Great Theoretical Ideas In Computer Science

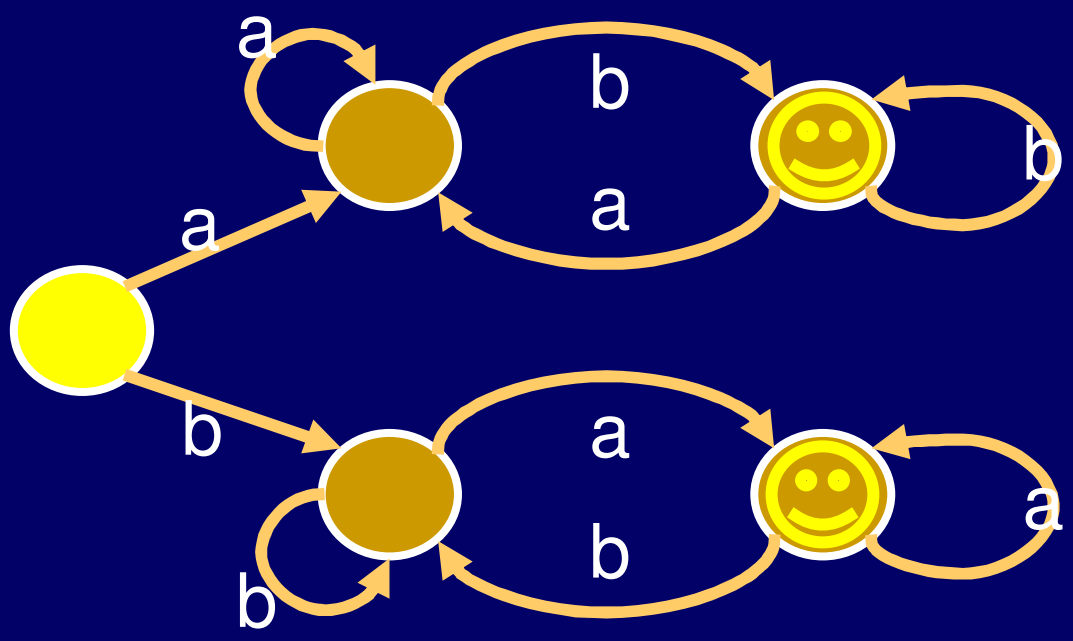
Steven Rudich

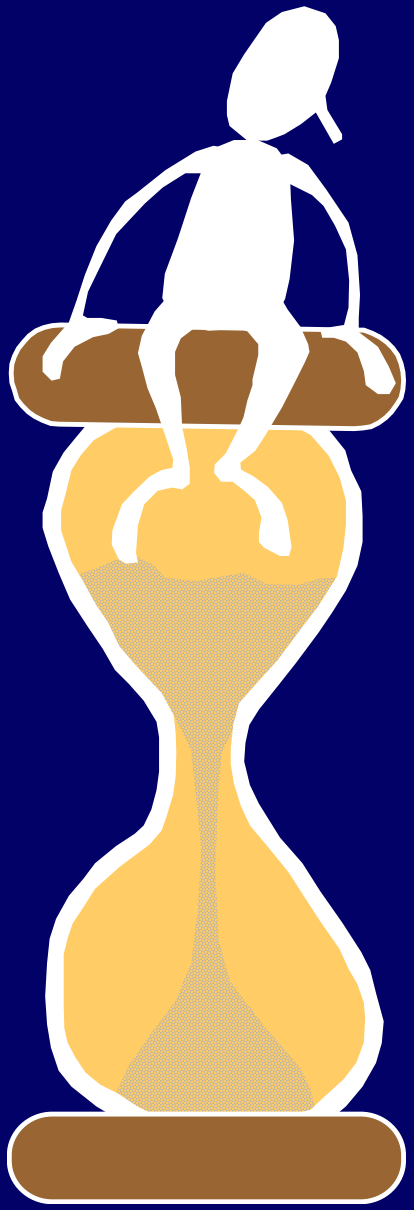
CS 15-251 Spring 2005

Lecture 9 Feb 8 2005

Carnegie Mellon University

One Minute To Learn Programming: Finite Automata





Let me teach you a programming language so simple that you can learn it in less than a minute.

