Great Theoretical Ideas In Computer Science

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CS 15-251 Spring 2003

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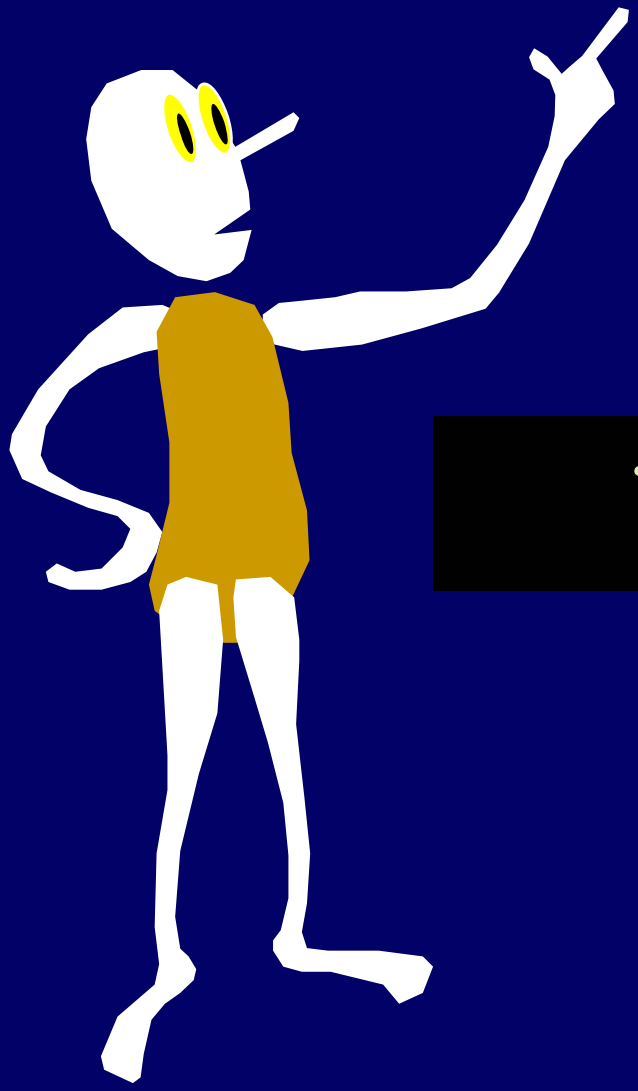
Carnegie Mellon University

Counting III: Pascal's Triangle, Polynomials, and Vector Programs

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$

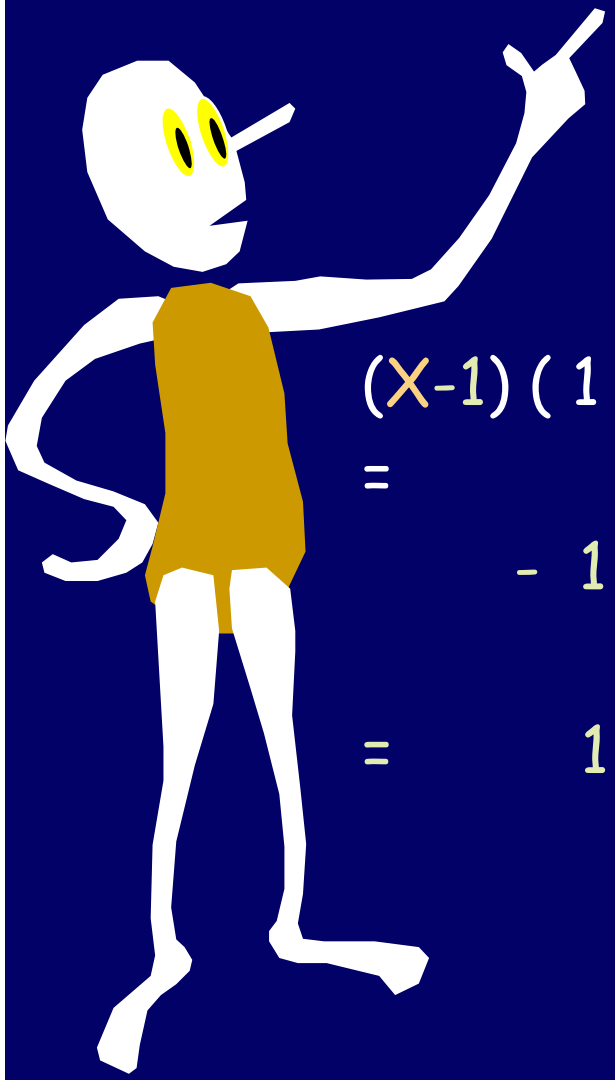


The Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots = \frac{1}{1 - X}$$

The Infinite Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots = \frac{1}{1 - X}$$



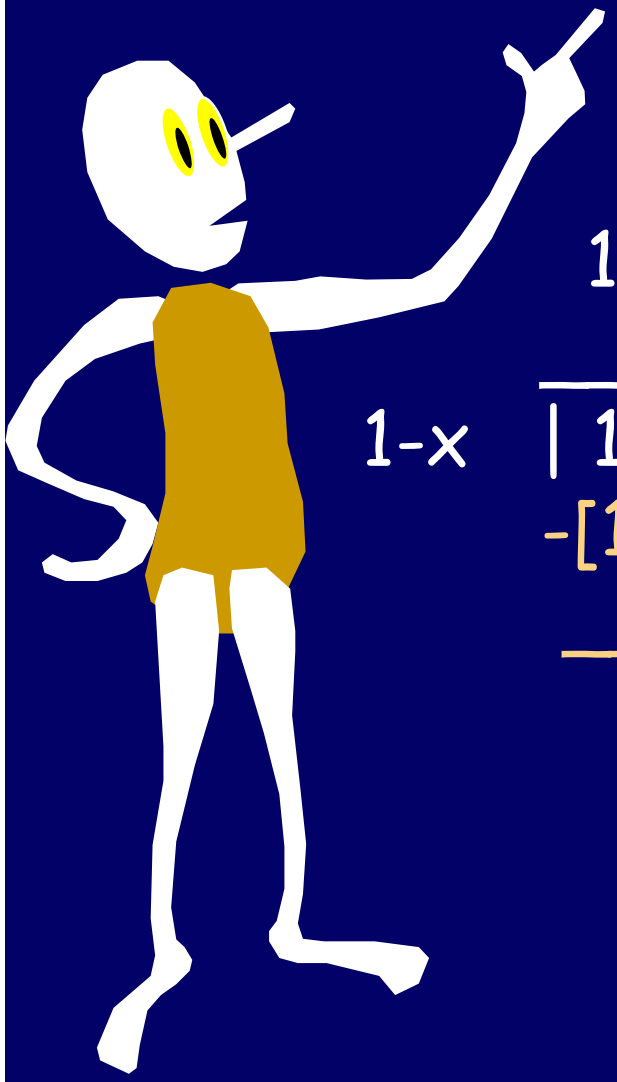
$$(X-1)\,(\,1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\,)$$

$$= \quad X^1 + X^2 + X^3 + \ldots \quad + X^n + X^{n+1} + \ldots.$$

$$- \; 1 - X^1 - X^2 - X^3 - \ldots - X^{n-1} - X^n - X^{n+1} - \ldots$$

$$= \quad 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots = \frac{1}{1 - X}$$



$$1 + x + \ldots$$

$$1-x \;\big|\; 1$$

$$-[1 - x]$$

$$x$$

$$-[x - x^2]$$

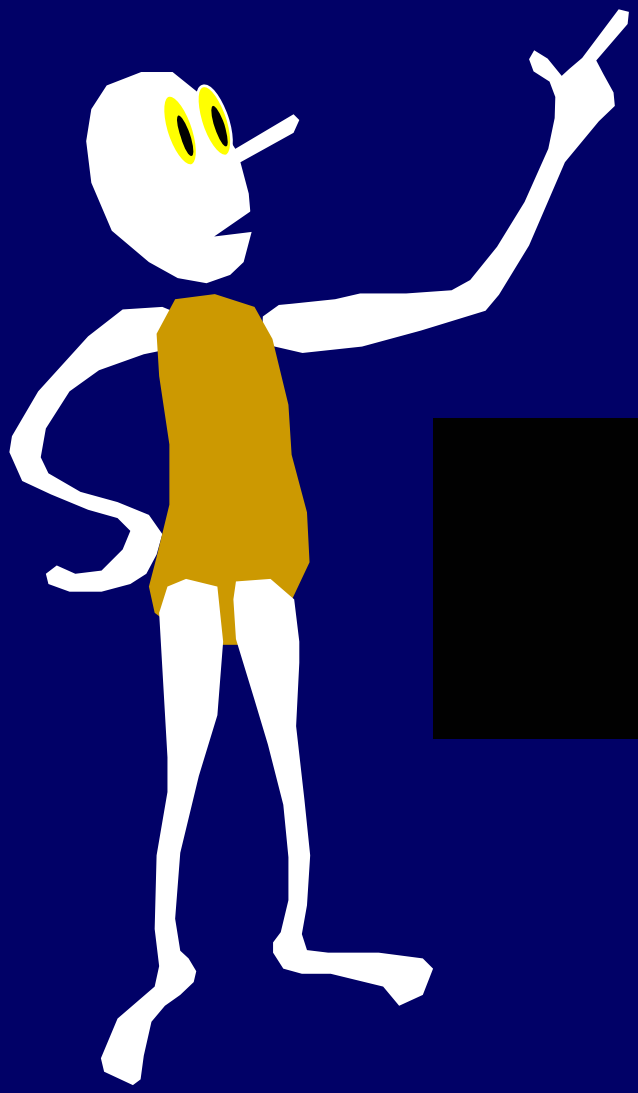
$$x^2$$

$$-\ldots$$

$$1 + aX^1 + a^2X^2 + a^3X^3 + \ldots + a^nX^n + \ldots = \frac{1}{1 - aX}$$

Geometric Series (Linear Form)

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$$



$$\frac{1}{(1 - aX)(1-bX)}$$

Geometric Series (Quadratic Form)

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..)\ (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$$

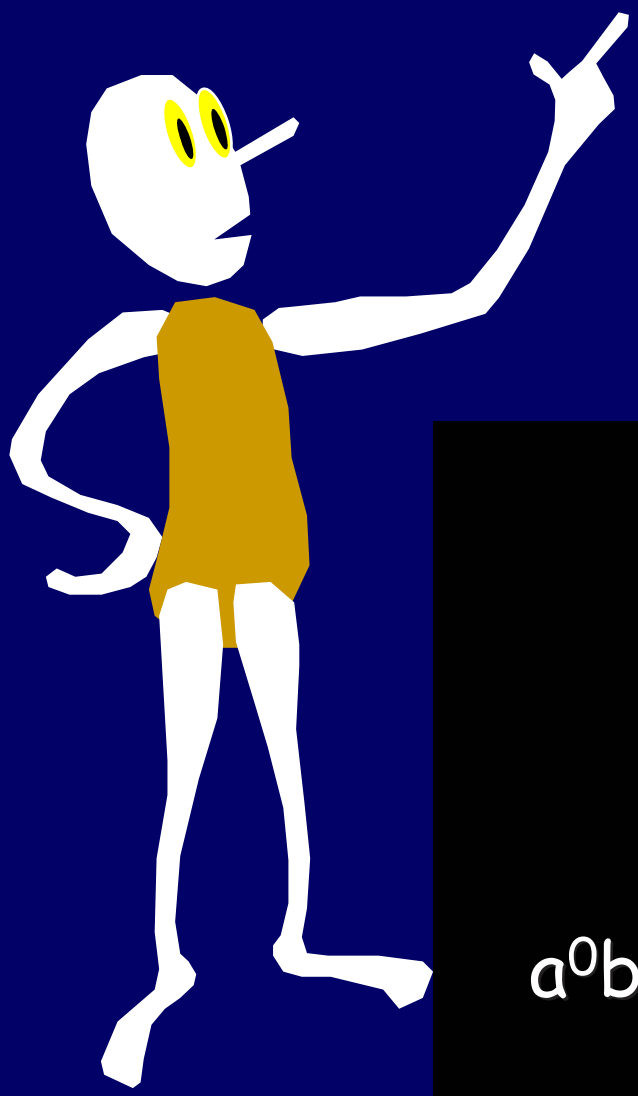
$$1 + c_1X^1 + .. + c_k\ X^k + \ldots$$

Suppose we multiply this out to get a single, infinite polynomial.

What is an expression for C_n?

