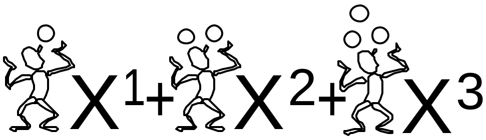
| Great Theoretical Ideas In Computer Science | | |
|---|---|---|
| Steven Rudich | | CS 15-251 Spring 2003 |
| Lecture 11 | Feb 14, 2004 | Carnegie Mellon University |

# Counting III: Pascal's Triangle, Polynomials, and Vector Programs

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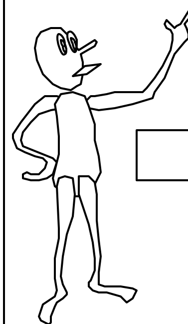
$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$



## The Geometric Series

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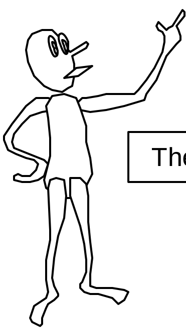
$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots.. = \frac{1}{1 - X}$$



## The Infinite Geometric Series

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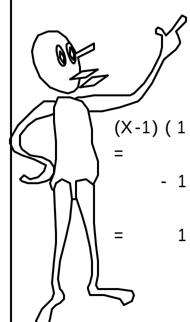
$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots.. = \frac{1}{1 - X}$$



$(X-1)\ (1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots)$

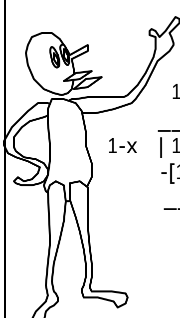
$= \quad X^1 + X^2 + X^3 + \ldots \quad + X^n + X^{n+1} + \ldots.$

$\quad - 1 - X^1 - X^2 - X^3 - \ldots - X^{n-1} - X^n - X^{n+1} - \ldots$

$= \quad 1$

---

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots.. = \frac{1}{1 - X}$$



```
        1 + x + .....
      ______________________
1-x   | 1
       -[1 – x]
       ______
            x
          -[x – x²]
             _______
                 x²
                 -....
```

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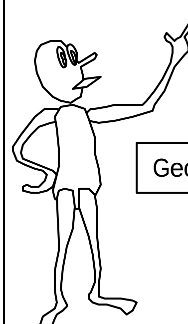
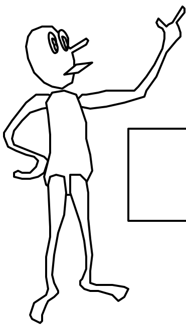
$$1 + aX^1 + a^2X^2 + a^3X^3 + \ldots + a^nX^n + \ldots.. = \frac{1}{1 - aX}$$



## Geometric Series (Linear Form)

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) =$

$$\frac{1}{(1 - aX)(1-bX)}$$



## Geometric Series (Quadratic Form)

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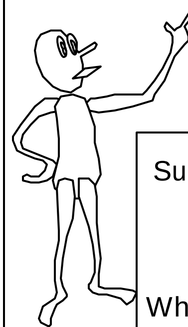
$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) =$

$1 + c_1X^1 + .. + c_k X^k + \ldots$



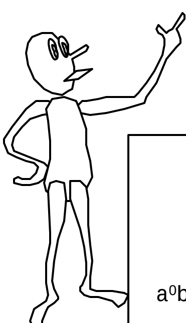
Suppose we multiply this out to get a single, infinite polynomial.

What is an expression for $C_n$?