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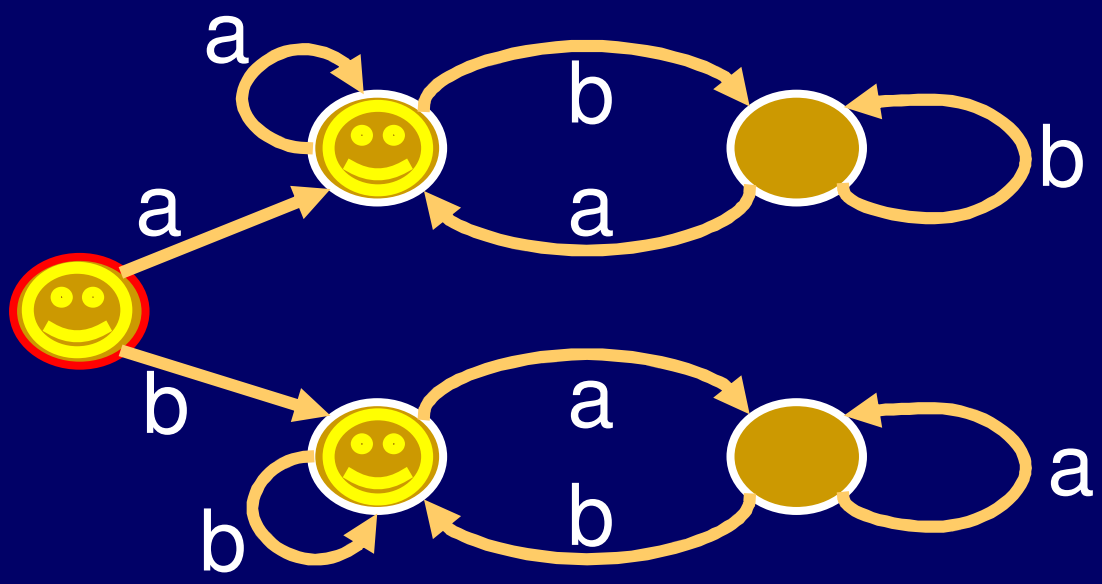
Meet "ABA" The Automaton!



Input String	Result
aba	Accept
aabb	Reject
aabba	Accept
ε	Accept

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The Simplest Interesting Machine:

Finite State Machine
OR
Finite Automaton

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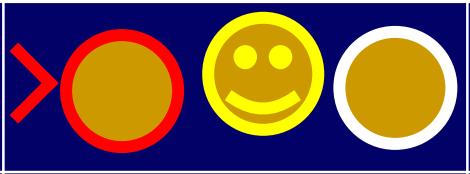
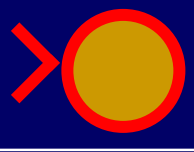
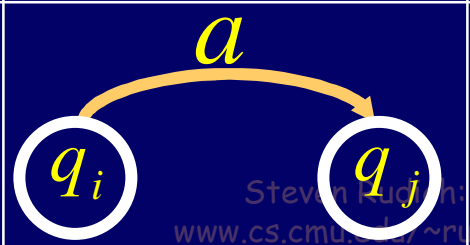
Finite Automaton

Finite set of states		$Q = \{q_o, q_1, q_2, \ldots, q_k\}$
A start state		q_o
A set of accepting states		$F = \{q_{i_1}, q_{i_2}, \ldots, q_{i_r}\}$
A finite alphabet	$a \;\; b \;\; \#$ $x \;\; 1$	Σ
State transition instructions		$\partial : Q \times \Sigma \to Q$ $\partial(q_i, a) = q_j$

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How Machine M operates.

