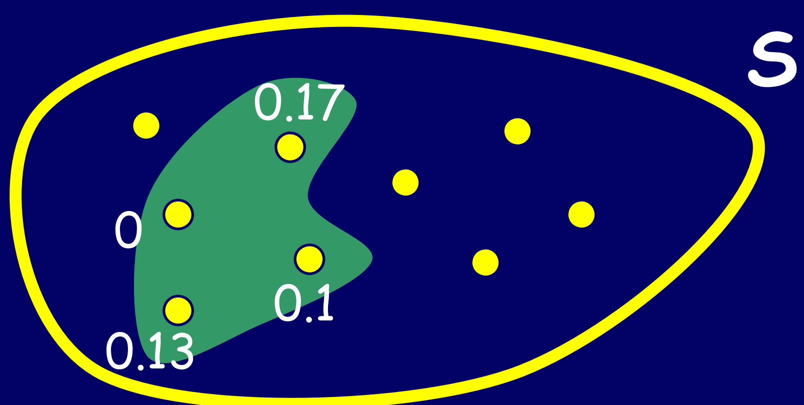
$\Pr_D[E] = 0.4$



Uniform Distribution

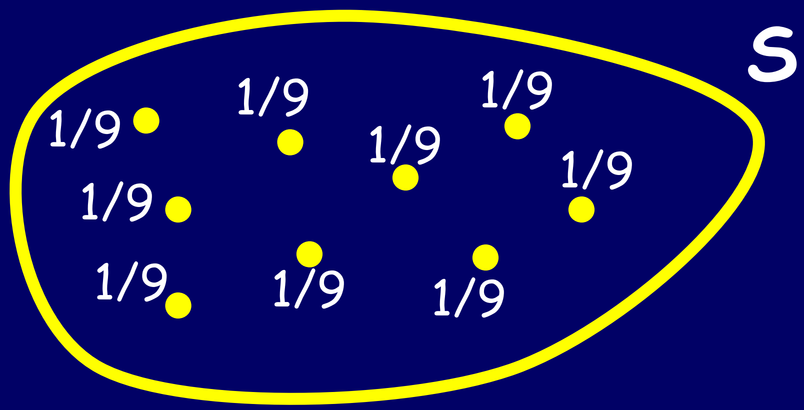
A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).

If each element has equal probability, the distribution is said to be uniform.

$$Pr_D[E] = \sum_{x \in E} p(x) = \frac{|E|}{|S|}$$

Uniform Distribution

A (finite) probability distribution D is a finite set S of elements, where each element x in S has a positive real weight, proportion, or probability p(x).



Each p(x) = 1/9.

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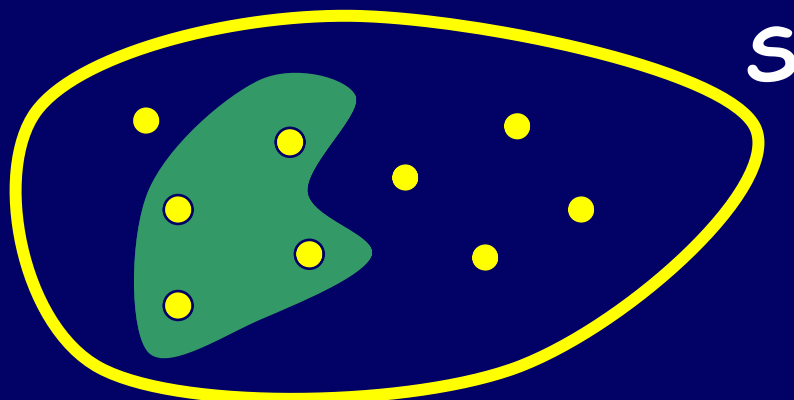
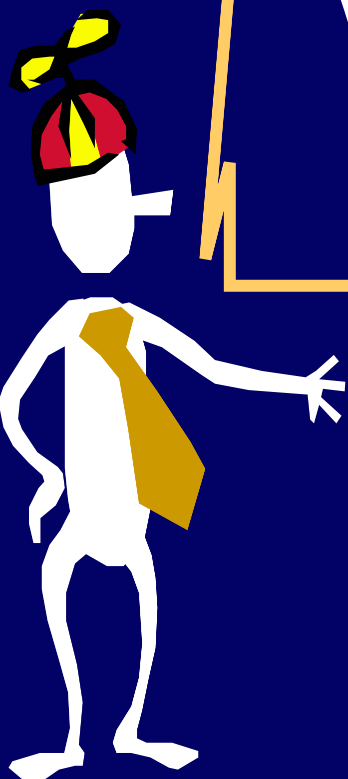


