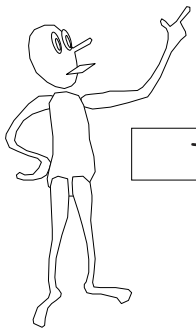
Great Theoretical Ideas In Computer Science		
Steven Rudich		CS 15-251 Spring 2003
Lecture 11	Feb 14, 2004	Carnegie Mellon University

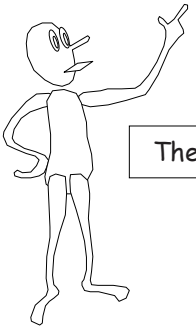
Counting III: Pascal's Triangle, Polynomials, and Vector Programs

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$



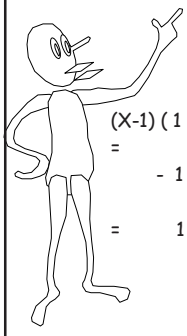
The Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$



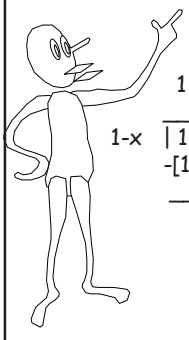
The Infinite Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$



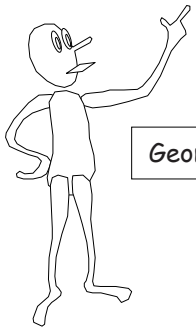
$$(X-1)\,(1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots)$$
$$= X^1 + X^2 + X^3 + \ldots + X^n + X^{n+1} + \ldots.$$
$$- 1 - X^1 - X^2 - X^3 - \ldots - X^{n-1} - X^n - X^{n+1} - \ldots$$
$$= 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$



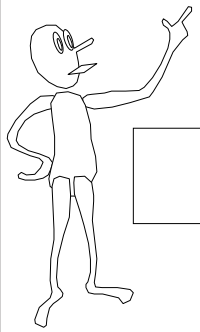
$$1 + x + \ldots\ldots$$
$$1-x \;\big|\; 1$$
$$-[1 - x]$$
$$x$$
$$-[x - x^2]$$
$$x^2$$
$$-\ldots.$$

$$1 + aX^1 + a^2X^2 + a^3X^3 + \ldots + a^nX^n + \ldots\ldots = \frac{1}{1 - aX}$$



Geometric Series (Linear Form)

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..) (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$

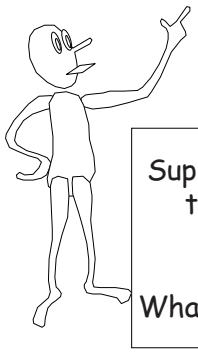


$$\frac{1}{(1 - aX)(1-bX)}$$

Geometric Series (Quadratic Form)

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..) (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$

$1 + c_1X^1 + .. + c_k X^k + \ldots$

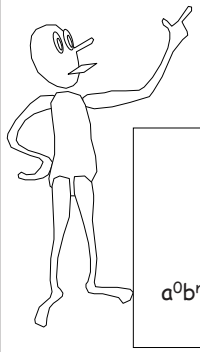


Suppose we multiply this out to get a single, infinite polynomial.

What is an expression for C_n?

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..) (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$

$1 + c_1X^1 + .. + c_k X^k + \ldots$

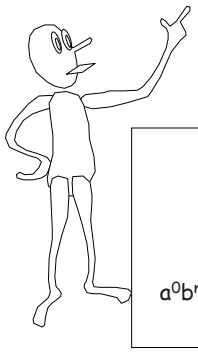


$c_n =$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..) (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$

$1 + c_1X^1 + .. + c_k X^k + \ldots$

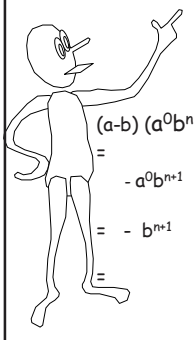


If $a = b$ then

$c_n = (n+1)(a^n)$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0 = \frac{a^{n+1} - b^{n+1}}{a - b}$



$(a-b)(a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0)$

$= a^1b^n + \ldots a^{i+1}b^{n-i} \ldots + a^nb^1 + a^{n+1}b^0$

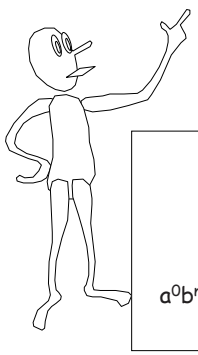
$- a^0b^{n+1} - a^1b^n \ldots a^{i+1}b^{n-i} \ldots - a^{n-1}b^2 - a^nb^1$