M "reads" one letter at a time from the input string (going from left to right)

M starts in state q_0.

If M is in state q_i reads the letter a then

If $\delta(q_i, a)$ is undefined then CRASH.

Otherwise M moves to state $\delta(q_i,a)$

Let $M=(Q,\Sigma,F,\delta)$ be a finite automaton.

M ***accepts*** the string *x* if when *M* reads *x* it ends in an accepting state.

M ***rejects*** the string *x* if when *M* reads *x* it ends in a non-accepting state.

M ***crashes*** on *x* if *M* crashes while reading *x*.

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The set (or language) accepted by M is:

$$L_M = \{x \in \Sigma^* \mid M \text{ accepts } x\}$$

$\Sigma^k \equiv$ All length k strings over the alphabet Σ

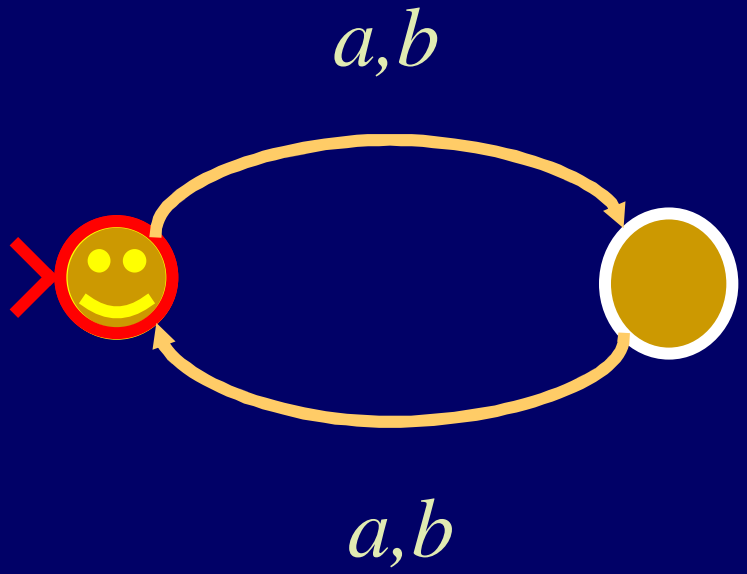
$$\Sigma^* \equiv \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \ldots$$

Notice that this is $\{\varepsilon\}$

What is the language accepted by this machine?

$L = \{a,b\}^*$ = all finite strings of *a*'s and *b*'s

What is the language accepted by this machine?



L = all even length strings of a's and b's

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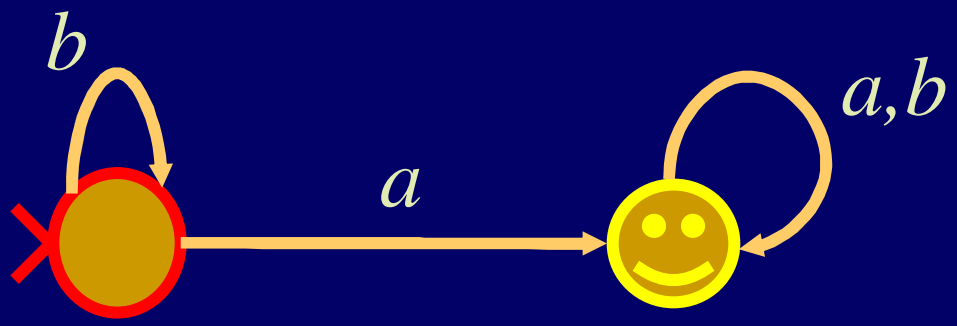
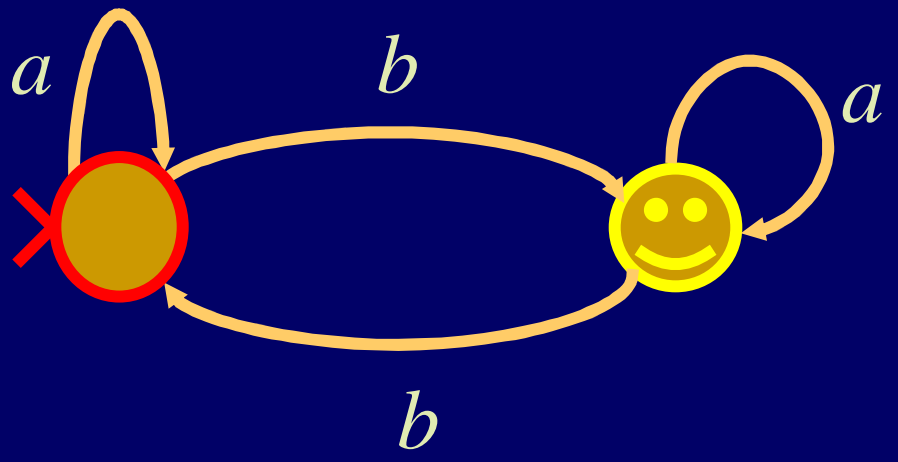
What machine accepts this language?