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..)\ (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$$

$$1 + c_1X^1 + .. + c_k\ X^k + \ldots$$

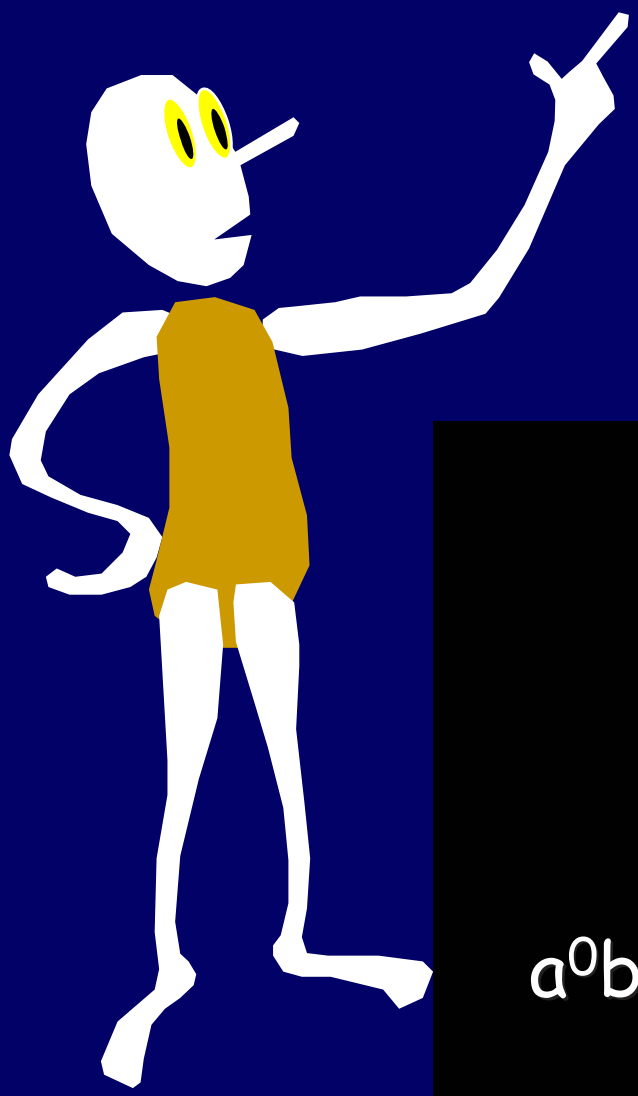


$$c_n =$$

$$a^0b^n + a^1b^{n-1} + \ldots\ a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$$

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..)\ (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$$

$$1 + c_1X^1 + .. + c_k\ X^k + \ldots$$

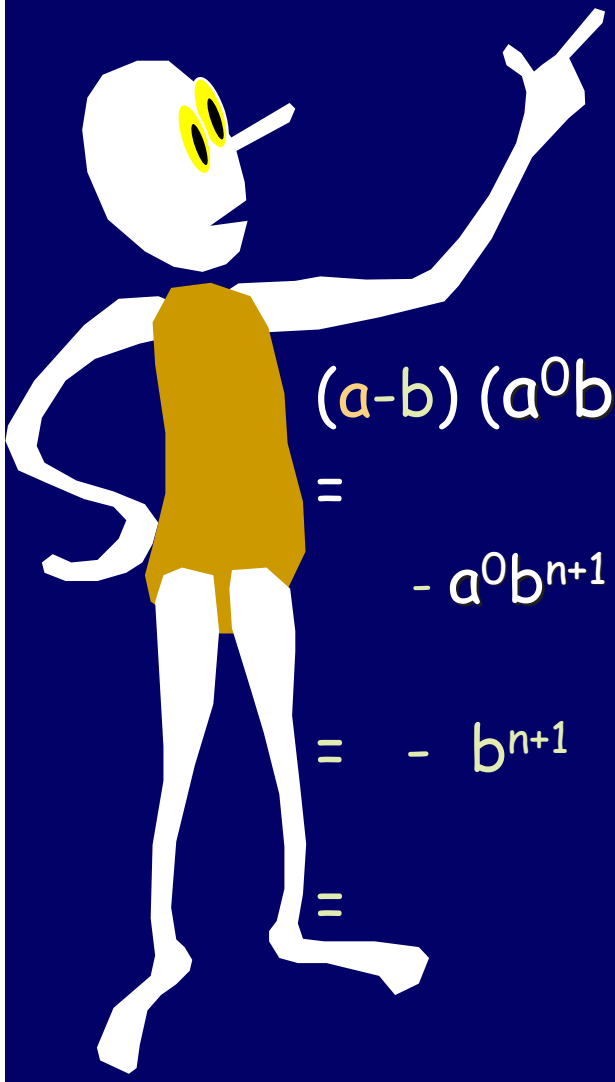


If a = b then

$$c_n = (n+1)(a^n)$$

$$a^0b^n + a^1b^{n-1} + \ldots\ a^ib^{n-i}\ldots + a^{n-1}b^1 + a^nb^0$$

$$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0 = \frac{a^{n+1} - b^{n+1}}{a - b}$$



$$(a-b)(a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0)$$

$$= \quad a^1b^n + \ldots a^{i+1}b^{n-i} \ldots \quad + a^nb^1 + a^{n+1}b^0$$

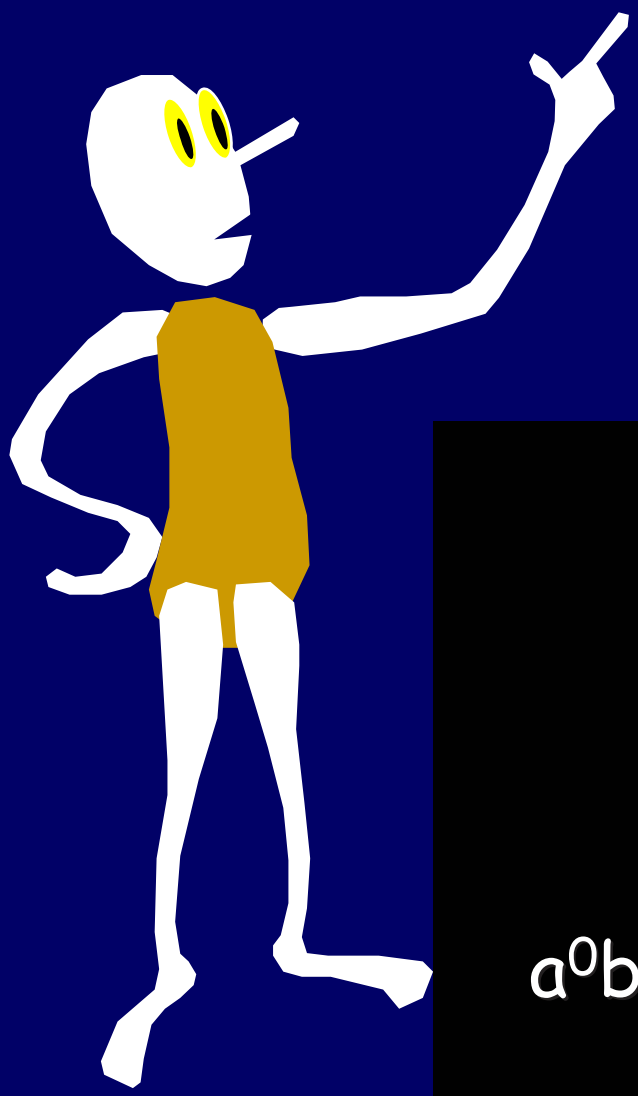
$$- a^0b^{n+1} \quad - a^1b^n \ldots \quad a^{i+1}b^{n-i} \ldots \quad - a^{n-1}b^2 - a^nb^1$$

$$= \quad - \quad b^{n+1} \quad + \quad a^{n+1}$$

$$= \quad a^{n+1} - b^{n+1}$$

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..) (1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$$

$$1 + c_1X^1 + .. + c_k X^k + \ldots$$

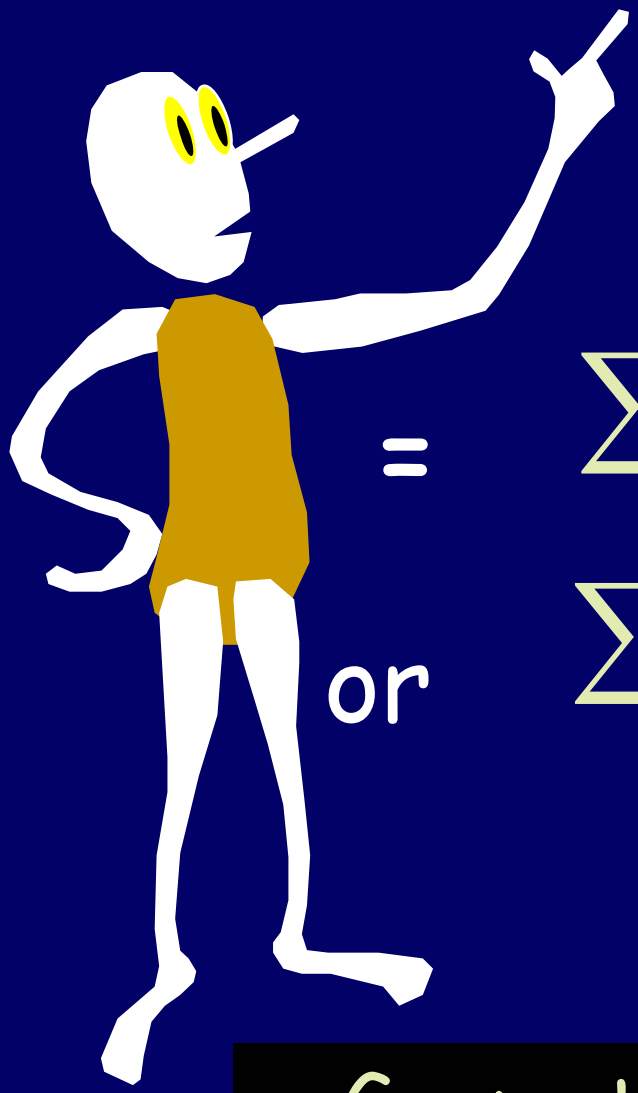


if $a \neq b$ then

$$c_n = \frac{a^{n+1} - b^{n+1}}{a - b}$$

$$a^0b^n + a^1b^{n-1} + \ldots a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$$

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots.)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots.) =$$



$$= \frac{1}{(1 - aX)(1-bX)}$$

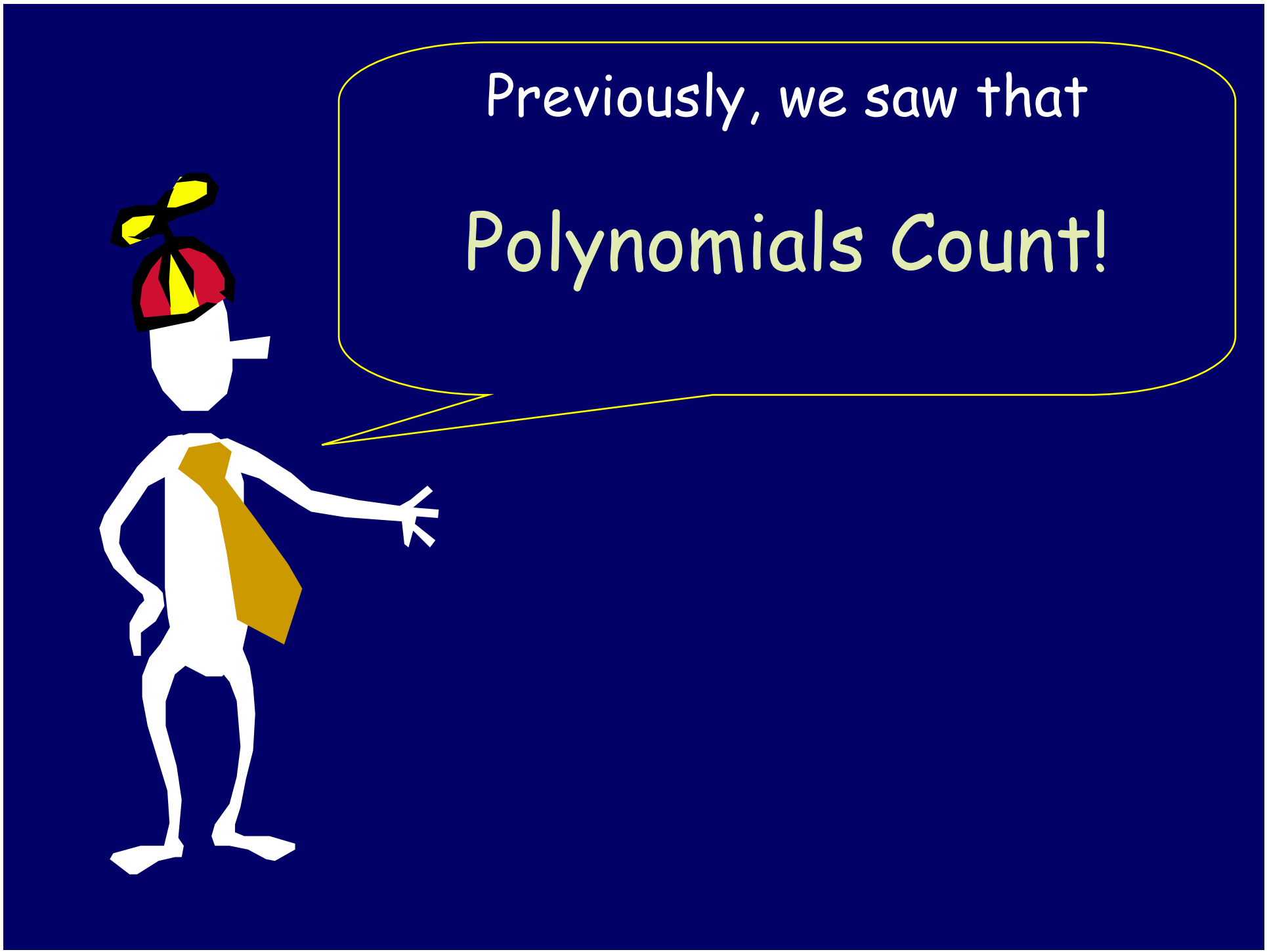
$$= \sum_{n=0..\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