$\Pr_D[E] = |E|/|S|$
$= 4/9$



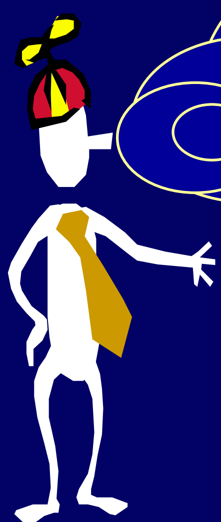
A fair coin is tossed 100 times in a row.

What is the probability that we get exactly half heads?

Using the Language

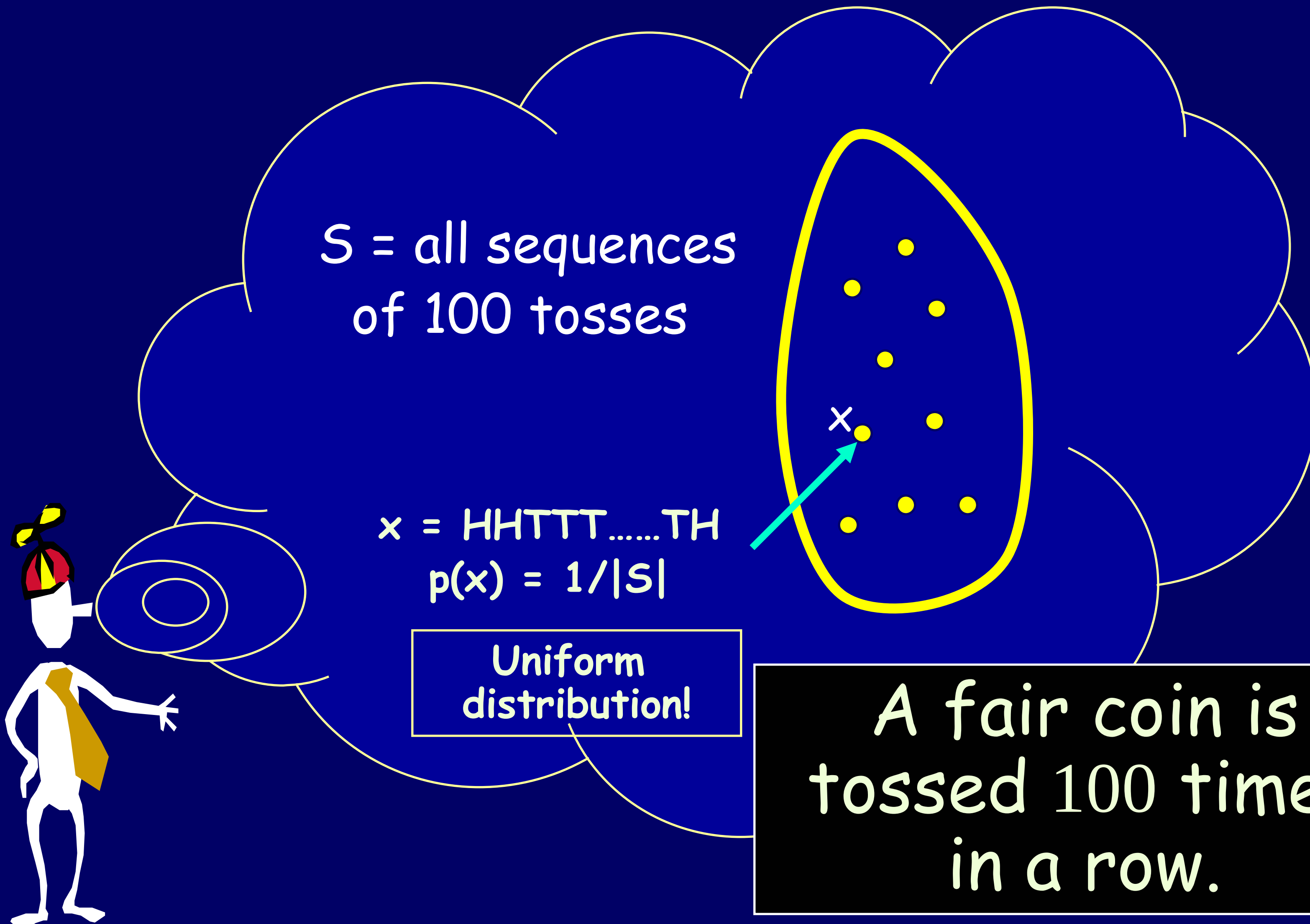
The sample space S is the set of all outcomes $\{H,T\}^{100}$.