L = all strings in $\{a,b\}^*$ that contain at least one a

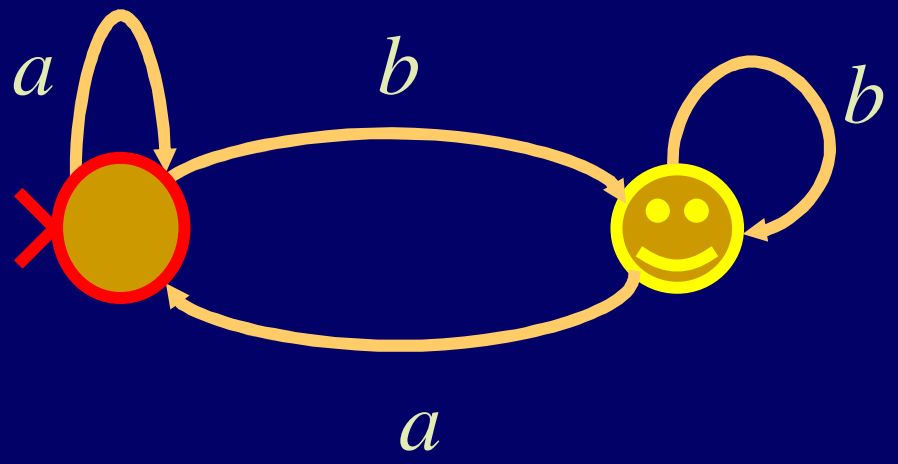


What machine accepts this language?