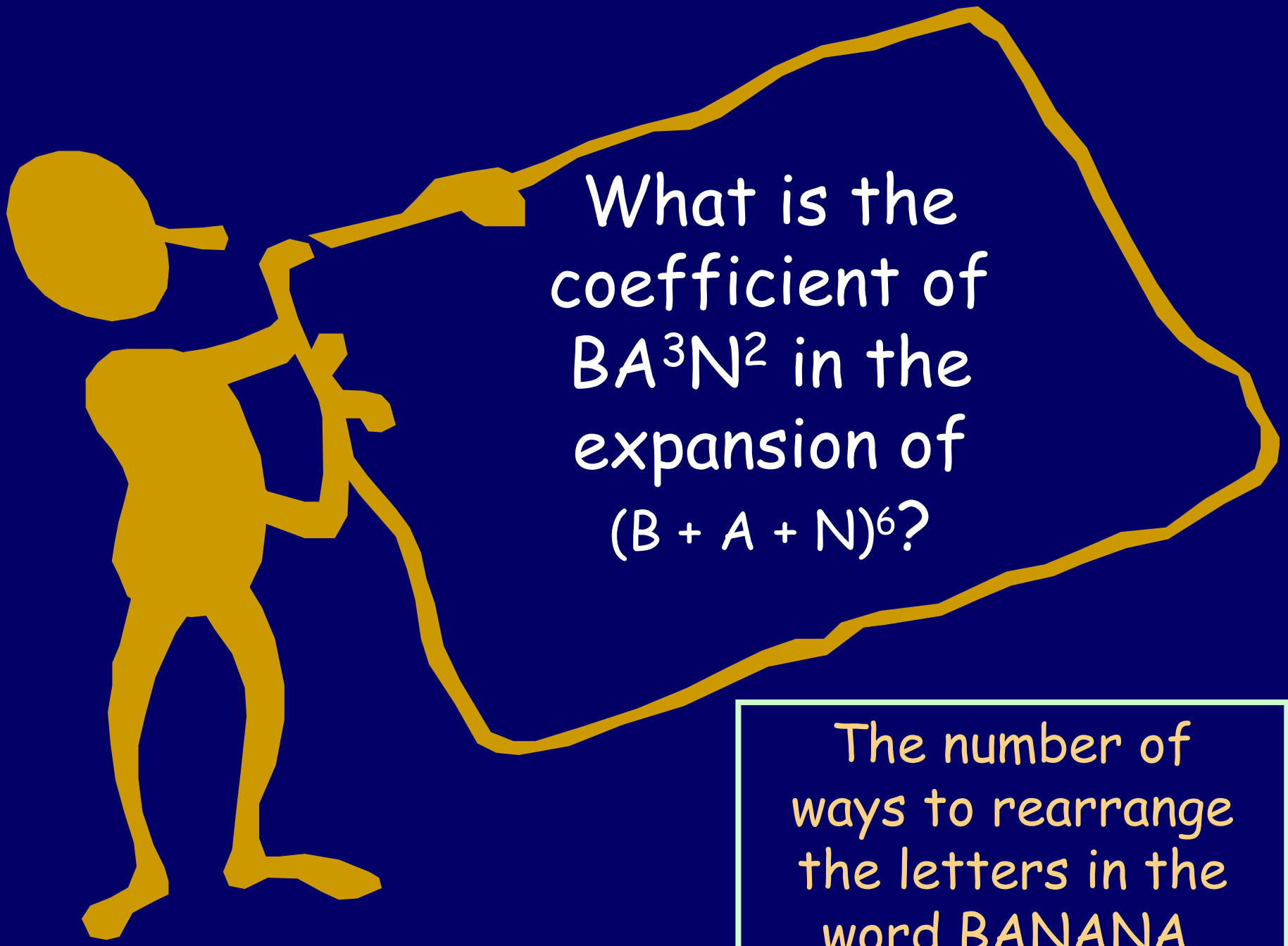
or

$$\sum_{n=0..\infty} (n+1)a^n X^n$$

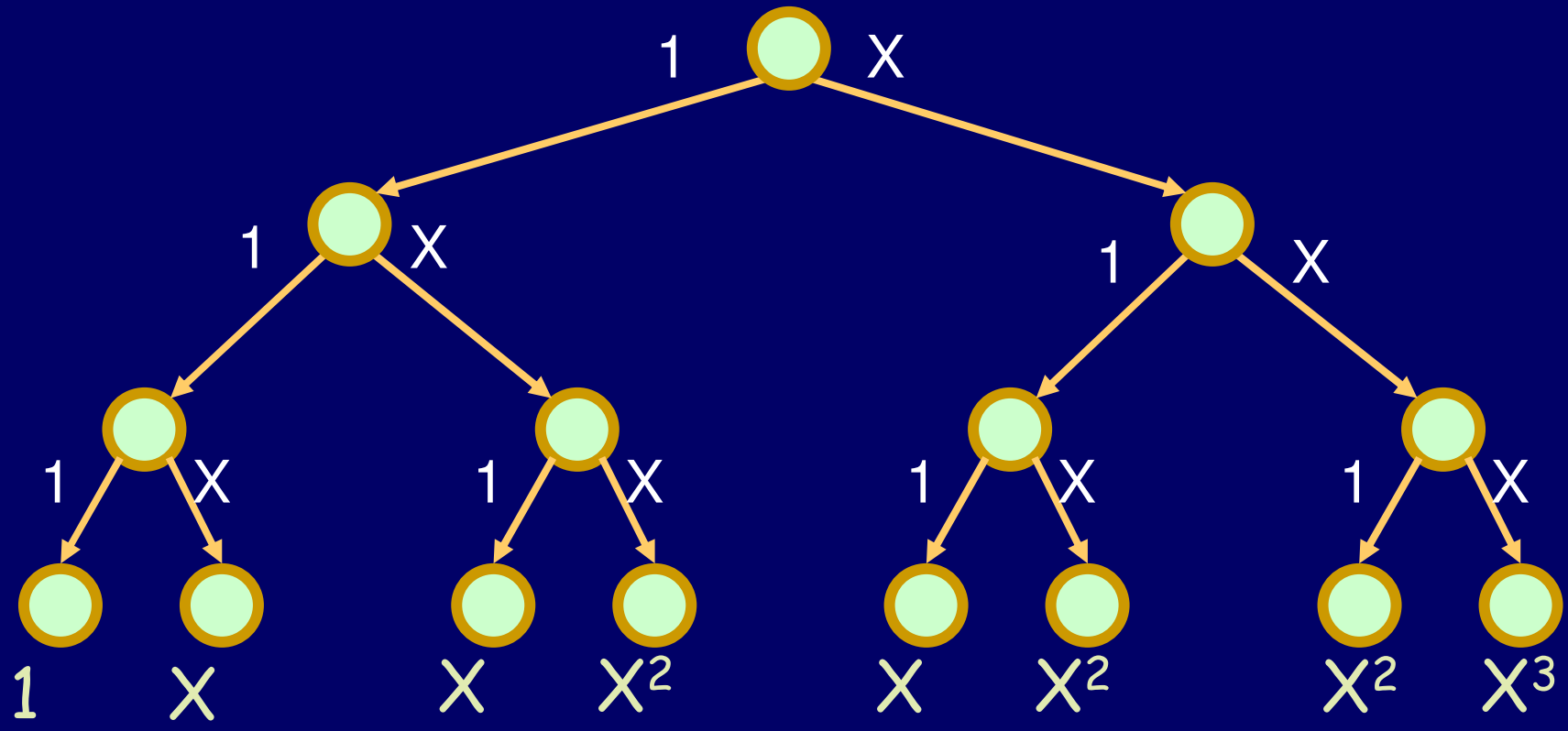
when a=b

Geometric Series (Quadratic Form)


Previously, we saw that
Polynomials Count!


What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?
The number of ways to rearrange the letters in the word BANANA.

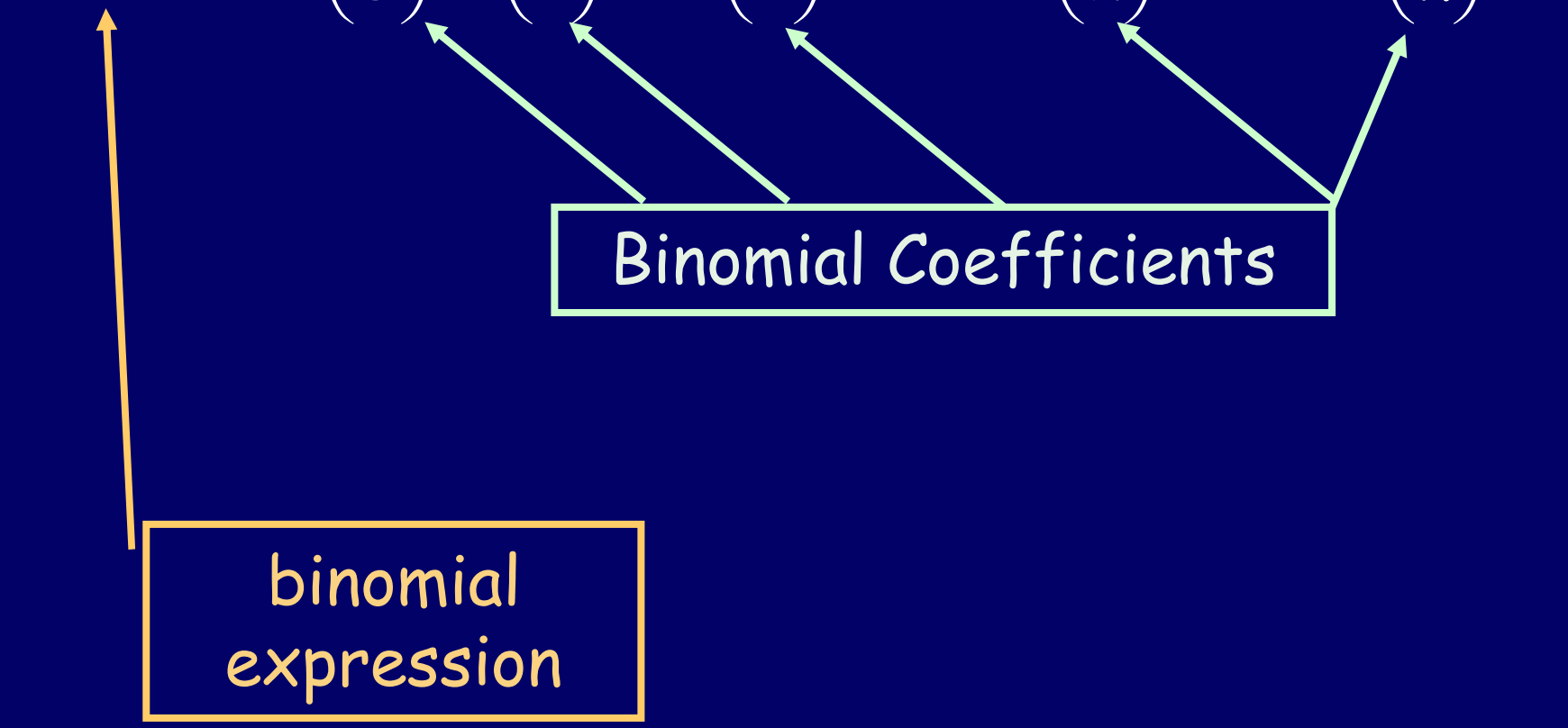
Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

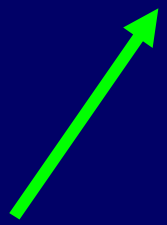


The Binomial Formula

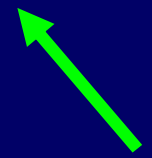
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

One polynomial, two representations

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



"Product form" or
"Generating form"



"Additive form" or
"Expanded form"

Power Series Representation

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k$$

"Closed form" or
"Generating form"

Since $\binom{n}{k} = 0$ if $k > n$

"Power series" ("Taylor series") expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x=1$.

$$2^n = \underbrace{\sum_{k=0}^{n} \binom{n}{k}}$$

The number of subsets of an n-element set

By varying x, we can discover new identities

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x = -1$.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