$= -b^{n+1} + a^{n+1}$

$= a^{n+1} - b^{n+1}$

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..) (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$

$1 + c_1X^1 + .. + c_k X^k + \ldots$

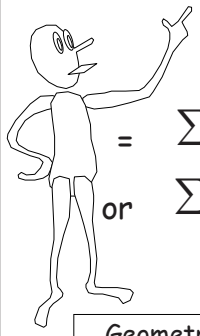


if $a \neq b$ then

$$c_n = \frac{a^{n+1} - b^{n+1}}{a - b}$$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots)(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) =$



$$= \frac{1}{(1 - aX)(1-bX)}$$

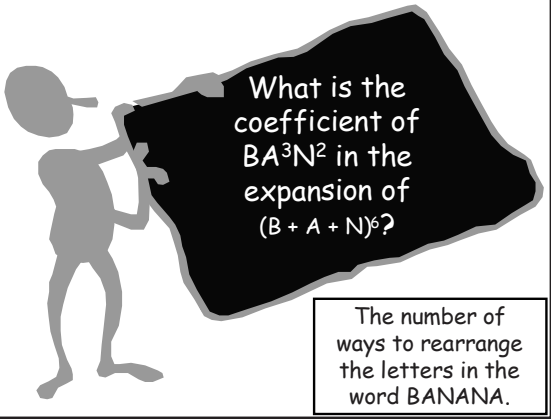
$$= \sum_{n=0..\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

or $\sum_{n=0..\infty} (n+1)a^n X^n$ when a=b

Geometric Series (Quadratic Form)

Previously, we saw that

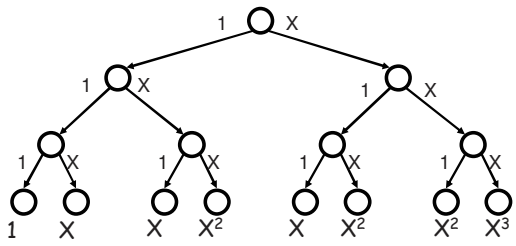
Polynomials Count!



What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?

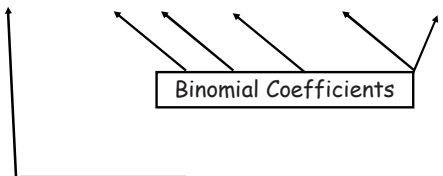
The number of ways to rearrange the letters in the word BANANA.

Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

The Binomial Formula



Binomial Coefficients

binomial expression

The Binomial Formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

One polynomial, two representations

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

"Product form" or "Generating form"

"Additive form" or "Expanded form"

Power Series Representation

$$\boxed{(1+x)^n} = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$= \boxed{\sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k}$$

Since $\binom{n}{k} = 0$ if $k > n$

"Closed form" or "Generating form"

"Power series" ("Taylor series") expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x=1$.

$$2^n = \underbrace{\sum_{k=0}^{n} \binom{n}{k}}$$

The number of subsets of an n-element set

By varying x, we can discover new identities

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let *x= -1*.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

The number of even-sized subsets of an *n* element set is the same as the number of odd-sized subsets.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let *x= -1*.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

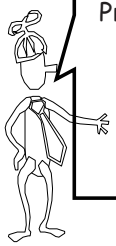
Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

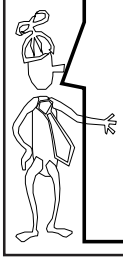
We could discover new identities by substituting in different numbers for X. One cool idea is to try complex roots of unity, however, the lecture is going in another direction.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



Proofs that work by manipulating algebraic forms are called "algebraic" arguments. Proofs that build a 1-1 onto correspondence are called "combinatorial" arguments.

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$



Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We gave an algebraic proof that

$|O_n| = |E_n|$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

A combinatorial proof must construct a one-to-one correspondence between O_n and E_n

An attempt at a correspondence

Let f_n be the function that takes an n-bit string and flips all its bits.

f_n is clearly a one-to-one and onto function

for odd n. *E.g.* in f_7 we have

0010011 → 1101100

1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100

101010 → 010101

Uh oh. Complementing maps evens to evens!