Each sequence in S is equally likely, and hence has probability $1/|S|=1/2^{100}$.



A fair coin is tossed 100 times in a row.

Using the Language: visually

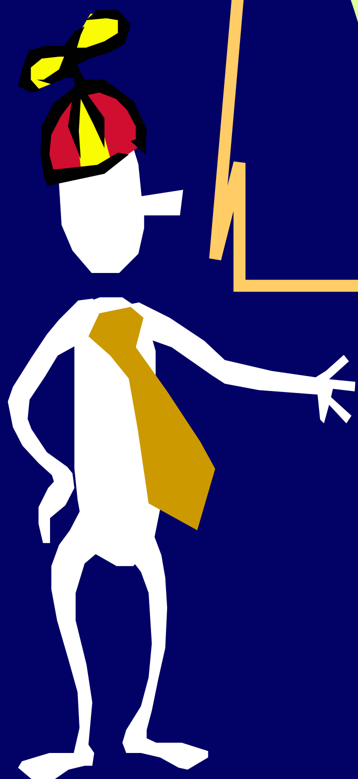


A fair coin is tossed 100 times in a row.

A fair coin is tossed 100 times in a row.

OK

What is the probability that we get exactly half heads?

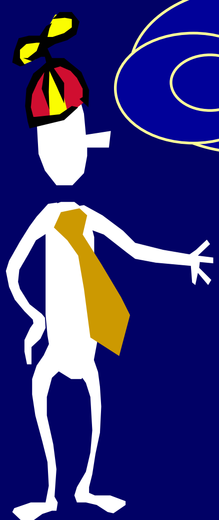


Using the Language

The event that we see half heads is

E = {x ∈ S | x has 50 heads}

And $|E| = \binom{100}{50}$

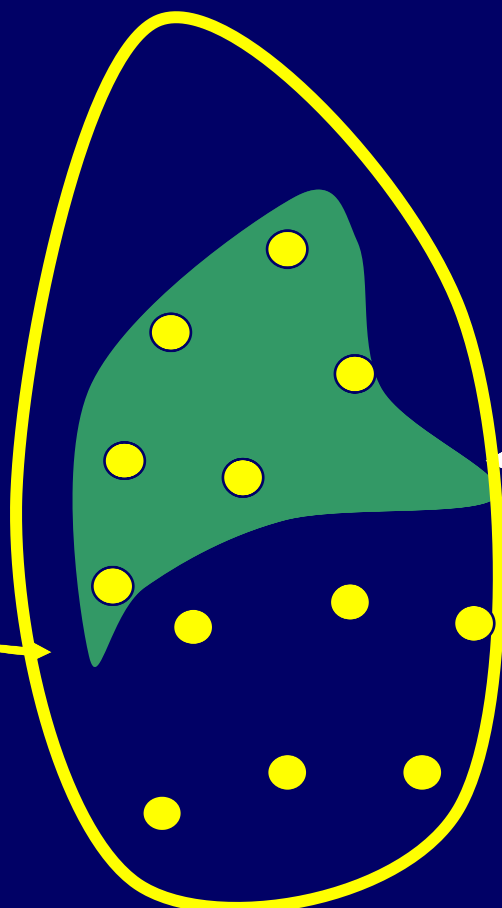


Probability of exactly half tails?

Picture

Event E = Set of sequences with 50 H's and 50 T's

Set of all 2^{100} sequences $\{H,T\}^{100}$



Probability of event E = proportion of E in S $= \dfrac{|E|}{|S|} = \dfrac{\binom{100}{50}}{2^{100}}$