L = strings with an odd number of b's and any number of a's

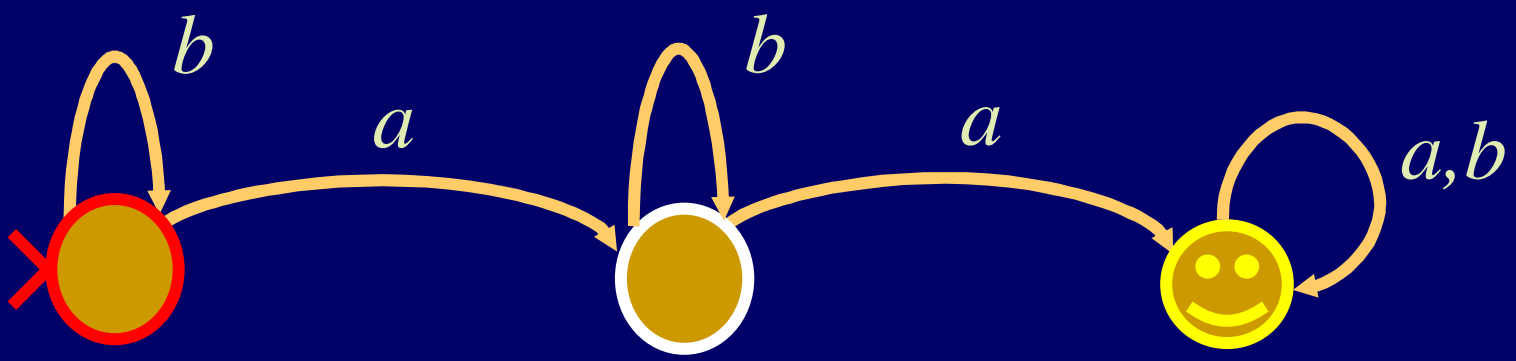


What is the language accepted by this machine?



L = any string ending with a *b*

What is the language accepted by this machine?



L = any string with at least two a's

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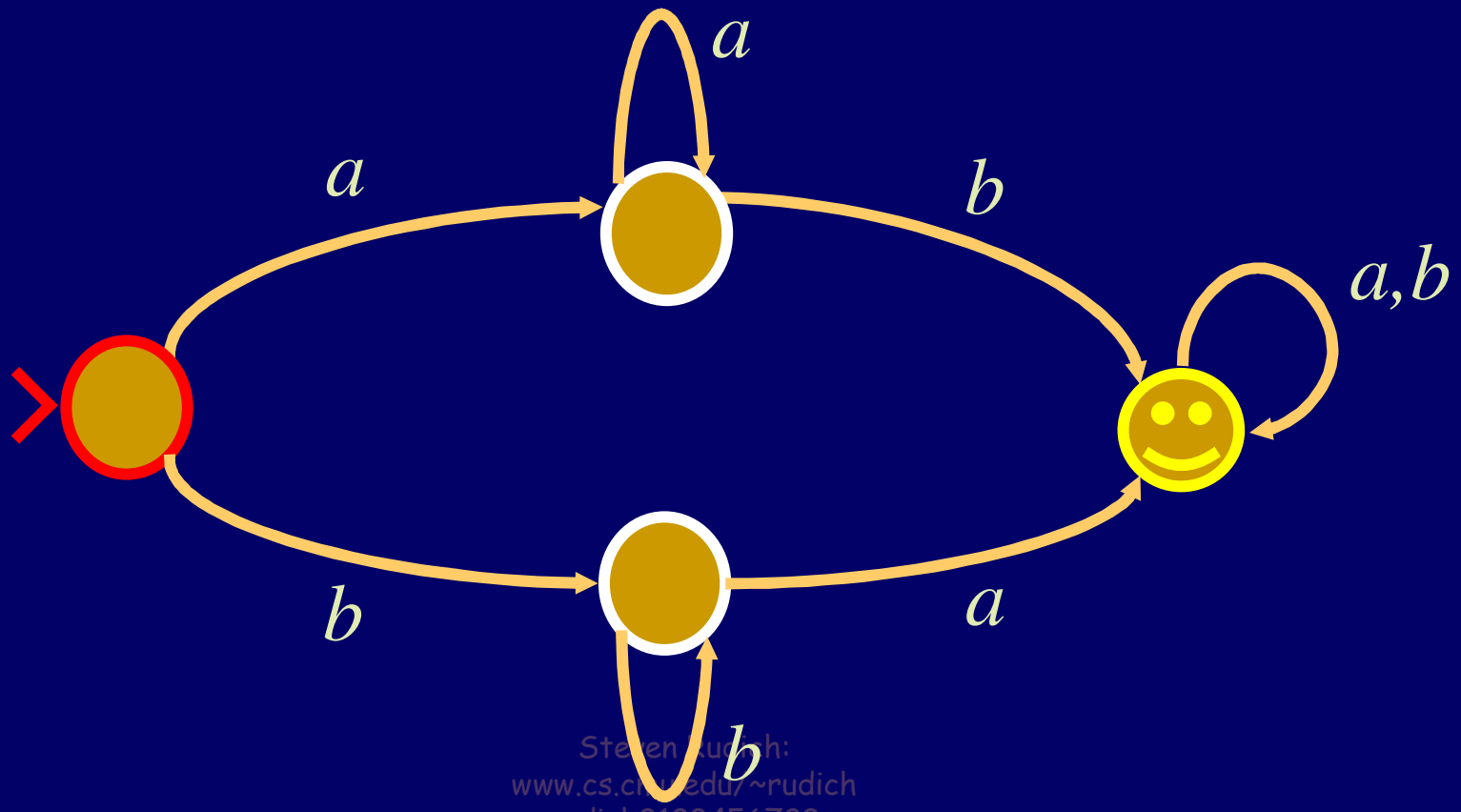
What machine accepts this language?