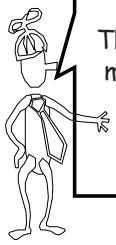
A correspondence that works for all n

Let f_n be the function that takes an n-bit string and flips only *the first bit*. For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle:
k^{th} row are the coefficients of $(1+X)^k$

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

k^{th} Row Of Pascal's Triangle:

$\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2} \cdots \binom{n}{k} \cdots \binom{n}{n}$

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Inductive definition of kth entry of nth row:
Pascal(n,0) = Pacal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n,k)

$(1+X)^0 = 1$

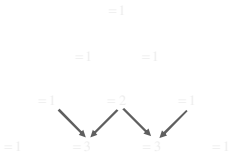
$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

"Pascal's Triangle"

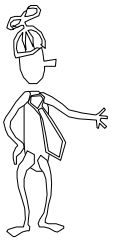


Al-Karaji, Baghdad 953-1029

Chu Shin-Chieh 1303
The Precious Mirror of the Four Elements
. . . Known in Europe by 1529

Blaise Pascal 1654

Pascal's Triangle



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

Summing The Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1 =1

1 + 1 =2

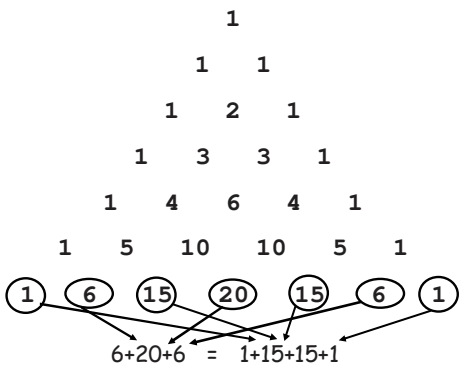
1 + 2 + 1 =4

1 + 3 + 3 + 1 =8

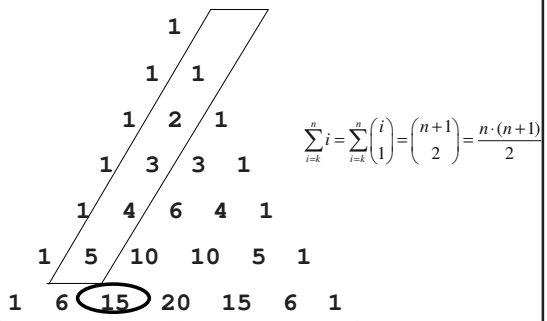
1 + 4 + 6 + 4 + 1 =16

1 + 5 + 10 + 10 + 5 + 1 =32

1 + 6 + 15 + 20 + 15 + 6 + 1 =64

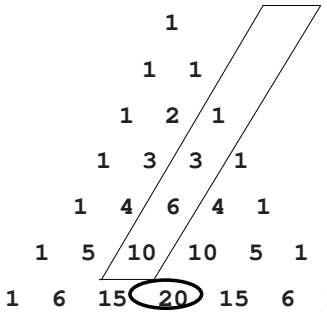


Summing on 1st Avenue

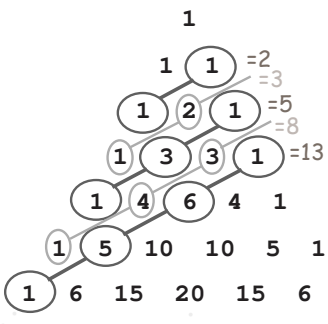


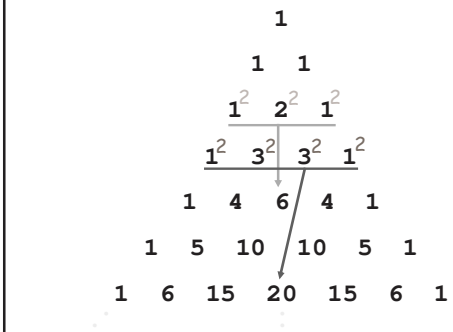
$$\sum_{i=k}^{n} i = \sum_{i=k}^{n} \binom{i}{1} = \binom{n+1}{2} = \frac{n \cdot (n+1)}{2}$$

Summing on k^th^ Avenue



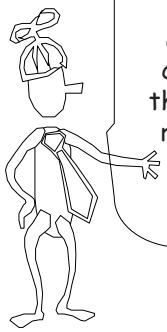
$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$





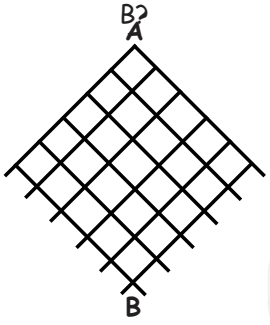
Al-Karaji Squares

```
                1                    =0
              1   1                  =1
            1   2 + 2*1              =4
          1   3 + 2*3   1            =9
        1   4 + 2*6   4   1          =16
      1   5 + 2*10   10   5   1      =25
    1   6 + 2*15   20   15   6   1=36
```