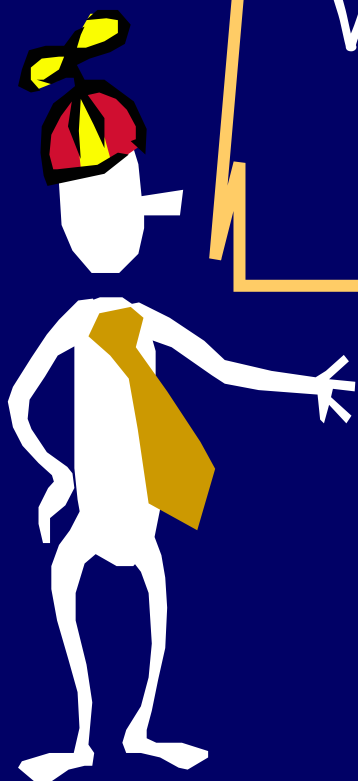
Using the Language

Answer:

Pr[E] = $|E|/|S|=|E|/2^{100}$

$$\frac{|E|}{|S|} = \frac{\binom{100}{50}}{2^{100}} \approx 0.0795$$

Probability of exactly half tails?



Suppose we roll a ***white*** die and a ***black*** die.

What is the probability that sum is 7 or 11?

Same methodology!

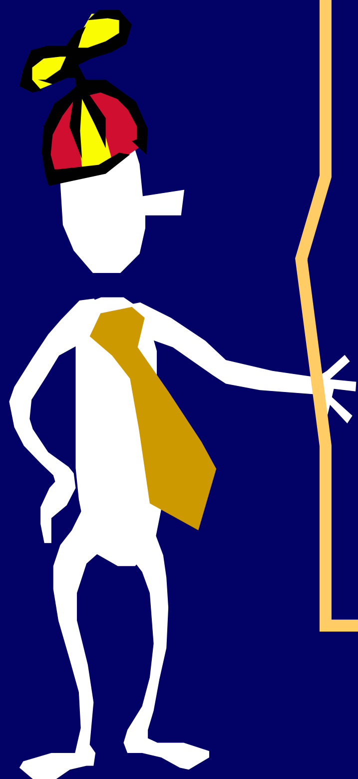
Sample space S =

{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

Pr(x) = 1/36
$\forall$ x $\in$ S

Event E = all (x,y) pairs with x+y = 7 or 11

Pr[E] = |E|/|S| = proportion of E in S = 8/36



23 people are in a room.

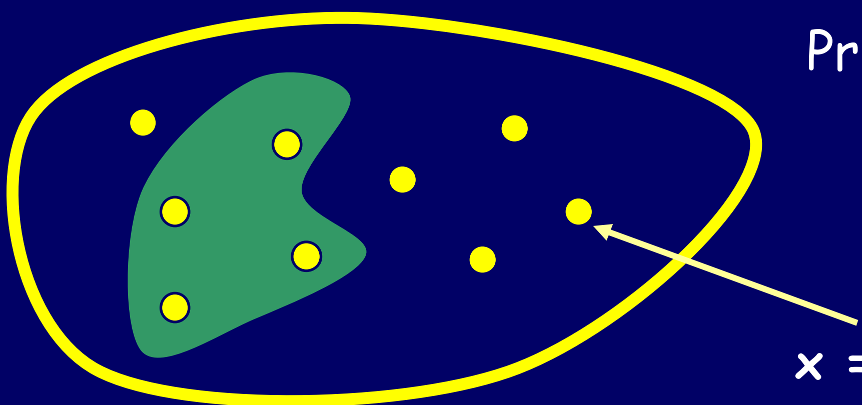
Suppose that all possible assignments of birthdays to the 23 people are equally likely.

What is the probability that two people will have the same birthday?

And the same methods again!

Sample space $\Omega = \{1, 2, 3, \ldots, 366\}^{23}$

Pretend it's always a leap year



$x = (17, 42, 363, 1, \ldots, 224, 177)$

$\underbrace{\hspace{6em}}_{\text{23 numbers}}$

Event $E = \{x \in \Omega \mid \text{two numbers in } x \text{ are same}\}$

What is $|E|$?

count $|\overline{E}|$ instead!