---

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) =$

$1 + c_1X^1 + .. + c_k X^k + \ldots$



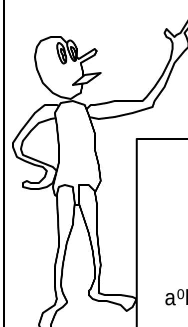
$c_n =$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$

---

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) =$

$1 + c_1X^1 + .. + c_k X^k + \ldots$



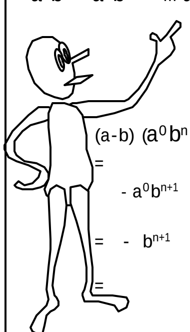
If $a = b$ then

$c_n = (n+1)(a^n)$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$

---

$$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0 = \frac{a^{n+1} - b^{n+1}}{a - b}$$



$(a-b)\,(a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0)$

$= \quad a^1b^n + \ldots\ a^{i+1}b^{n-i} \ldots \quad + a^nb^1 + a^{n+1}b^0$

$\quad - a^0b^{n+1} - a^1b^n \ldots\ a^{i+1}b^{n-i} \ldots\ - a^{n-1}b^2 - a^nb^1$

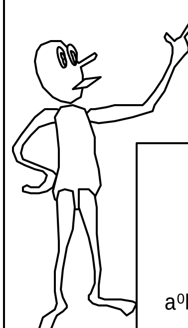
$= \ - b^{n+1} + a^{n+1}$

$= a^{n+1} - b^{n+1}$

---

$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots) =$

$1 + c_1X^1 + .. + c_k X^k + \ldots$

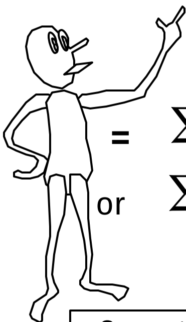


if $a \neq b$ then

$$c_n = \frac{a^{n+1} - b^{n+1}}{a - b}$$

$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$

$$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots)\ (1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots) =$$

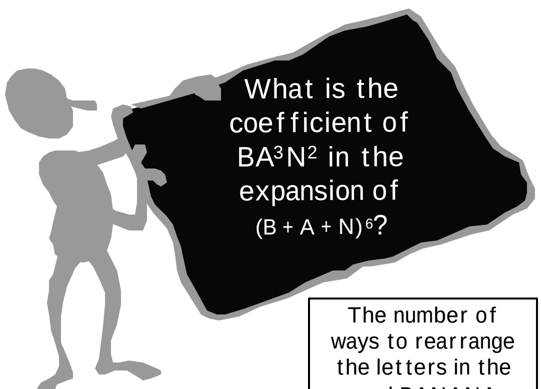


$$= \frac{1}{(1 - aX)(1-bX)}$$

$$= \sum_{n=0..\,1} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

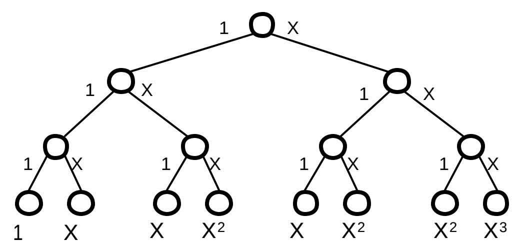
or $\sum_{n=0..\,1} (n+1)a^n\ X^n$ when a=b

Geometric Series (Quadratic Form)


The number of ways to rearrange the letters in the word BANANA.

Choice tree for terms of $(1+X)^3$




Combine like terms to get $1 + 3X + 3X^2 + X^3$

## The Binomial Formula

Binomial Coefficients

binomial expression

## The Binomial Formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

## One polynomial, two representations

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

“Product form” or “Generating form”

“Additive form” or “Expanded form”

## Power Series Representation

$$\boxed{(1+x)^n} = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k = \boxed{\sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k}$$

“Closed form” or “Generating form”

Since $\binom{n}{k} = 0$ if $k > n$

“Power series” (“Taylor series”) expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x=1$.

$$2^n = \underbrace{\sum_{k=0}^{n} \binom{n}{k}}$$

The number of subsets of an *n*-element set

By varying x, we can discover new identities

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x = -1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

The number of even-sized subsets of an *n* element set is the same as the number of odd-sized subsets.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x = -1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

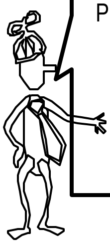
Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

We could discover new identities by substituting in different numbers for X. One cool idea is to try complex roots of unity, however, the lecture is going in another direction.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



Proofs that work by manipulating algebraic forms are called "algebraic" arguments. Proofs that build a 1-1 onto correspondence are called "combinatorial" arguments.