The number of even-sized subsets of an *n* element set is the same as the number of odd-sized subsets.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

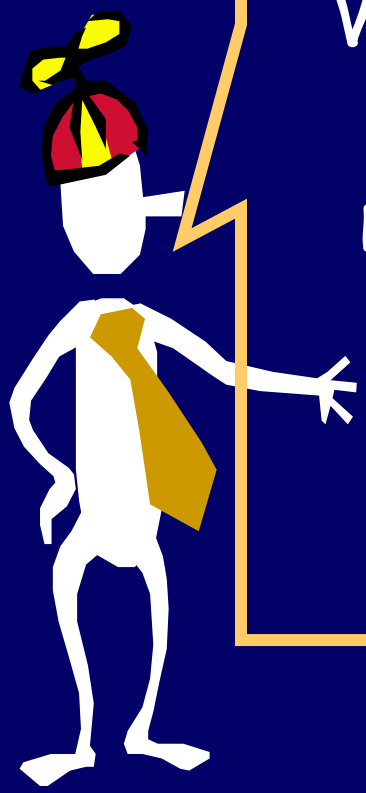
Let *x= -1.*

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

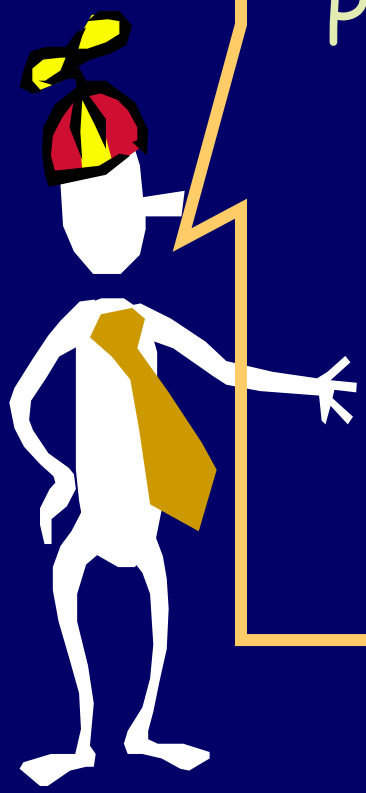
$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



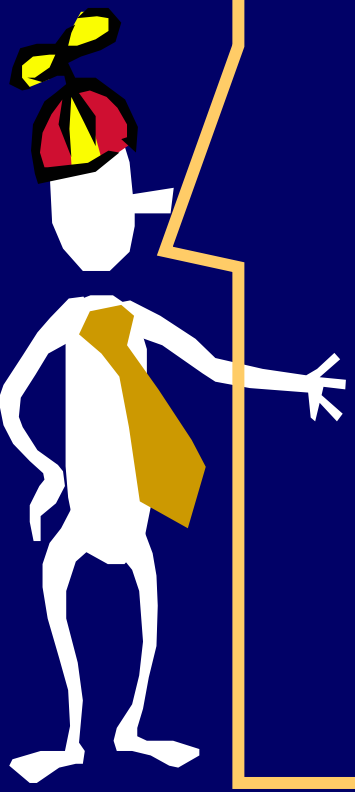
We could discover new identities by substituting in different numbers for X. One cool idea is to try complex roots of unity, however, the lecture is going in another direction.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



Proofs that work by manipulating algebraic forms are called "algebraic" arguments. Proofs that build a 1-1 onto correspondence are called "combinatorial" arguments.

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$



Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We gave an algebraic proof that

$$|O_n| = |E_n|$$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

A combinatorial proof must construct a one-to-one correspondence between O_n and E_n

An attempt at a correspondence

Let f_n be the function that takes an n-bit string and flips all its bits.

f_n is clearly a one-to-one and onto function

for odd *n*. *E.g.* in f_7 we have

0010011 → 1101100

1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100

101010 → 010101

Uh oh. Complementing maps evens to evens!

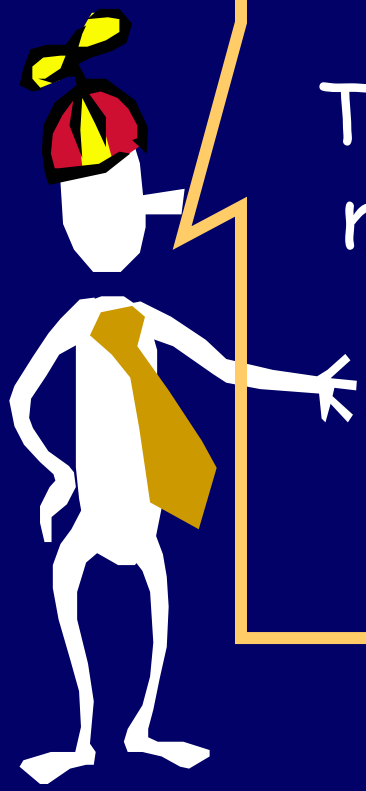
A correspondence that works for all *n*

Let f_n be the function that takes an n-bit string and flips only *the first bit*. For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

The Binomial Formula

$$(1+X)^0 = 1$$

$$(1+X)^1 = 1 + 1X$$

$$(1+X)^2 = 1 + 2X + 1X^2$$

$$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$$

$$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$$

Pascal's Triangle:
k^{th} row are the coefficients of $(1+X)^k$

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

k^{th} Row Of Pascal's Triangle:

$$\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{k},\ldots\binom{n}{n}$$

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Inductive definition of kth entry of nth row:

Pascal(n,0) = Pacal (n,n) = 1;

Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n,k)

$(1+X)^0 = 1$

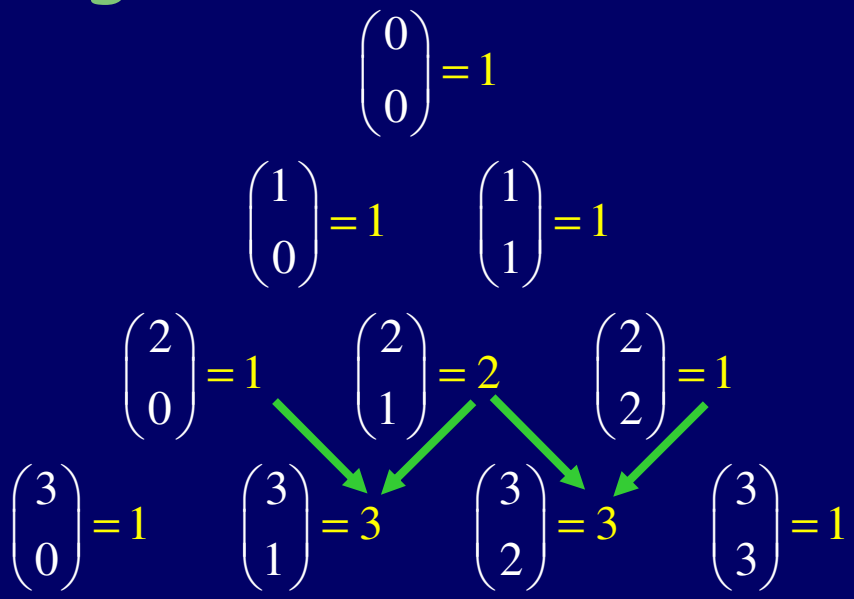
$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

"Pascal's Triangle"

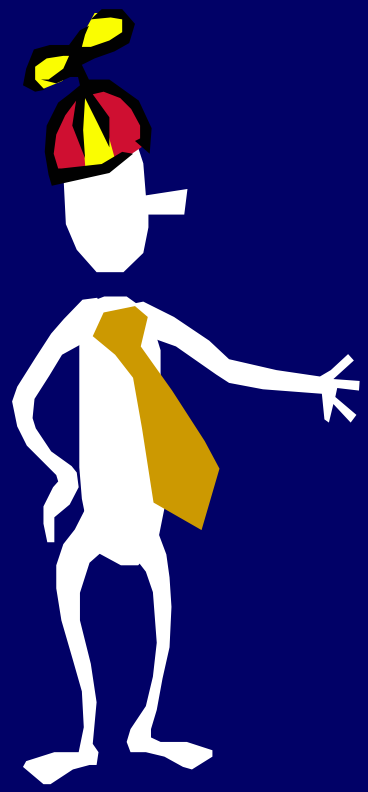


Al-Karaji, Baghdad 953-1029

Chu Shin-Chieh 1303
The Precious Mirror of the Four Elements
. . . Known in Europe by 1529

Blaise Pascal 1654

Pascal's Triangle



1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

. . .