$\overline{E}$ = all sequences in Ω that have <u>no</u> repeated numbers

$$|\bar{E}| = 366 \cdot 365 \cdots 344$$

$$\frac{|\bar{E}|}{|\Omega|} = \frac{366 \cdots 344}{366^{23}} \approx .494$$

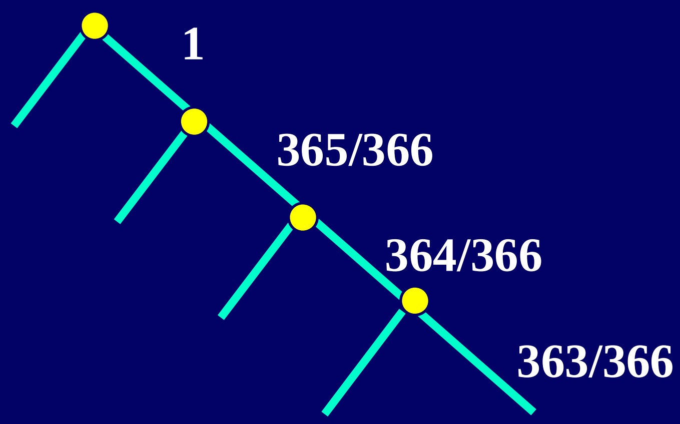
$$\frac{|E|}{|\Omega|} \approx .51$$

Another way to calculate Pr(no collision)

Pr(1st person doesn't collide) = 1.
Pr(2nd doesn't | no collisions yet) = 365/366.
Pr(3rd doesn't | no collisions yet) = 364/366.
Pr(4th doesn't | no collisions yet) = 363/366.

…

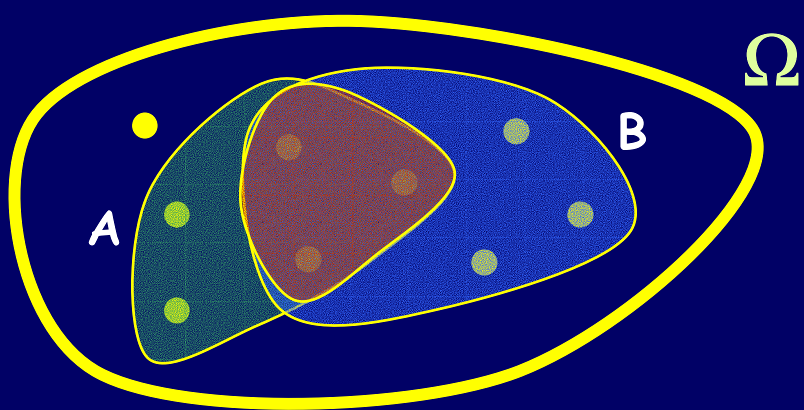
Pr(23rd doesn't| no collisions yet) = 344/366.



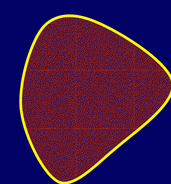
More Language Of Probability

The probability of event A <u>given event B</u> is written Pr[A | B]

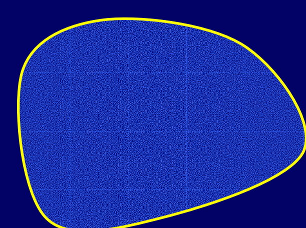
and is defined to be = $\dfrac{\Pr[A \cap B]}{\Pr[B]}$

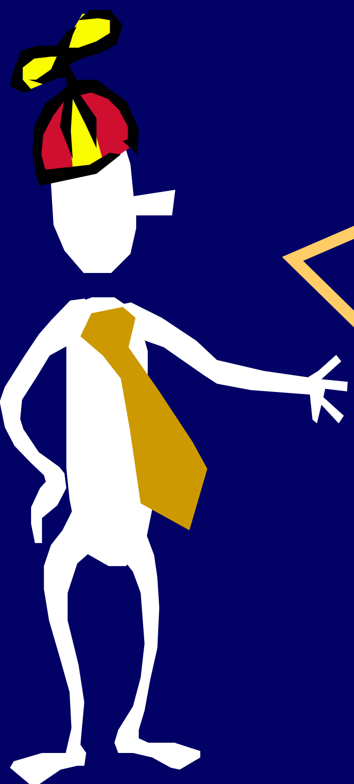


proportion of $A \cap B$



to B





Suppose we roll a ***white*** die and ***black*** die.

What is the probability that the white is 1 <u>given that the total is 7</u>?

event *A* = {white die = 1}

event B = {total = 7}

Sample space S =

$$\{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$$
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$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

event A = {white die = 1}

event B = {total = 7}

$$\frac{|A \cap B|}{|B|} = \frac{Pr[A \mid B]}{} = \frac{Pr[A \cap B]}{Pr[B]} = \frac{1/36}{1/6}$$

Can do this because Ω is uniformly distributed.

This way does not care about the distribution.

Independence!