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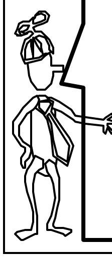
$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$



Let $O_n$ be the set of binary strings of length $n$ with an odd number of ones.

Let $E_n$ be the set of binary strings of length $n$ with an even number of ones.

We gave an <u>algebraic</u> proof that

$|O_n| = |E_n|$

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## A Combinatorial Proof

Let $O_n$ be the set of binary strings of length $n$ with an odd number of ones.

Let $E_n$ be the set of binary strings of length $n$ with an even number of ones.

A <u>combinatorial</u> proof must construct a one-to-one correspondence between $O_n$ and $E_n$

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## An attempt at a correspondence

*Let $f_n$ be the function that takes an n-bit string and flips all its bits.*

$f_n$ is clearly a one-to-one and onto function

for odd $n$. *E.g.* in $f_7$ we have

0010011 → 1101100

1001101 → 0110010

*...but do even n work? In $f_6$ we have*

*110011 → 001100*

*101010 → 010101*

*Uh oh. Complementing maps evens to evens!*

---

## A correspondence that works for all $n$

Let $f_n$ be the function that takes an n-bit string and flips only *the first bit*. For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

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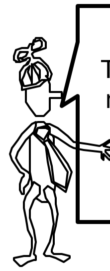
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge...

## The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

## Pascal's Triangle: $k^{th}$ row are the coefficients of $(1+X)^k$

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

## $k^{th}$ Row Of Pascal's Triangle:

$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{k}+\dots+\binom{n}{n}$

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

## Inductive definition of kth entry of nth row:

Pascal(n,0) = Pacal (n,n) = 1;

Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n,k)

$(1+X)^0 = 1$

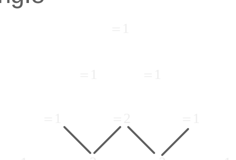
$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

## "Pascal's Triangle"



Al-Karaji, Baghdad 953-1029

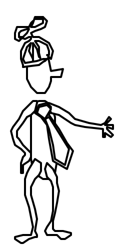
Chu Shin-Chieh 1303

The Precious Mirror of the Four Elements

. . . Known in Europe by 1529

Blaise Pascal 1654

## Pascal's Triangle



| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 1 | | | | | | |
| | | | | | 1 | | 1 | | | | | |
| | | | | 1 | | 2 | | 1 | | | | |
| | | | 1 | | 3 | | 3 | | 1 | | | |
| | | 1 | | 4 | | 6 | | 4 | | 1 | | |
| | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | |
| 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 |

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

### Summing The Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

| | |
|---|---|
| 1 | =1 |
| 1 + 1 | =2 |
| 1 + 2 + 1 | =4 |
| 1 + 3 + 3 + 1 | =8 |
| 1 + 4 + 6 + 4 + 1 | =16 |
| 1 + 5 + 10 + 10 + 5 + 1 | =32 |
| 1 + 6 + 15 + 20 + 15 + 6 + 1 | =64 |

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

6+20+6 = 1+15+15+1

### Summing on $1^{st}$ Avenue

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$\sum_{i=k}^{n} i = \sum_{i=k}^{n} \binom{i}{1} = \binom{n+1}{2} = \frac{n \cdot (n+1)}{2}$$

### Summing on $k^{th}$ Avenue

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

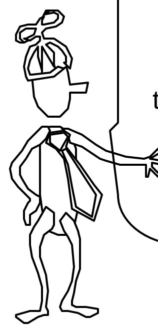
$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

1
1 1 =2
=3
1 2 1 =5
=8
1 3 3 1 =13
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1
1 1
$1^2$ $2^2$ $1^2$
$1^2$ $3^2$ $3^2$ $1^2$
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

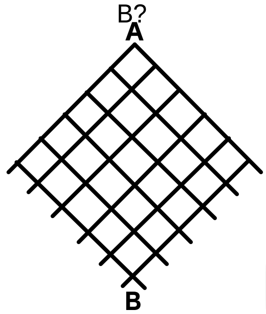
## Al-Karaji Squares

```
                 1                =0
              1     1             =1
           1    2 + 2*1           =4
        1    3 + 2*3    1         =9
     1    4 + 2*6    4    1       =16
  1    5 + 2*10    10    5    1   =25
1    6 + 2*15    20    15    6   1=36
```