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

Summing The Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$1 = 1$

$1 + 1 = 2$

$1 + 2 + 1 = 4$

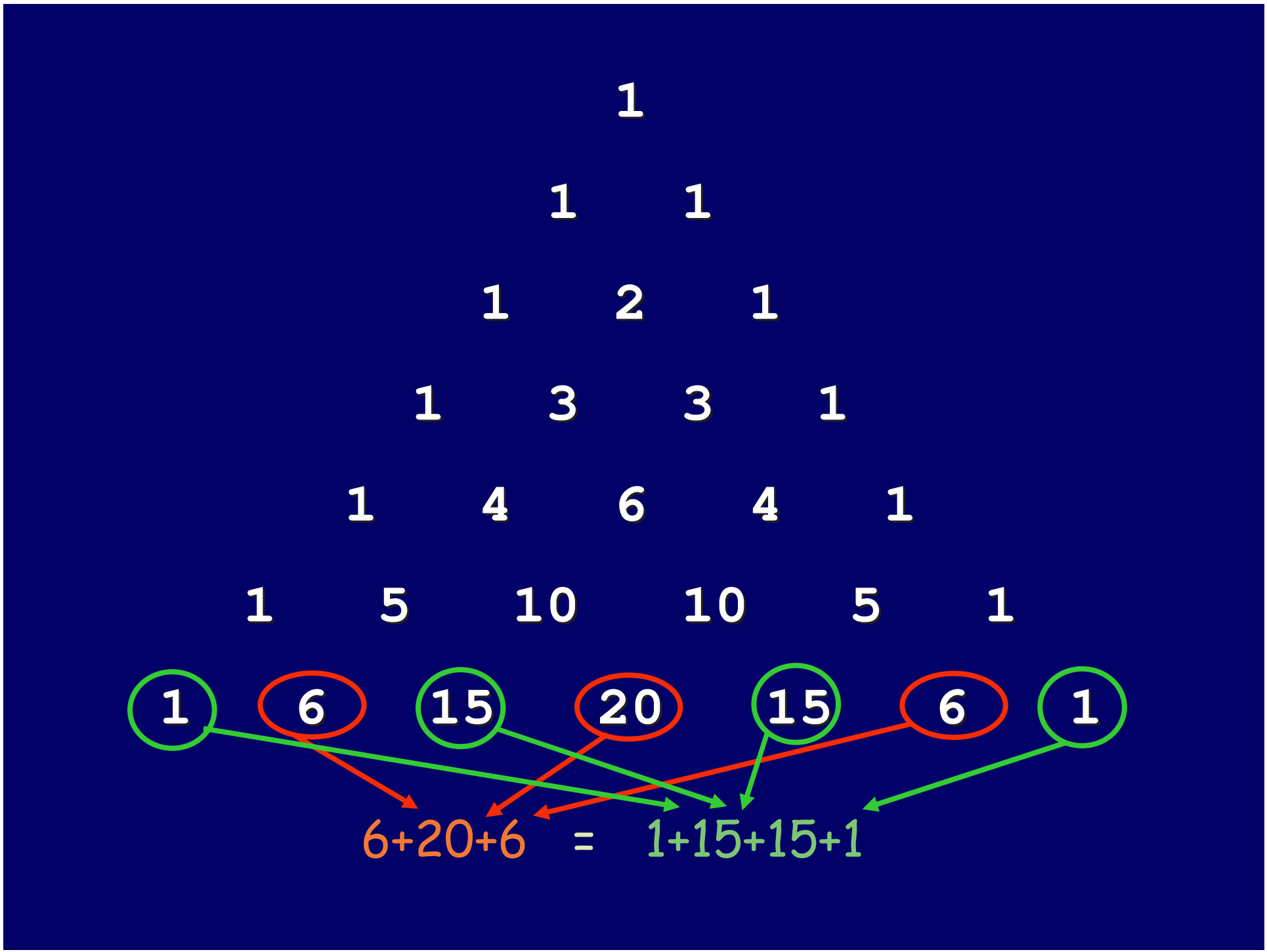
$1 + 3 + 3 + 1 = 8$

$1 + 4 + 6 + 4 + 1 = 16$

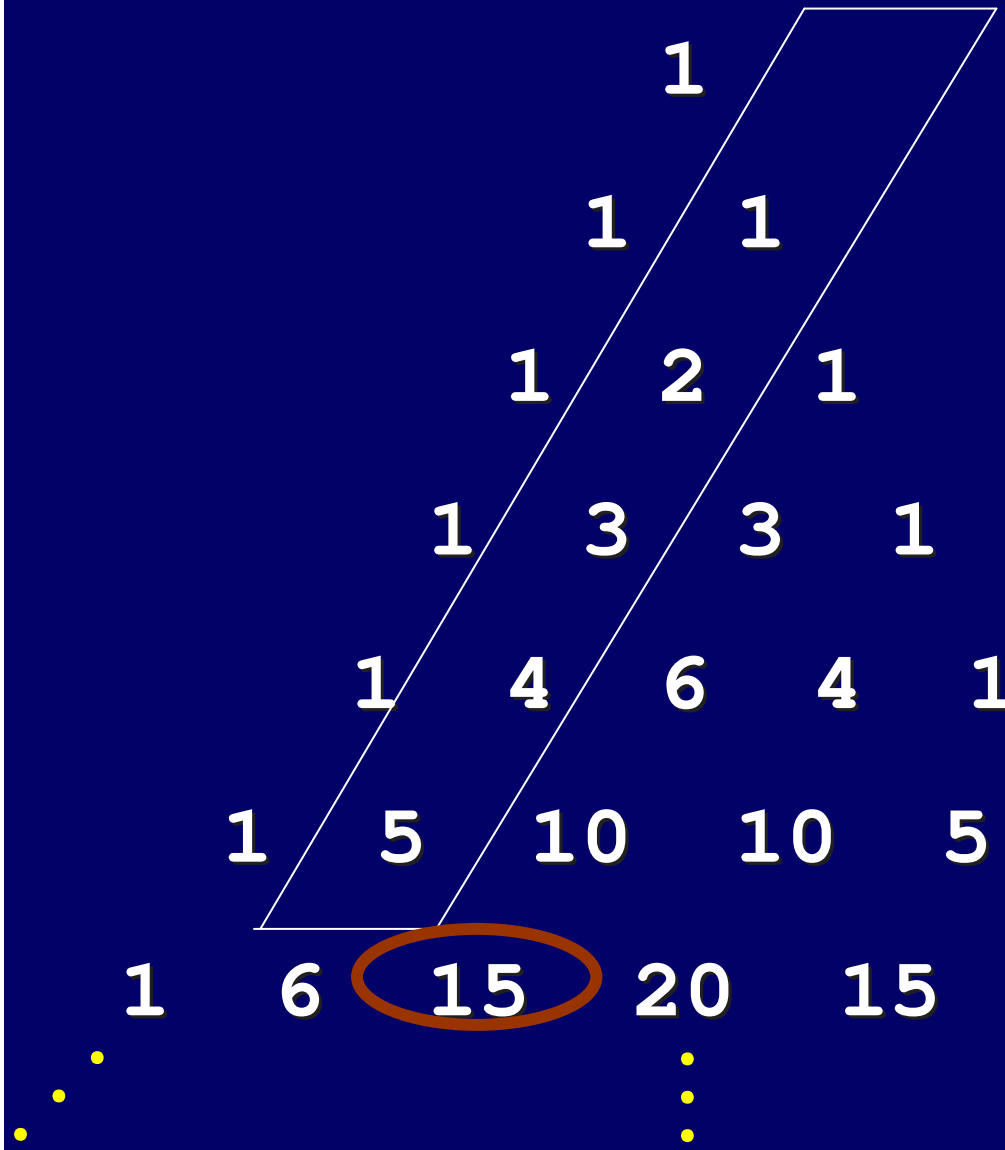
$1 + 5 + 10 + 10 + 5 + 1 = 32$

$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$

⋮ ⋮ ⋮


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
6+20+6 = 1+15+15+1

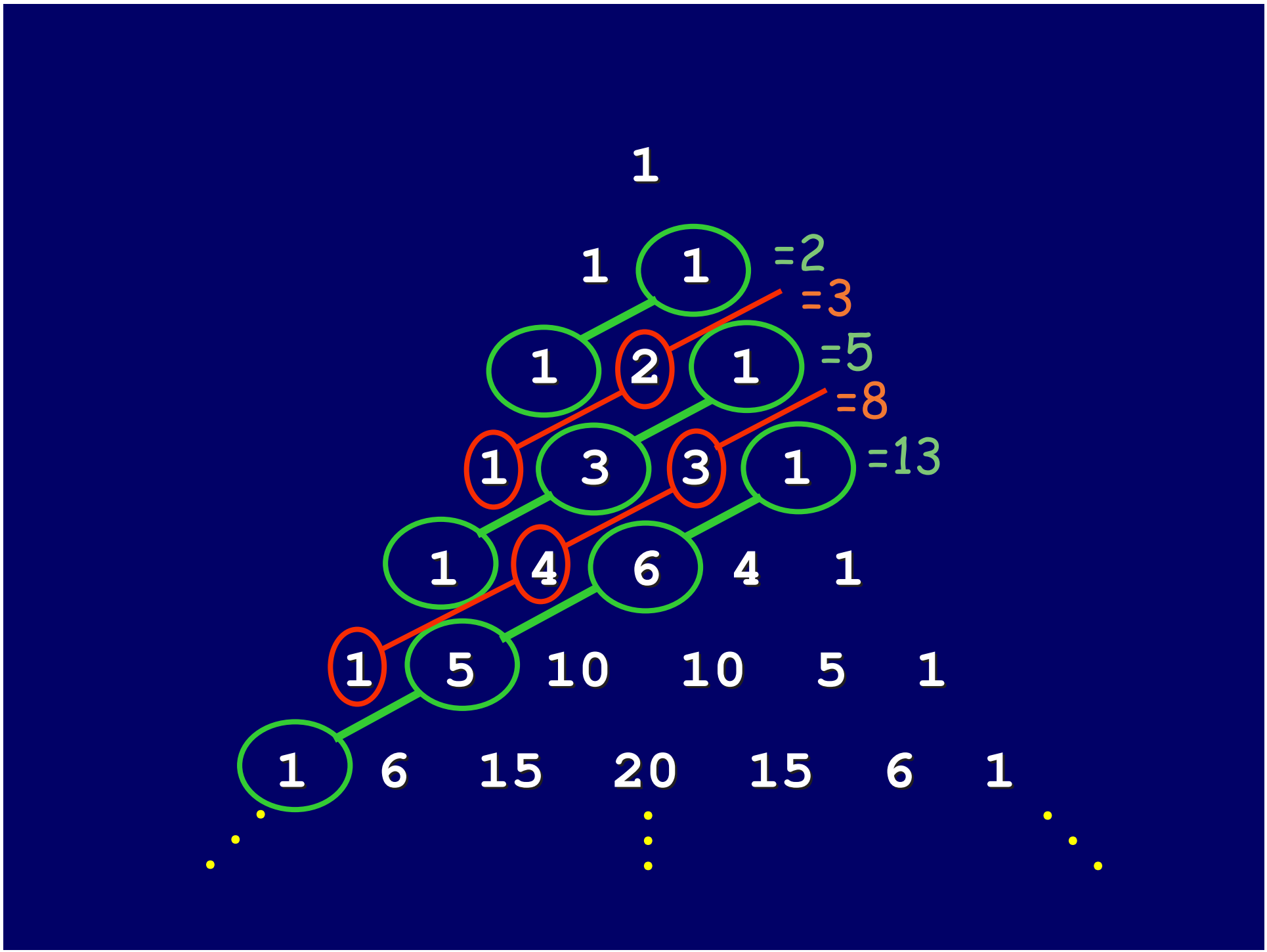
Summing on 1st Avenue

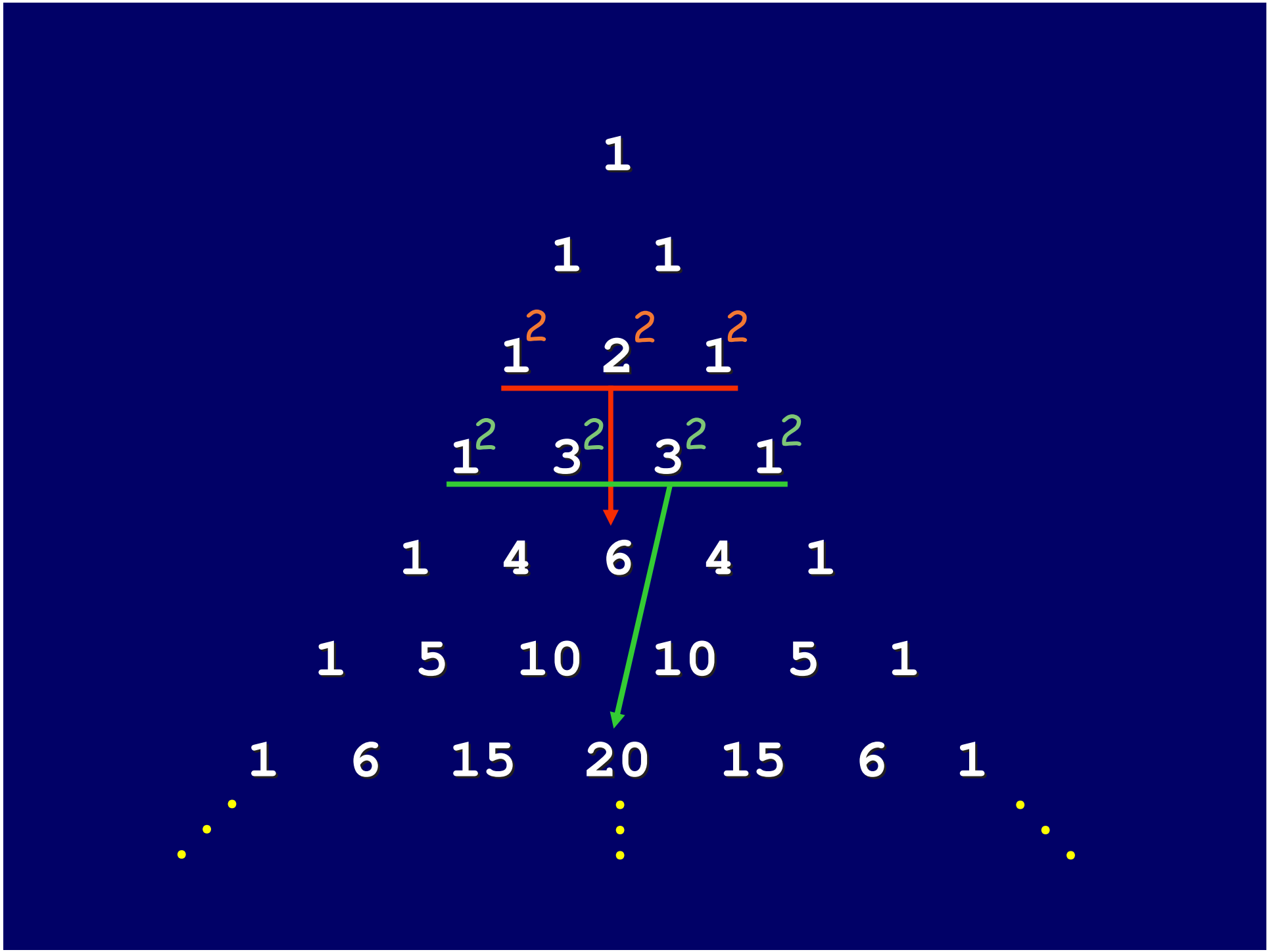


$$\sum_{i=k}^{n} i = \sum_{i=k}^{n} \binom{i}{1} = \binom{n+1}{2} = \frac{n \cdot (n+1)}{2}$$