A and B are <u>independent events</u> if

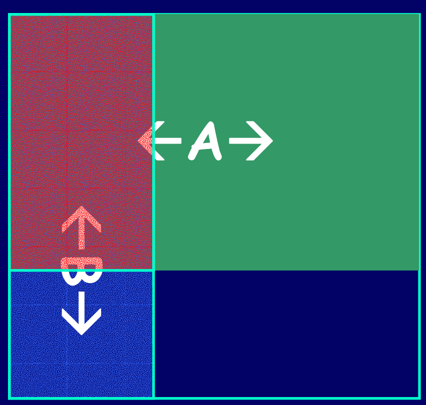
$$\Pr[\ A \mid B\ ] = \Pr[\ A\ ]$$

$$\Leftrightarrow$$

$$\Pr[\ A \cap B\ ] = \Pr[\ A\ ]\Pr[\ B\ ]$$

$$\Leftrightarrow$$

$$\Pr[\ B \mid A\ ] = \Pr[\ B\ ]$$



What about $\Pr[\ A \mid \text{not}(B)\ ]$?

Independence!

$A_1, A_2, \ldots, A_k$ are independent events if knowing if some of them occurred does not change the probability of any of the others occurring.

E.g., {A_1, A_2, A_3} are independent events if:

$$\Pr[A_1 \mid A_2 \cap A_3] = \Pr[A_1]$$
$$\Pr[A_2 \mid A_1 \cap A_3] = \Pr[A_2]$$
$$\Pr[A_3 \mid A_1 \cap A_2] = \Pr[A_3]$$

$$\Pr[A_1 \mid A_2] = \Pr[A_1] \qquad \Pr[A_1 \mid A_3] = \Pr[A_1]$$
$$\Pr[A_2 \mid A_1] = \Pr[A_2] \qquad \Pr[A_2 \mid A_3] = \Pr[A_2]$$
$$\Pr[A_3 \mid A_1] = \Pr[A_3] \qquad \Pr[A_3 \mid A_2] = \Pr[A_3]$$

Independence!

$A_1, A_2, \ldots, A_k$ are independent events if knowing if some of them occurred does not change the probability of any of the others occurring.

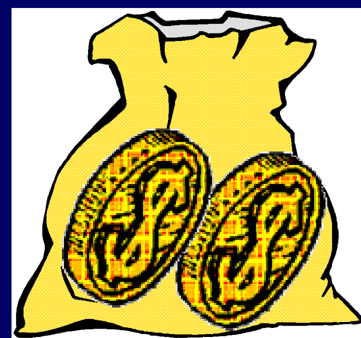
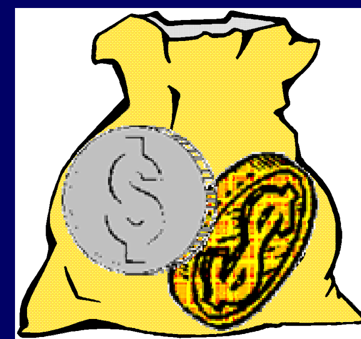
$\Pr[A|X] = \Pr[A]$

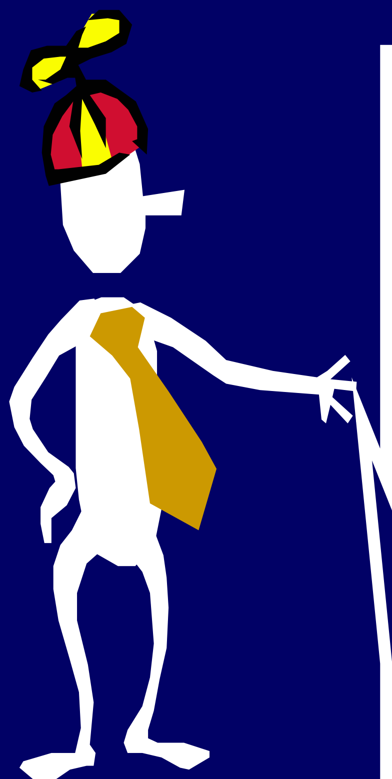
$\forall\ A \in \{A_i\}$

$\forall$ X a conjunction of any of the others

(e.g., A_2 and A_6 and A_7)

Silver and Gold



One bag has two silver coins, another has two gold coins, and the third has one of each.

One of the three bags is selected at random. Then one coin is selected at random from the two in the bag. It turns out to be gold.