L = any string with an a and a b



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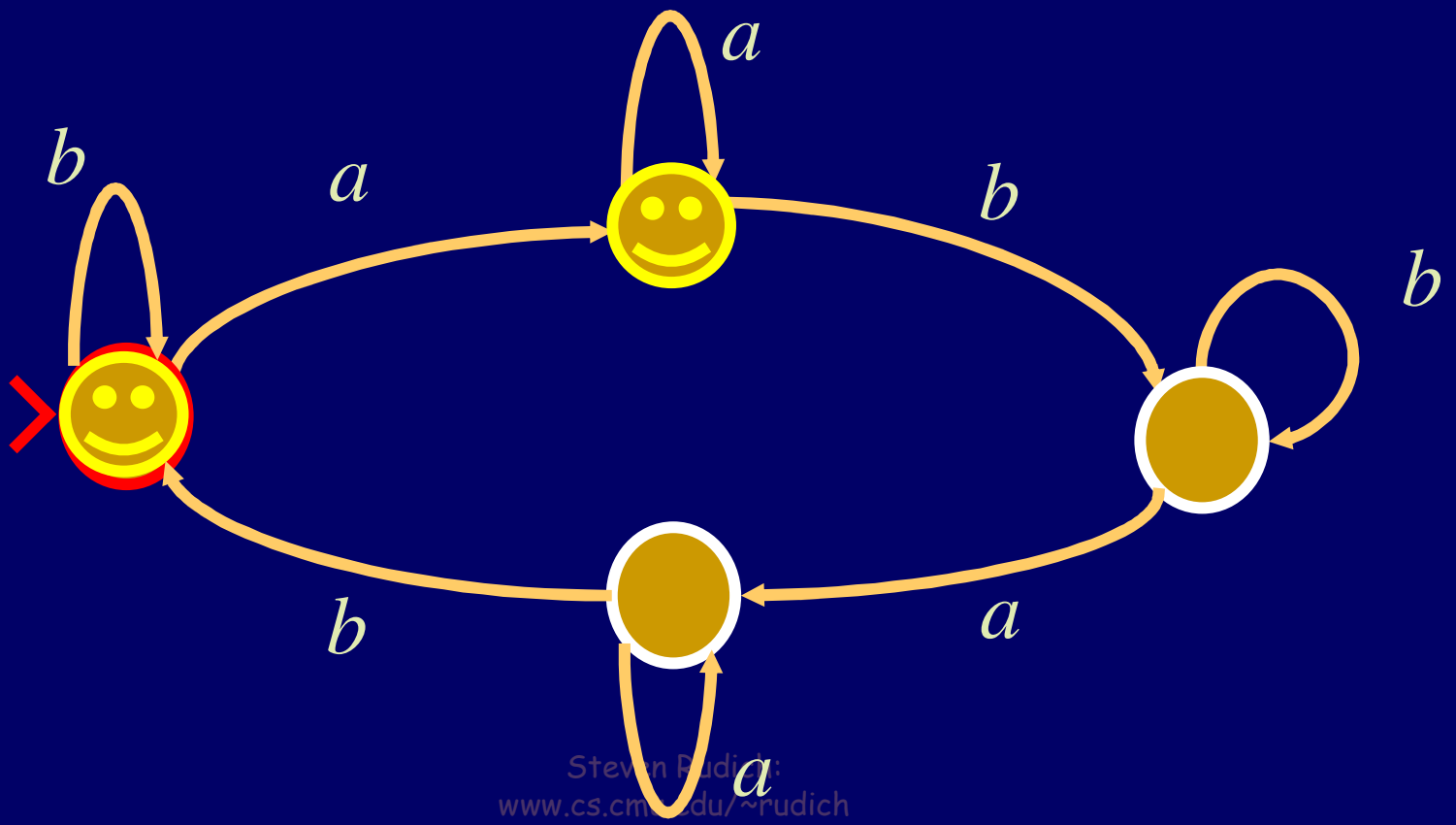
What machine accepts this language?