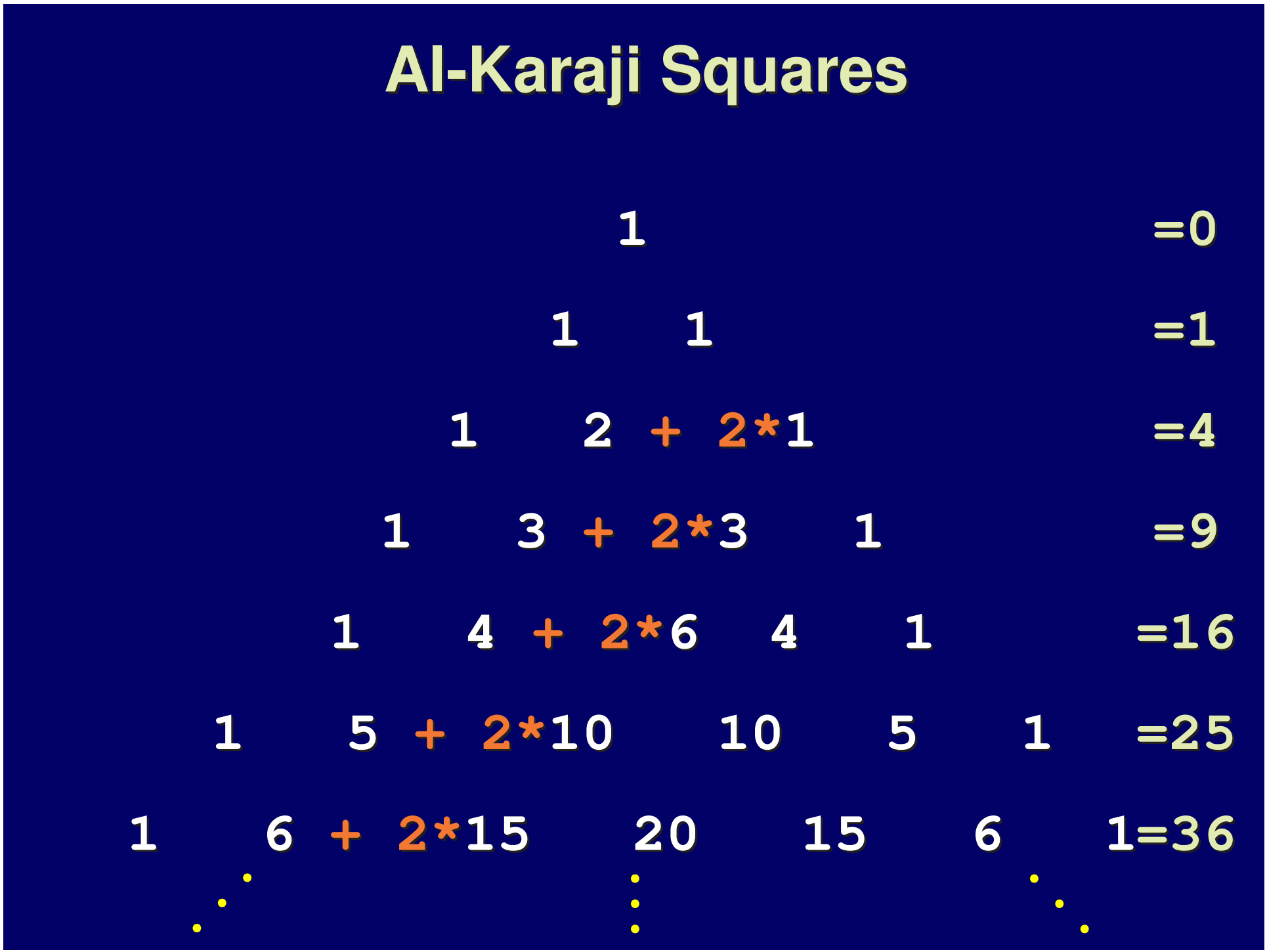
Summing on k^{th} Avenue

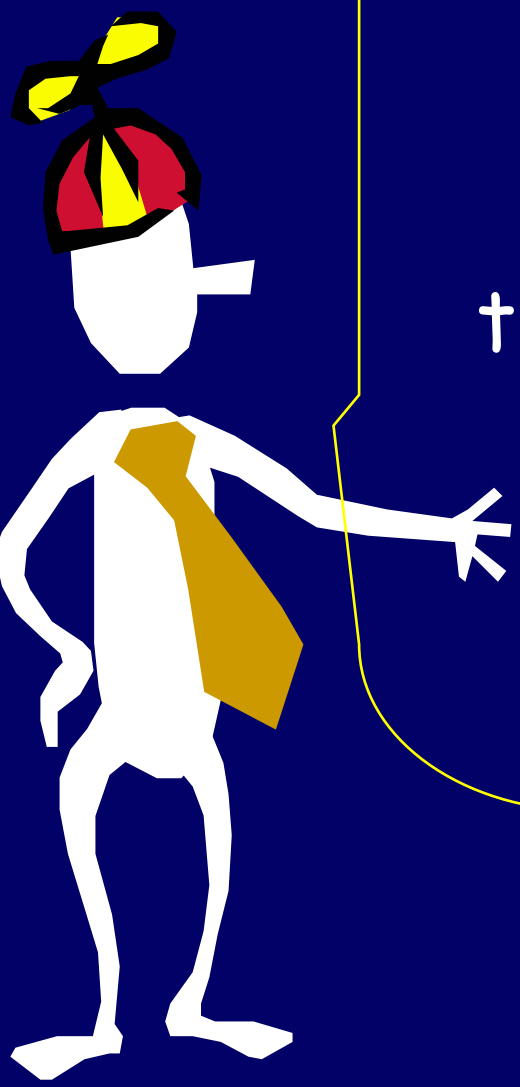
$$\sum_{i=k}^{n}\binom{i}{k}=\binom{n+1}{k+1}$$


1
1 1
=2
=3
1 2 1
=5
=8
1 3 3 1
=13
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1


1
1 1
1² 2² 1²
1² 3² 3² 1²
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

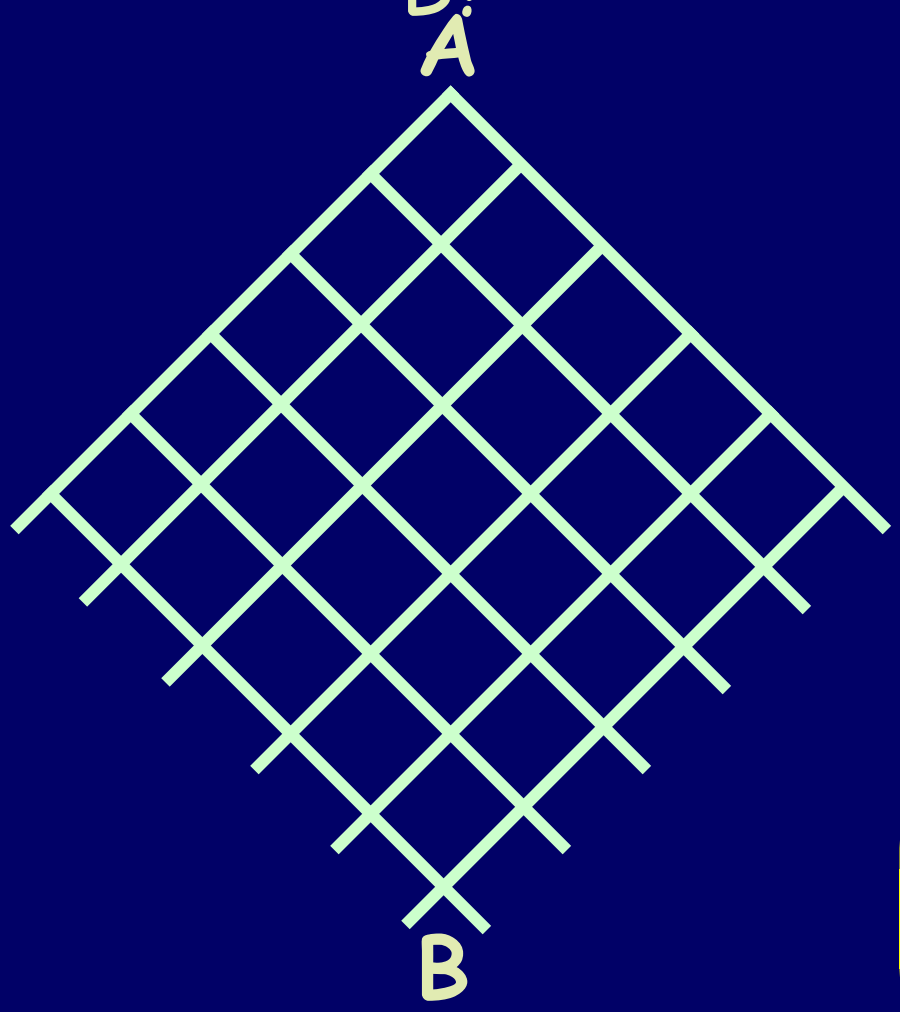
Al-Karaji Squares





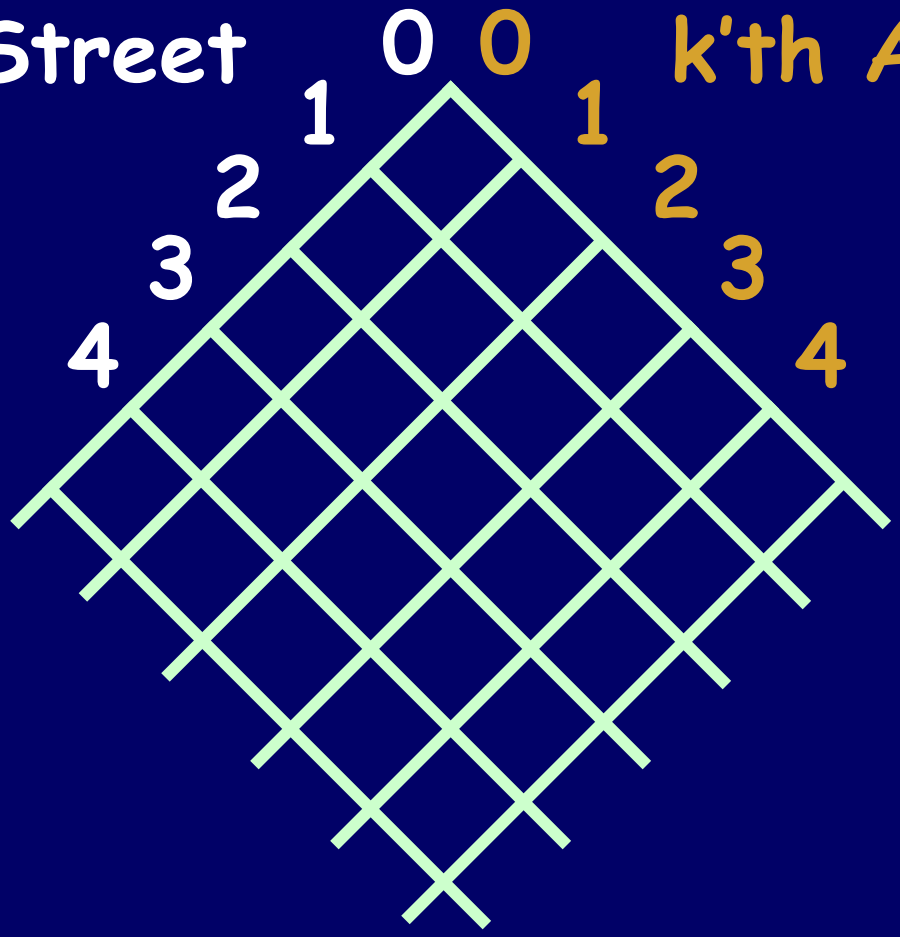
All these properties can be proved inductively and algebraically. We will give combinatorial proofs using the Manhattan block walking representation of binomial coefficients.

How many shortest routes from A to B?