What is the probability that the other coin is gold?

Let G_1 be the event that the first coin is gold.

$Pr[G_1] = 1/2$

Let G_2 be the event that the second coin is gold.

$$Pr[G_2 \mid G_1] = Pr[G_1 \text{ and } G_2] \,/\, Pr[G_1]$$

$$= (1/3) \,/\, (1/2)$$

$$= 2/3$$

Note: G_1 and G_2 are not independent.

Monty Hall problem

- Announcer hides prize behind
 one of 3 doors at random.
- You select some door.
- Announcer opens one of others with no prize.
- You can decide to keep or switch.

What to do?

Monty Hall problem

- Sample space Ω =
  { prize behind door 1,
  prize behind door 2,
  prize behind door 3 }.

Each has probability 1/3.

Staying

we win if we choose the correct door

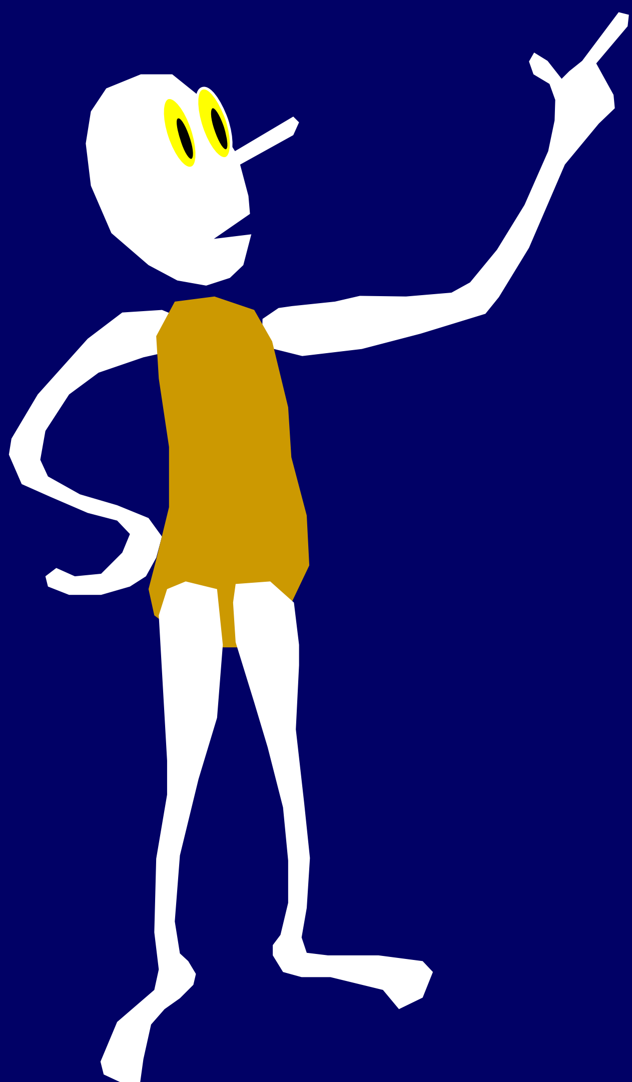
Pr[choosing correct door] = 1/3.

Switching

we win if we choose the <u>incorrect</u> door

Pr[choosing incorrect door] = 2/3.

why was this tricky?