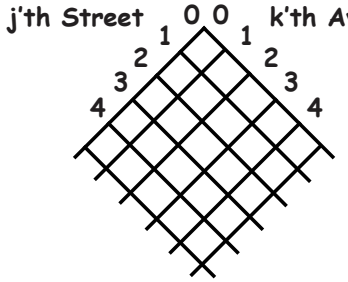


All these properties can be proved inductively and algebraically. We will give combinatorial proofs using the Manhattan block walking representation of binomial coefficients.

How many shortest routes from A to B?

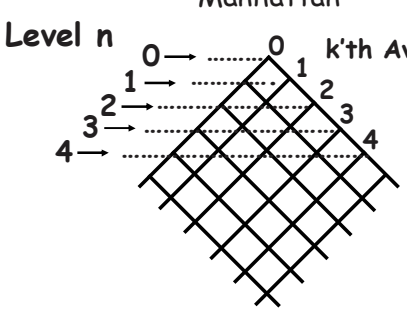


Manhattan



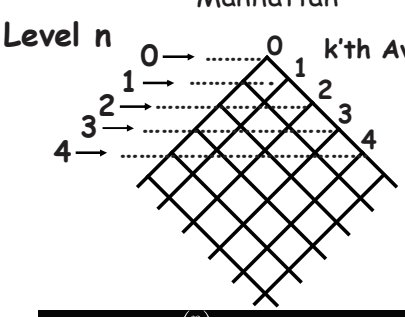
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k).

Manhattan

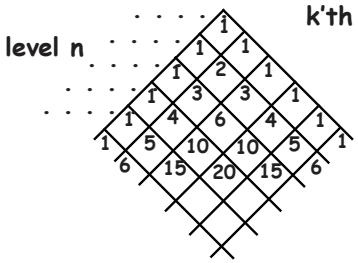


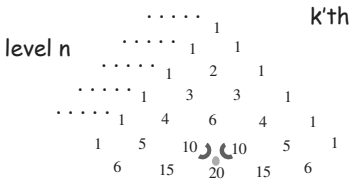
There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k).

Manhattan

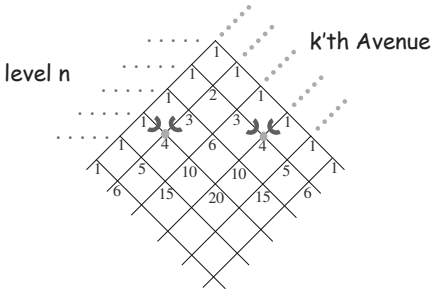


There are $\binom{n}{k}$ shortest routes from (0,0) to Level n and k^{th} Avenue.

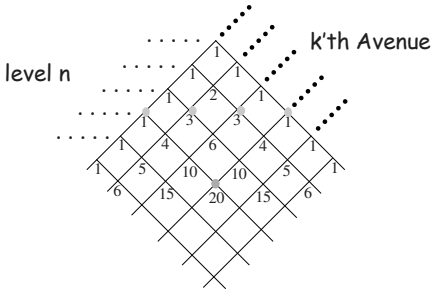




$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



$$\sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}$$



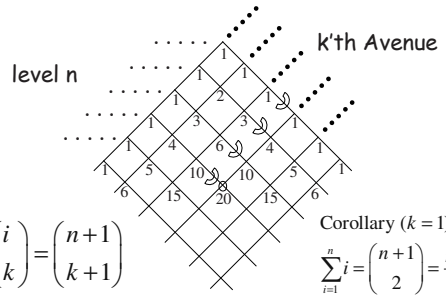
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

By convention:

$0! = 1$ (empty product = 1)

$$\binom{n}{k} = 1 \quad \text{if } k = 0$$

$$\binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n$$



$$\sum_{i=1}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Corollary ($k = 1$)

$$\sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

Application (Al-Karaji):

$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$
$$= (1\cdot 0 + 1) + (2\cdot 1 + 2) + (3\cdot 2 + 3) + \cdots + (n(n-1) + n)$$
$$= 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + n(n-1) + \sum_{i=1}^{n} i$$
$$= 2\left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n}{2}\right] + \binom{n+1}{2}$$
$$= 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{(2n+1)(n+1)n}{6}$$

Vector Programs

Let's define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable $V^{\rightarrow}$ can be thought of as:

< * , * , * , * , * , * , ... >

0 1 2 3 4 5 . . . . . .