L = strings with an even number of ab pairs



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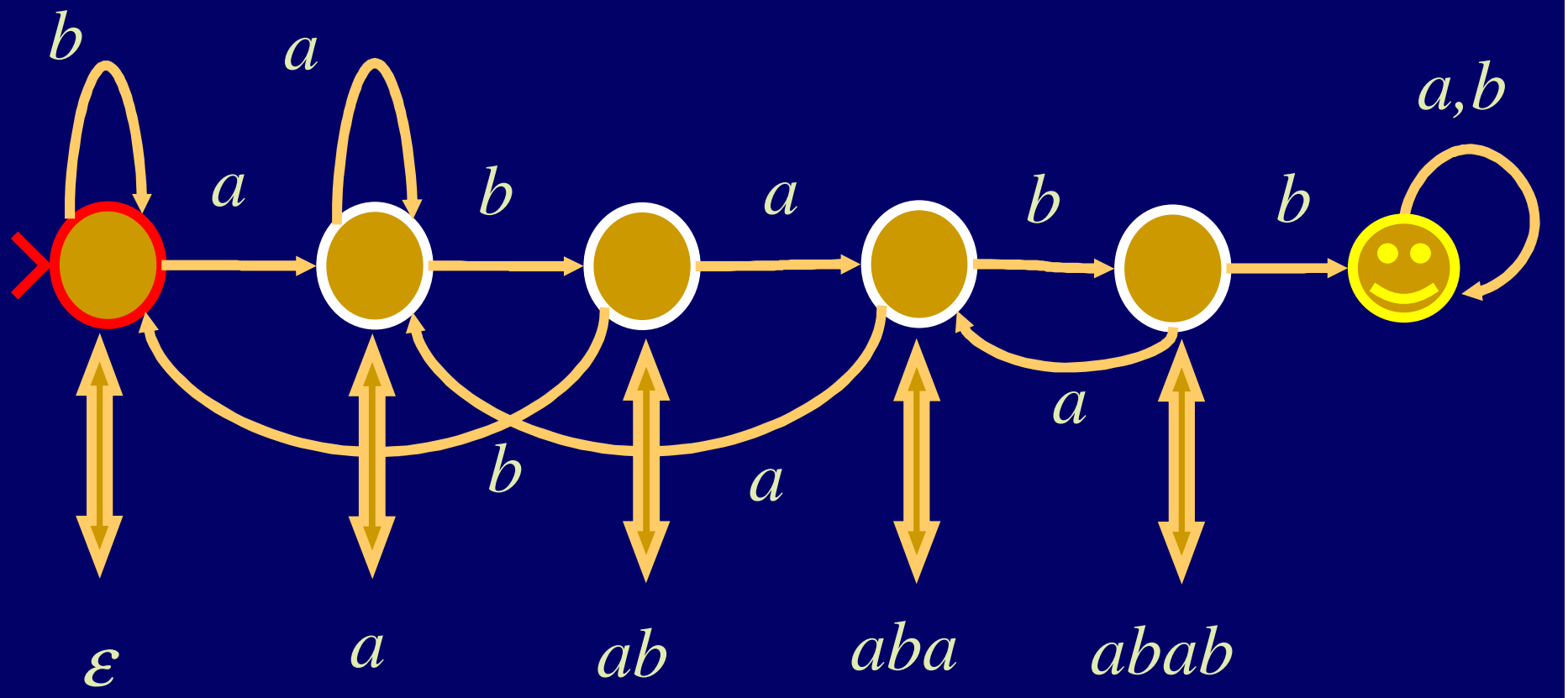
L = all strings containing *ababb* as a consecutive substring