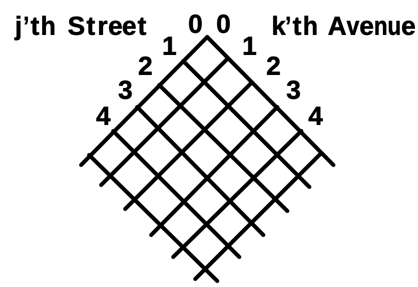
All these properties can be proved inductively and algebraically. We will give combinatorial proofs using the Manhattan block walking representation of binomial coefficients.

How many shortest routes from A to B?



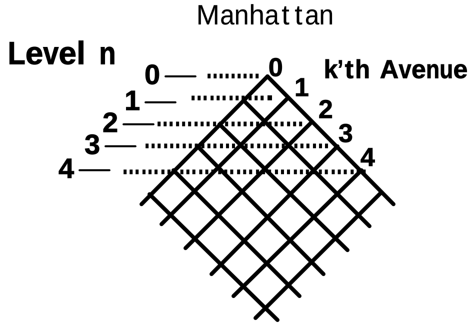

$\binom{10}{5}$

Manhattan



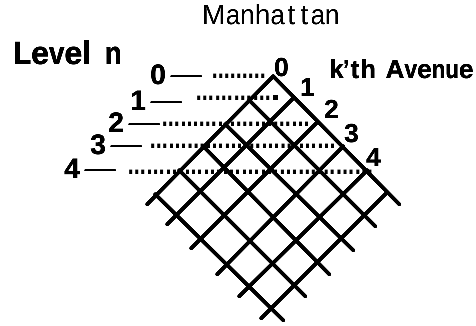

There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k).

Manhattan

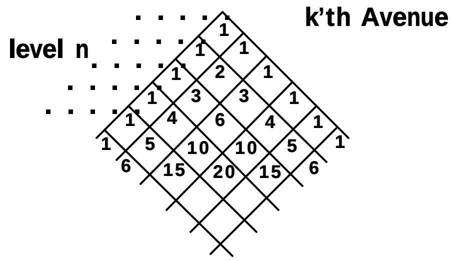


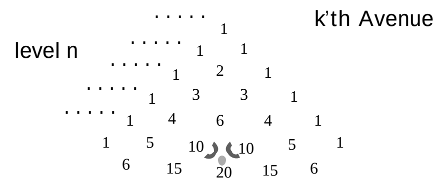

There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k).

Manhattan

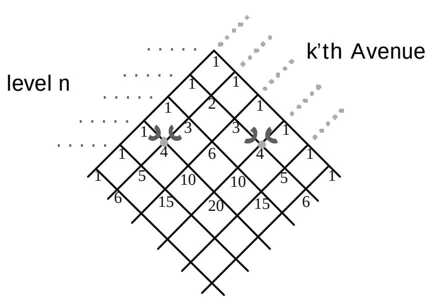



There are $\binom{n}{k}$ shortest routes from (0,0) to Level $n$ and $k^{th}$ Avenue.

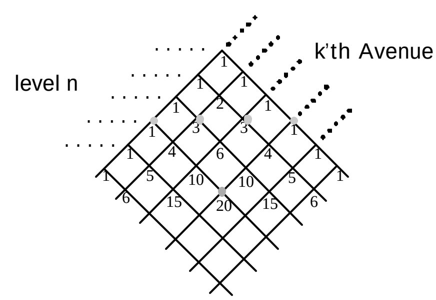



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$




$$\sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}$$



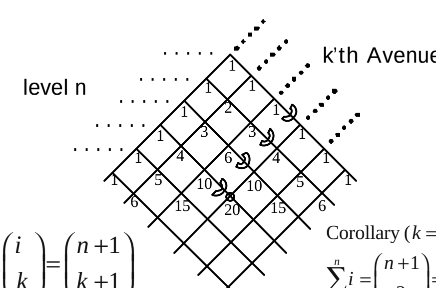

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

By convention:

$$0! = 1 \qquad \text{(empty product = 1)}$$

$$\binom{n}{k} = 1 \quad \text{if } k = 0$$

$$\binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n$$




$$\sum_{i=1}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Corollary ($k = 1$)

$$\sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

Application (Al-Karaji):