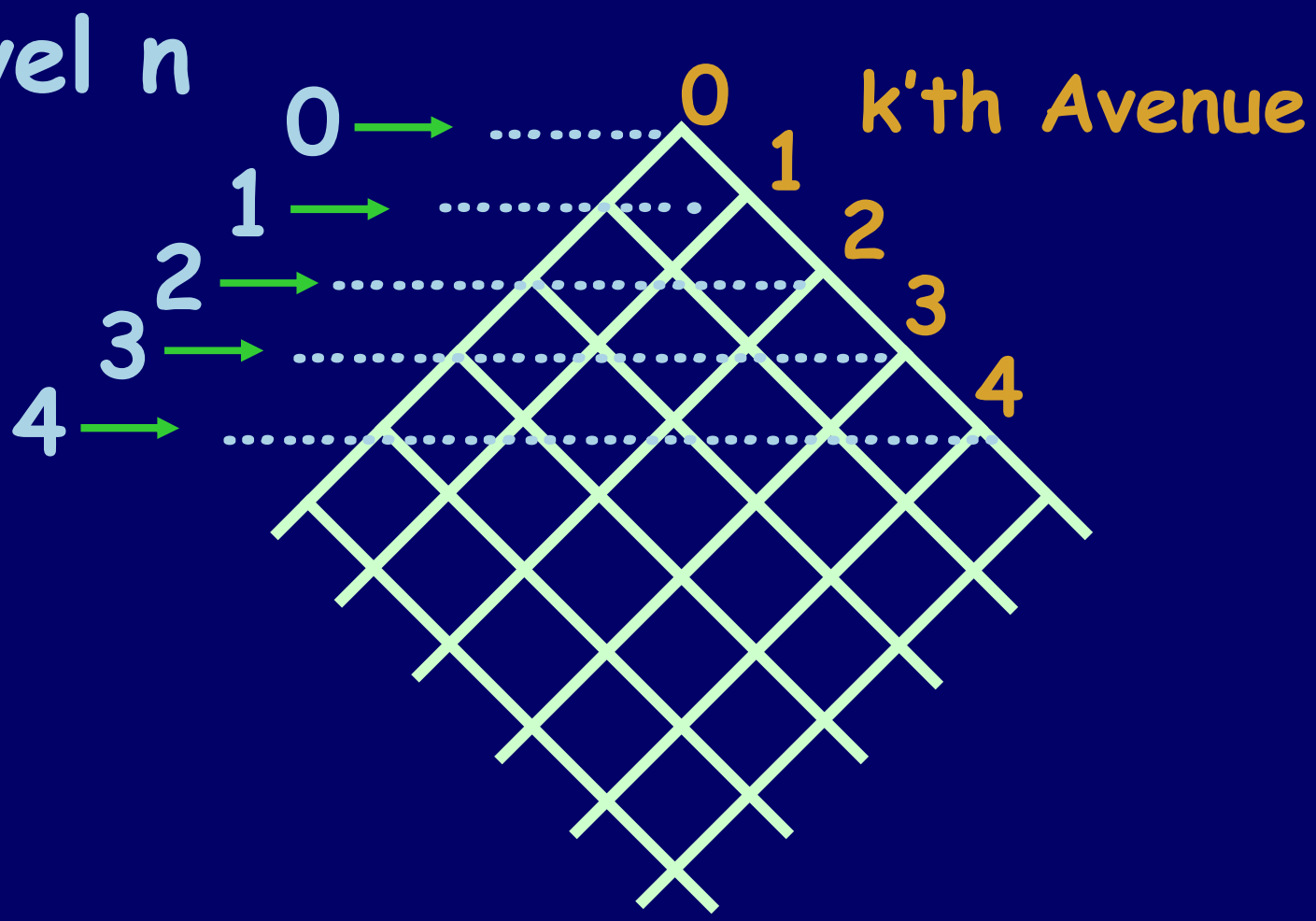
$$\binom{10}{5}$$

Manhattan



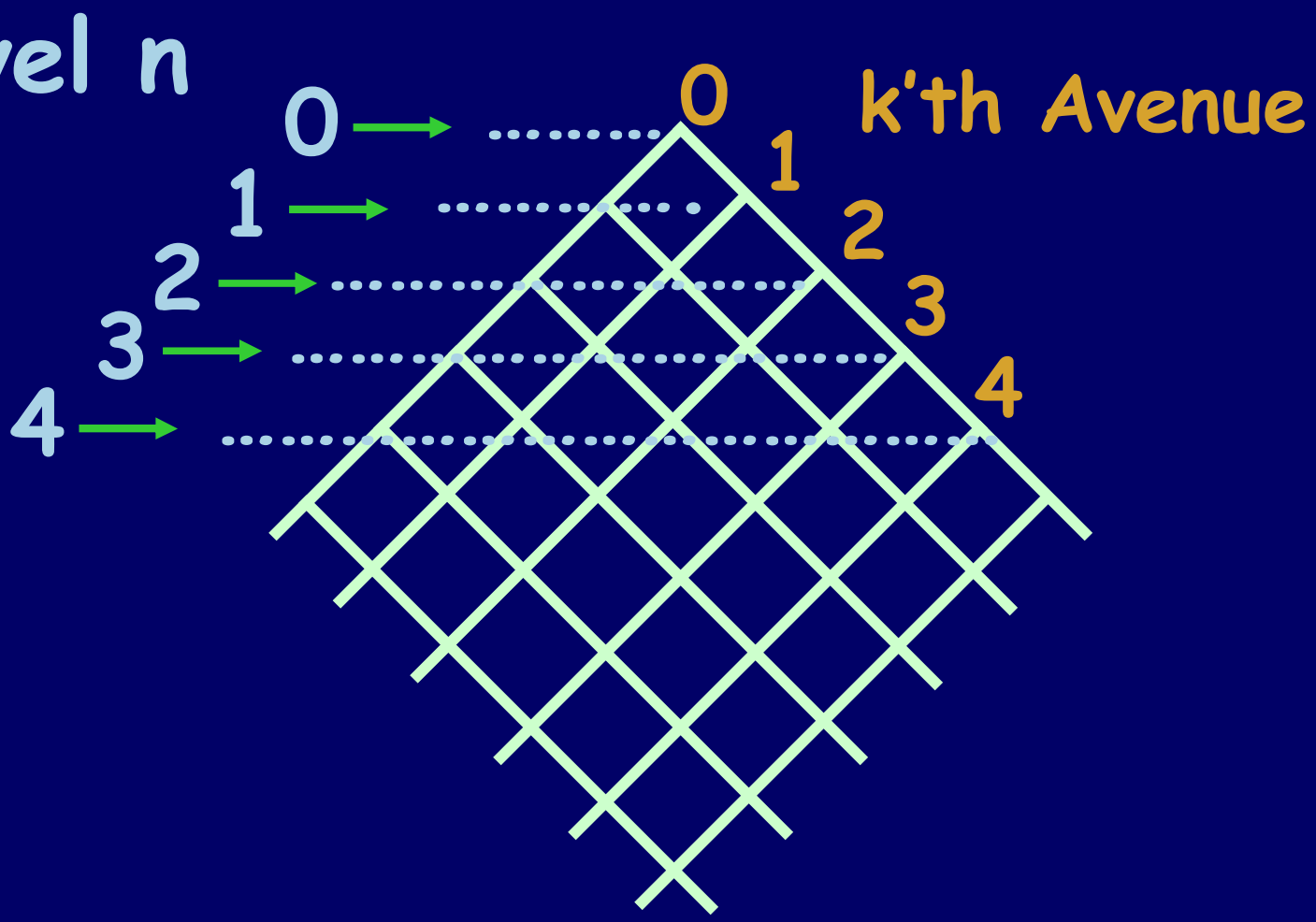
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k).

Manhattan

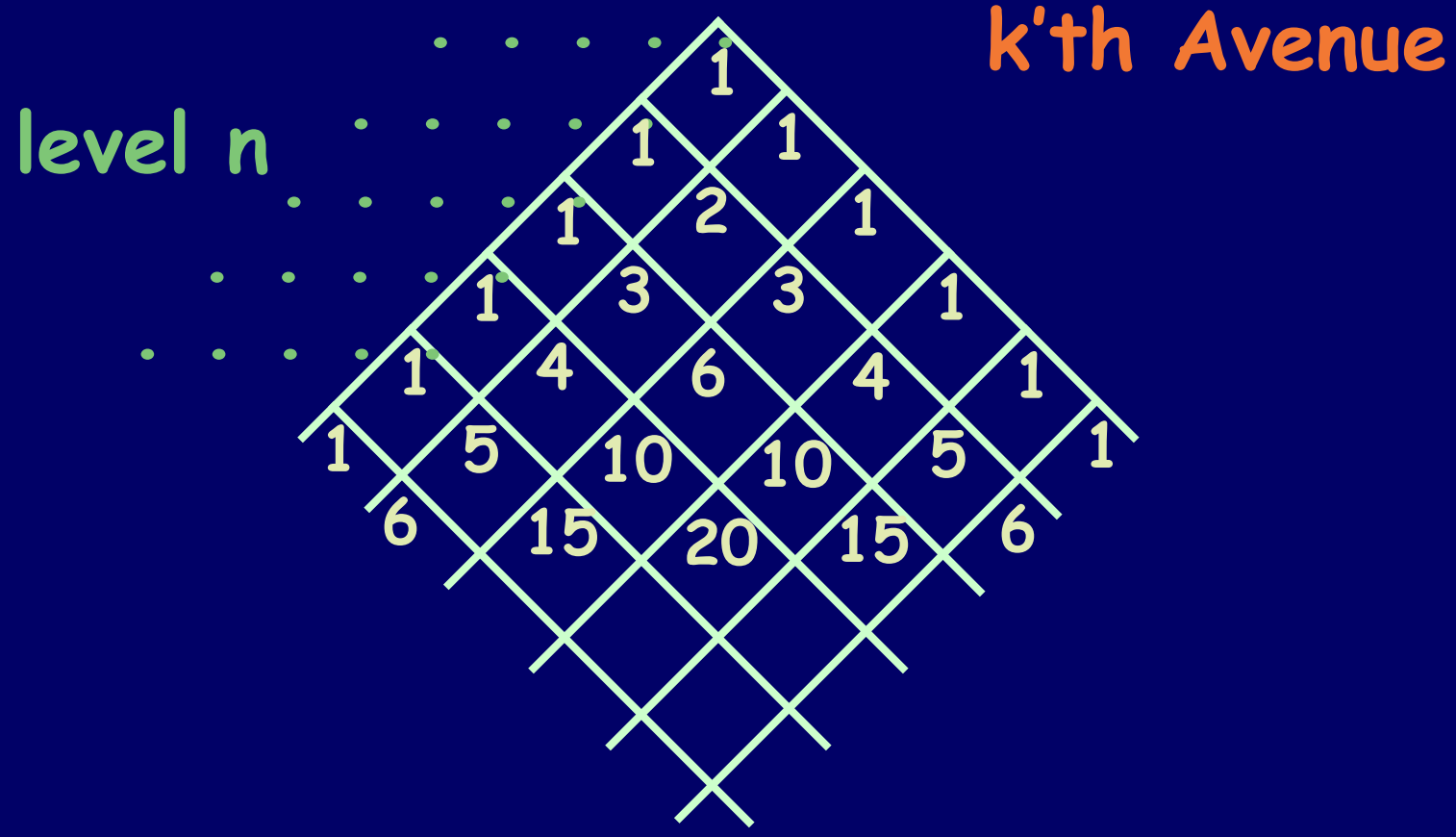


There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k).

Manhattan

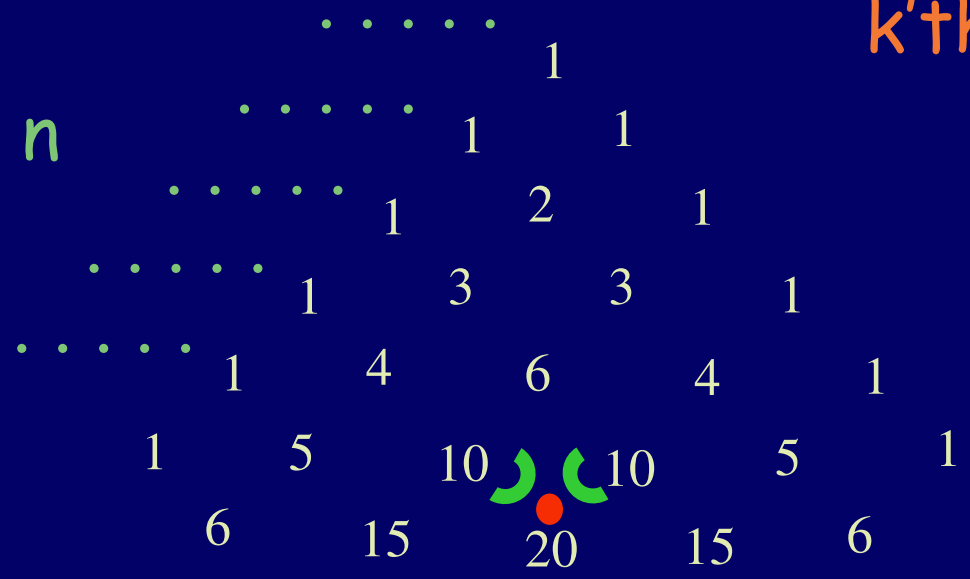


There are $\binom{n}{k}$ shortest routes from (0,0) to Level n and k^{th} Avenue.