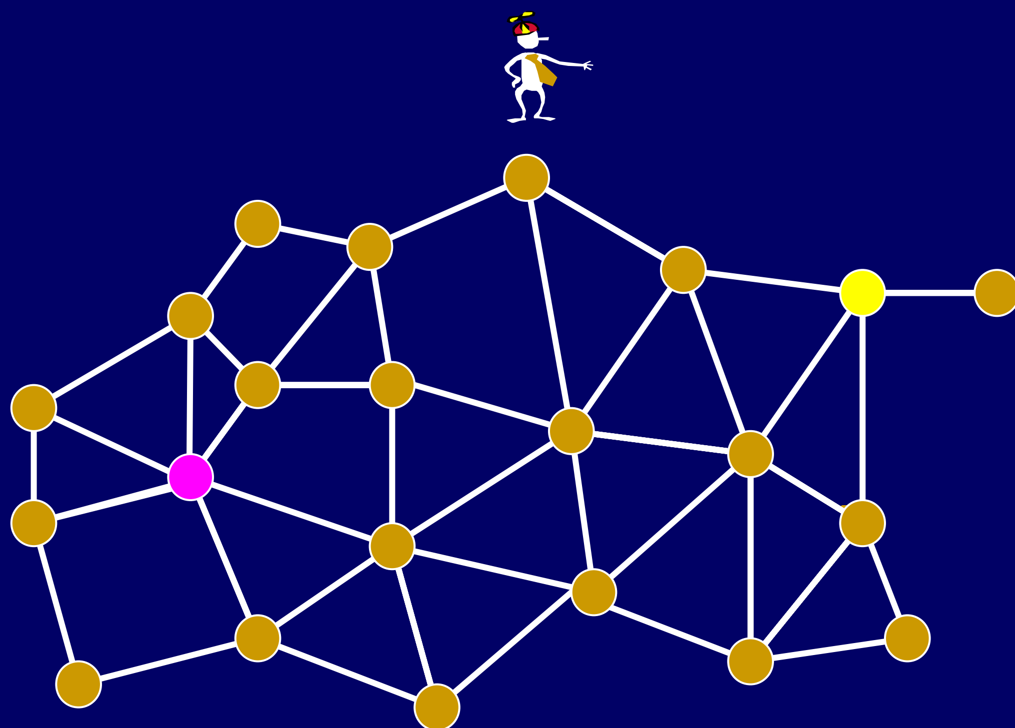
We are inclined to think:

"After one door is opened, others are equally likely…"

But his action is not independent of yours!

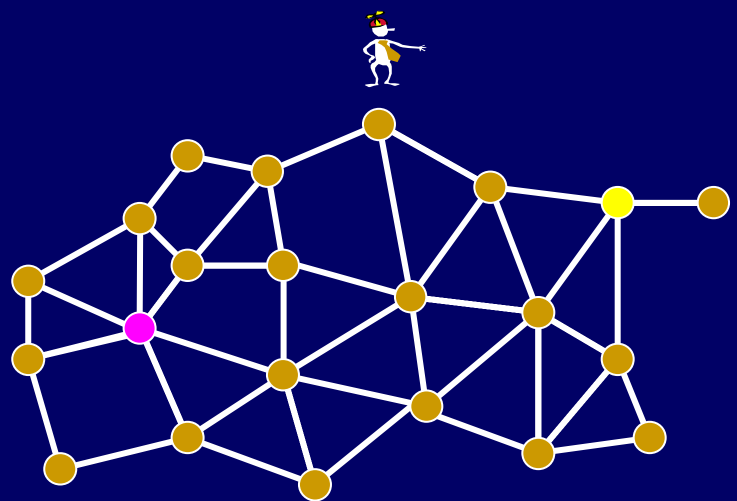
Random walks and electrical networks

What is chance I reach yellow before magenta?



Same as voltage if edges are resistors and we put 1-volt battery between yellow and magenta.

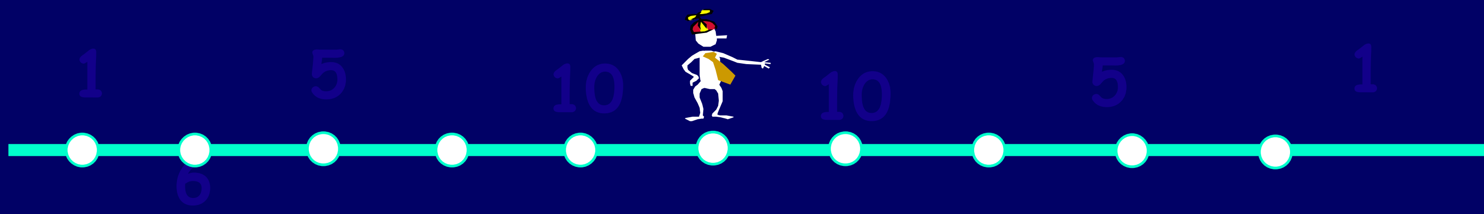
Random walks and electrical networks



- p_x = Pr(reach yellow first starting from x)
- $p_{yellow} = 1$, $p_{magenta} = 0$, and for the rest,
- $p_x = \text{Average}_{y \in \text{Nbr}(x)}(p_y)$

Same as equations for voltage if edges all have same resistance!

Random walks come up all the time



- Model stock as: each day has 50/50 chance of going up by \$1, or down by \$1.
- If currently \$k, what is chance will reach \$100 before \$0?
- Ans: k/100.
- Will see other ways of analyzing later…