Vector Programs

Let k stand for a scalar constant
<k> will stand for the vector <k,0,0,0,…>

<0> = <0,0,0,0,....>
<1> = <1,0,0,0,…>

$V^{\rightarrow}$+$T^{\rightarrow}$means to add the vectors position-wise.

<4,2,3,…> + <5,1,1,....> = <9,3,4,…>

Vector Programs

RIGHT($V^{\rightarrow}$) means to shift every number in $V^{\rightarrow}$ one position to the right and to place a 0 in position 0.

RIGHT(<1,2,3, …>) = <0,1,2,3,. …>

Vector Programs

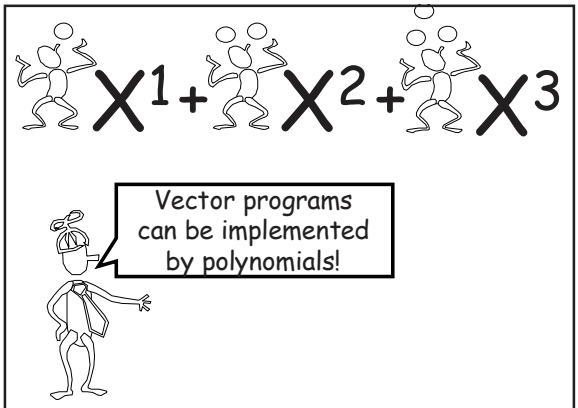
Example:	Stare
$V^{\rightarrow}$:= <6>;	$V^{\rightarrow}$ = <6,0,0,0,..>
$V^{\rightarrow}$:= RIGHT($V^{\rightarrow}$) + <42>;	$V^{\rightarrow}$ = <42,6,0,0,..>
$V^{\rightarrow}$:= RIGHT($V^{\rightarrow}$) + <2>;	$V^{\rightarrow}$ = <2,42,6,0,..>
$V^{\rightarrow}$:= RIGHT($V^{\rightarrow}$) + <13>;	$V^{\rightarrow}$= <13,2,42,6,..>

$V^{\rightarrow}$ = < 13, 2, 42, 6, 0, 0, 0, . . . >

Vector Programs

Example:	Stare
$V^{\rightarrow}$:= <1>;	$V^{\rightarrow}$ = <1,0,0,0,..>
Loop n times:	$V^{\rightarrow}$ = <1,1,0,0,..>
$V^{\rightarrow}$:= $V^{\rightarrow}$ + RIGHT($V^{\rightarrow}$);	$V^{\rightarrow}$ = <1,2,1,0,..>
	$V^{\rightarrow}$= <1,3,3,1,.>

$V^{\rightarrow}$ = n^{th} row of Pascal's triangle.



Programs -----> Polynomials

The vector $V^{\rightarrow} = \langle a_0, a_1, a_2, \ldots \rangle$ will be represented by the polynomial:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

Formal Power Series

The vector $V^{\rightarrow} = \langle a_0, a_1, a_2, \ldots \rangle$ will be represented by the formal power series:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

$V^{\rightarrow} = \langle a_0, a_1, a_2, \ldots \rangle$

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

<0>is represented by	0
<k>is represented by	k
$V^{\rightarrow} + T^{\rightarrow}$ is represented by	$(P_V + P_T)$
RIGHT($V^{\rightarrow}$) is represented by	$(P_V X)$

Vector Programs

Example:

$V^{\rightarrow}$:= <1>;	$P_V := 1$;
Loop n times:	
$V^{\rightarrow} := V^{\rightarrow}$ + RIGHT($V^{\rightarrow}$);	$P_V := P_V + P_V X$;

$V^{\rightarrow}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

$V^{\rightarrow}$:= <1>;	$P_V := 1$;
Loop n times:	
$V^{\rightarrow} := V^{\rightarrow}$ + RIGHT($V^{\rightarrow}$);	$P_V := P_V (1+ X)$;

$V^{\rightarrow}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

$V^{\rightarrow}$:= <1>;

Loop n times:
$V^{\rightarrow}$:= $V^{\rightarrow}$ + RIGHT($V^{\rightarrow}$);

$P_V = (1+X)^n$

$V^{\rightarrow}$ = n^{th} row of Pascal's triangle.