$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$
$$= (1\cdot 0 + 1) + (2\cdot 1 + 2) + (3\cdot 2 + 3) + \cdots + (n(n-1) + n)$$
$$= 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + n(n-1) + \sum_{i=1}^{n} i$$
$$= 2\left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n}{2}\right] + \binom{n+1}{2}$$
$$= 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{(2n+1)(n+1)n}{6}$$

## Vector Programs

Let's define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable V! can be thought of as:

< * , * , * , * , *, *, . . . . . . . . . >

0 1 2 3 4 5 . . . . . . . . .

## Vector Programs

Let k stand for a scalar constant
<k> will stand for the vector <k,0,0,0,...>

<0> = <0,0,0,0,....>
<1> = <1,0,0,0,...>

V! +T! means to add the vectors position-wise.

<4,2,3,...> + <5,1,1,....> = <9,3,4,...>

## Vector Programs

RIGHT(V!) means to shift every number in V! one position to the right and to place a 0 in position 0.

RIGHT( <1,2,3, ...> ) = <0,1,2,3,. ...>

## Vector Programs

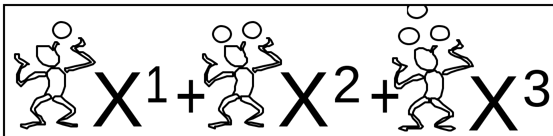
| Example: | Stare |
|---|---|
| V! := <6>; | V! = <6,0,0,0,..> |
| V! := RIGHT(V!) + <42>; | V! = <42,6,0,0,..> |
| V! := RIGHT(V!) + <2>; | V! = <2,42,6,0,..> |
| V! := RIGHT(V!) + <13>; | V! = <13,2,42,6,.> |

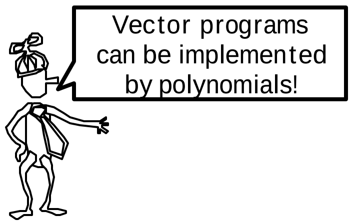
V! = < 13, 2, 42, 6, 0, 0, 0, . . . >

## Vector Programs

| Example: | Stare |
|---|---|
| V! := <1>; | V! = <1,0,0,0,..> |
| Loop n times: | V! = <1,1,0,0,..> |
| V! := V! + RIGHT(V!); | V! = <1,2,1,0,..> |
| | V! = <1,3,3,1,.> |

V! = $n^{th}$ row of Pascal's triangle.




Programs -----> Polynomials

The vector V! = < $a_0$, $a_1$, $a_2$, . . . > will be represented by the polynomial:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

Formal Power Series

The vector V! = < $a_0$, $a_1$, $a_2$, . . . > will be represented by the formal power series:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

V! = < $a_0$, $a_1$, $a_2$, . . . >

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

| | |
|---|---|
| <0>is represented by | 0 |
| <k>is represented by | k |
| V! + T! is represented by | $(P_V + P_T)$ |
| RIGHT(V!) is represented by | $(P_V X)$ |

Vector Programs

Example:

V! := <1>; $\qquad$ $P_V := 1;$

Loop n times:
V! := V! + RIGHT(V!); $\qquad$ $\mathbf{P_V := P_V + P_V\ X;}$

V! = $n^{th}$ row of Pascal's triangle.

Vector Programs

Example:

V! := <1>; $\qquad$ $P_V := 1;$

Loop n times:
V! := V! + RIGHT(V!); $\qquad$ $P_V := P_V (1+ X);$

V! = $n^{th}$ row of Pascal's triangle.

## Vector Programs

Example:

V! := <1>;

Loop n times:
 V! := V! + RIGHT(V!);

$P_V = (1 + X)^n$

V! = $n^{th}$ row of Pascal's triangle.