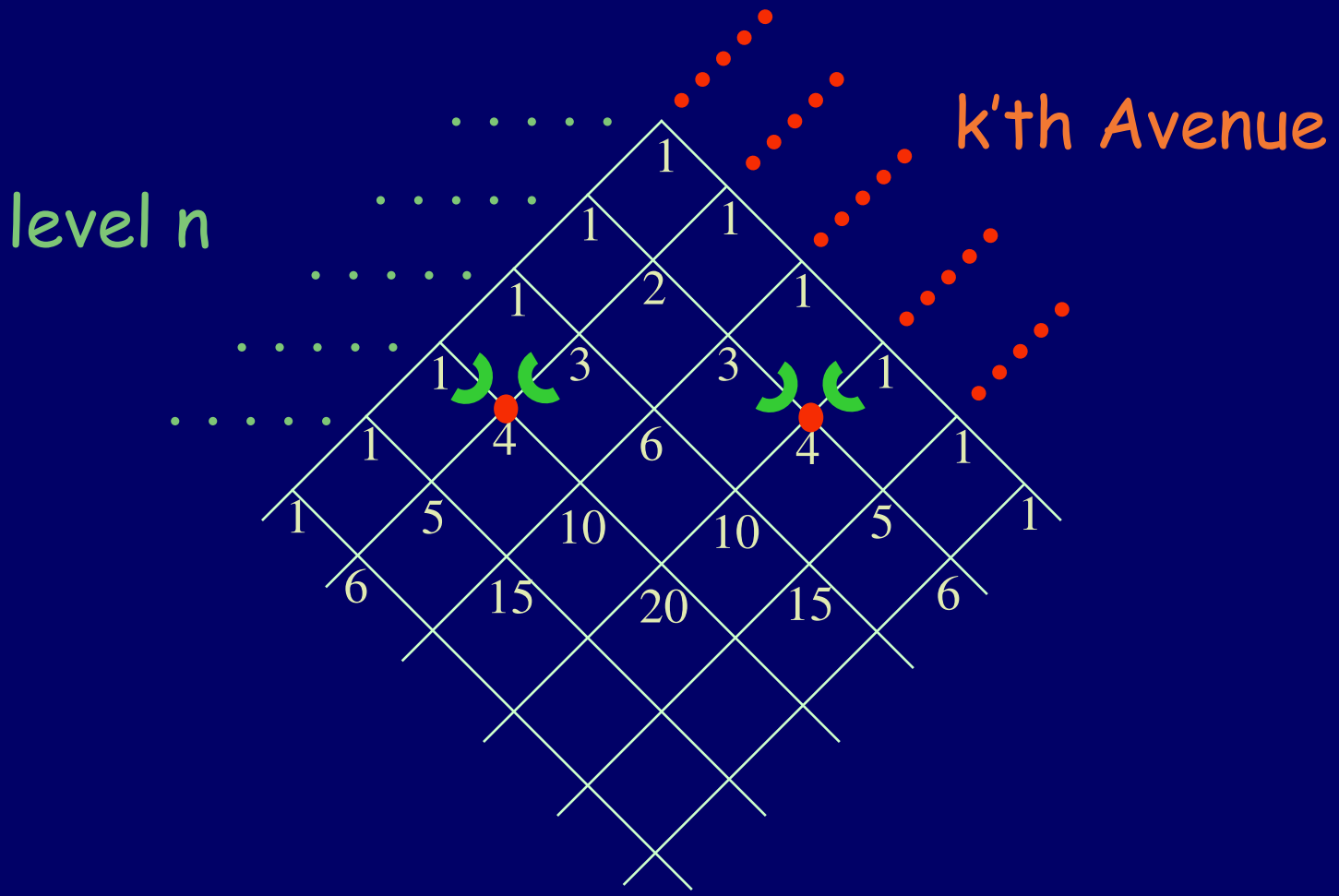

level n
k'th Avenue
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6

level n

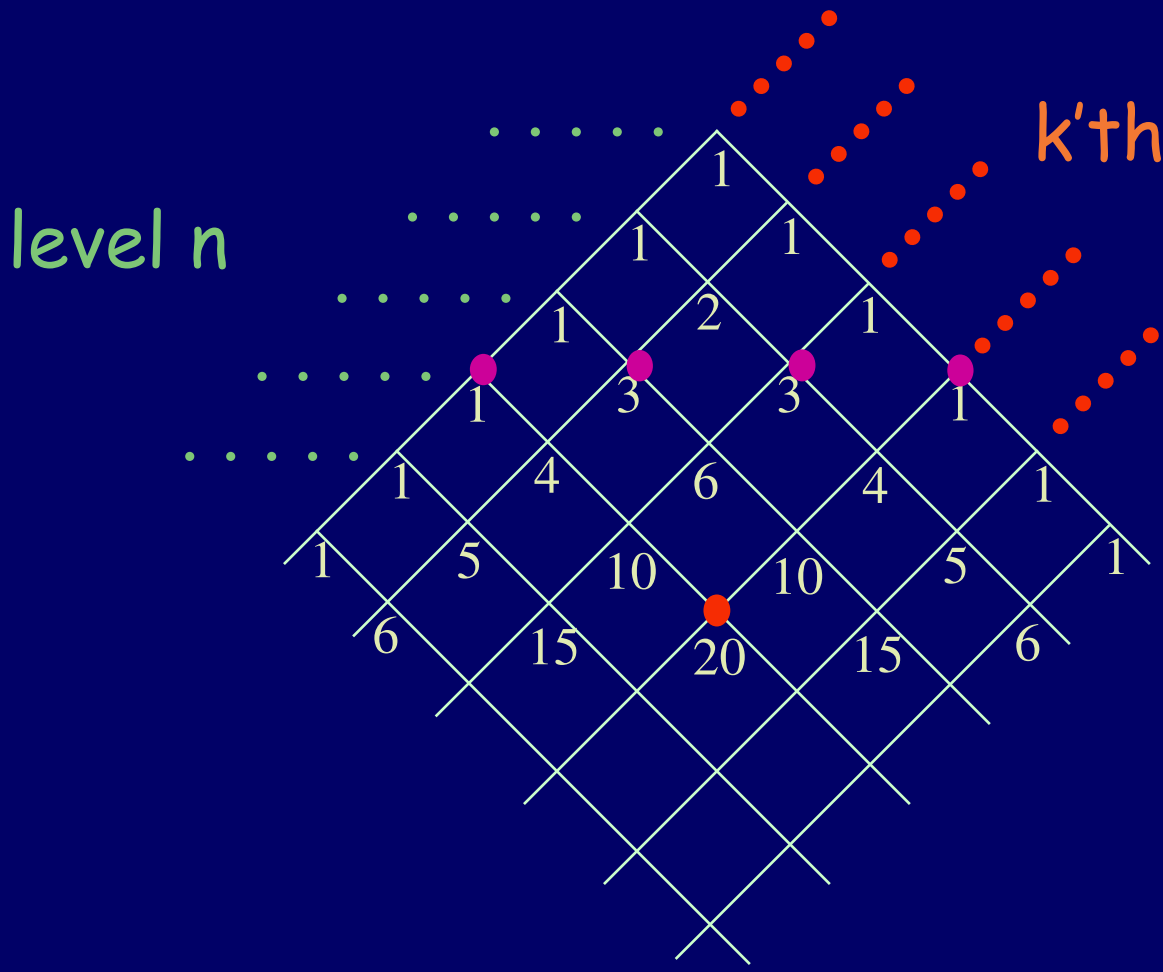
k'th Avenue



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



$$\sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}$$



$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

By convention:

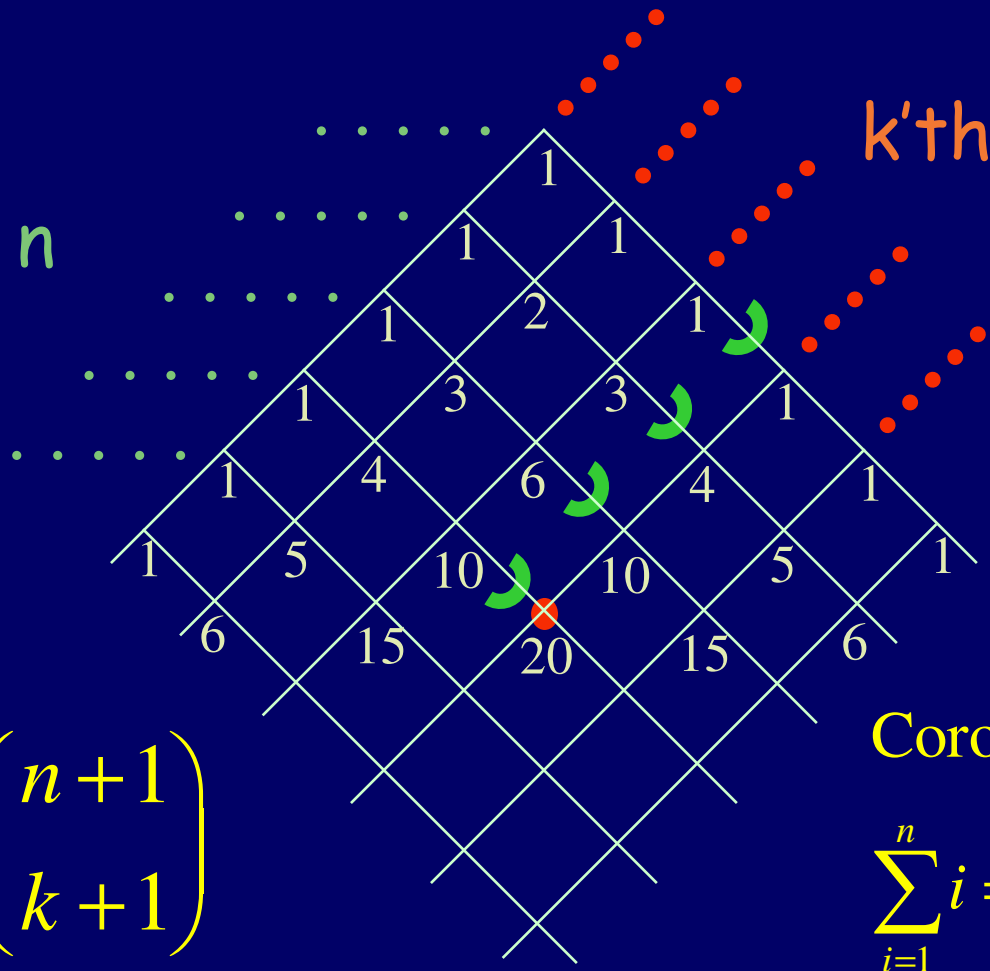
$$0! = 1 \qquad \text{(empty product = 1)}$$

$$\binom{n}{k} = 1 \quad \text{if } k = 0$$

$$\binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n$$

level n

k'th Avenue



$$\sum_{i=1}^{n}\binom{i}{k}=\binom{n+1}{k+1}$$

Corollary ($k=1$)

$$\sum_{i=1}^{n} i=\binom{n+1}{2}=\frac{n(n+1)}{2}$$

Application (Al-Karaji):

$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$= (1\cdot 0 + 1) + (2\cdot 1 + 2) + (3\cdot 2 + 3) + \cdots + (n(n-1) + n)$$

$$= 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + n(n-1) + \sum_{i=1}^{n} i$$

$$= 2\left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n}{2}\right] + \binom{n+1}{2}$$

$$= 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{(2n+1)(n+1)n}{6}$$

Vector Programs

Let's define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable $V^{\rightarrow}$ can be thought of as:

< * , * , * , * , * , * , ... >

0 1 2 3 4 5 . . . . . .

Vector Programs

Let k stand for a scalar constant

<k> will stand for the vector <k,0,0,0,…>

<0> = <0,0,0,0,….>

<1> = <1,0,0,0,…>

$V^{\rightarrow}+T^{\rightarrow}$ means to add the vectors position-wise.

<4,2,3,…> + <5,1,1,….> = <9,3,4,…>

Vector Programs

RIGHT($V^{\rightarrow}$) means to shift every number in $V^{\rightarrow}$ one position to the right and to place a 0 in position 0.

RIGHT(<1,2,3, ...>) = <0,1,2,3,. ...>

Vector Programs

Example:

Stare

$V^{\rightarrow}$:= <6>;

$V^{\rightarrow}$ = <6,0,0,0,..>

$V^{\rightarrow}$:= RIGHT($V^{\rightarrow}$) + <42>;

$V^{\rightarrow}$ = <42,6,0,0,..>

$V^{\rightarrow}$:= RIGHT($V^{\rightarrow}$) + <2>;

$V^{\rightarrow}$ = <2,42,6,0,..>

$V^{\rightarrow}$:= RIGHT($V^{\rightarrow}$) + <13>;

$V^{\rightarrow}$= <13,2,42,6,.>

$$V^{\rightarrow} = \langle 13, 2, 42, 6, 0, 0, 0, \ldots \rangle$$

Vector Programs

Example:

$V^{\rightarrow}$:= <1>;

Loop n times:
 $V^{\rightarrow}$:= $V^{\rightarrow}$ + RIGHT($V^{\rightarrow}$);

Stare

$V^{\rightarrow}$ = <1,0,0,0,..>

$V^{\rightarrow}$ = <1,1,0,0,..>
$V^{\rightarrow}$ = <1,2,1,0,..>
$V^{\rightarrow}$= <1,3,3,1,.>

$V^{\rightarrow}$ = n^{th} row of Pascal's triangle.

Vector programs
can be implemented
by polynomials!

Programs -----> Polynomials

The vector $V^{\rightarrow}$ = < a_0, a_1, a_2, . . . > will be represented by the polynomial:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

Formal Power Series

The vector $V^{\rightarrow}$ = < a_0, a_1, a_2, . . . > will be represented by the formal power series:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

$$V^{\rightarrow} = \langle a_0, a_1, a_2, \ldots \rangle$$

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

<0>is represented by	0
<k>is represented by	k
$V^{\rightarrow}$+ $T^{\rightarrow}$ is represented by	$(P_V + P_T)$
RIGHT($V^{\rightarrow}$) is represented by	$(P_V\ X)$

Vector Programs

Example:

$V^{\rightarrow}$:= <1>;

$P_V := 1;$

Loop n times:
 $V^{\rightarrow} := V^{\rightarrow} + \mathrm{RIGHT}(V^{\rightarrow});$

$P_V := P_V + P_V X;$

$V^{\rightarrow}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

$V^{\rightarrow}$:= <1>;

$P_V := 1;$

Loop n times:

$V^{\rightarrow} := V^{\rightarrow} + \mathrm{RIGHT}(V^{\rightarrow});$

$P_V := P_V (1+ X);$

$V^{\rightarrow}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

$V^{\rightarrow}$:= <1>;

Loop n times:
 $V^{\rightarrow}$:= $V^{\rightarrow}$ + RIGHT($V^{\rightarrow}$);

$P_V = (1+X)^n$

$V^{\rightarrow}$ = n^{th} row of Pascal's triangle.