Invariant: I am state s exactly when s is the longest suffix of the input (so far) that forms a prefix of *ababb*.

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The "grep" Problem

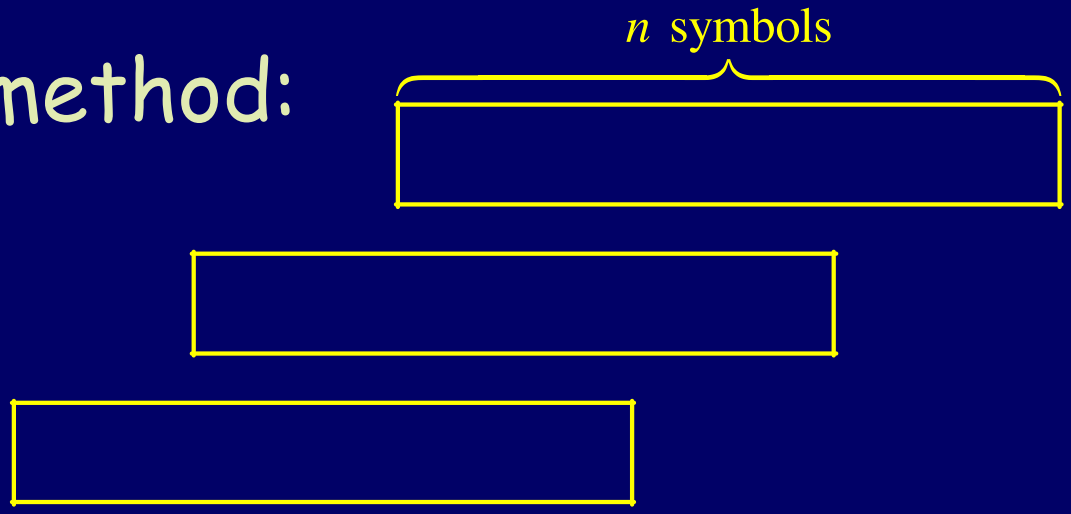
Input:

- text T of length t
- string S of length n

Problem:

Does the string S appear inside the text T?

Naïve method:



$$a_1, a_2, a_3, \ldots\ldots\ldots\ldots\ldots, a_t$$

Cost: $O(nt)$ comparisons

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Automata Solution

- Build a machine *M* that accepts any string with *S* as a consecutive substring.
- Feed the text to *M*.
- Cost: *t* comparisons + time to build *M*.
- As luck would have it, the Knuth, ***Morris***, Pratt algorithm builds *M* quickly.
- By the way, it can be done with fewer than *t* comparisons in the worst case!

Real-life uses of finite state machines

- grep
- coke machines
- thermostats (fridge)
- elevators
- train track switches
- lexical analyzers for parsers

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Any $L \subseteq \Sigma^*$ is defined to be a language.

L is just a set of strings. It is called a language for historical reasons.

Let $L \subseteq \Sigma^*$ be a language.

L is called a regular language if there is some finite automaton that accepts L.

In this lecture we have seen many regular languages.

- Σ^*
- even length strings
- strings containing *ababb*

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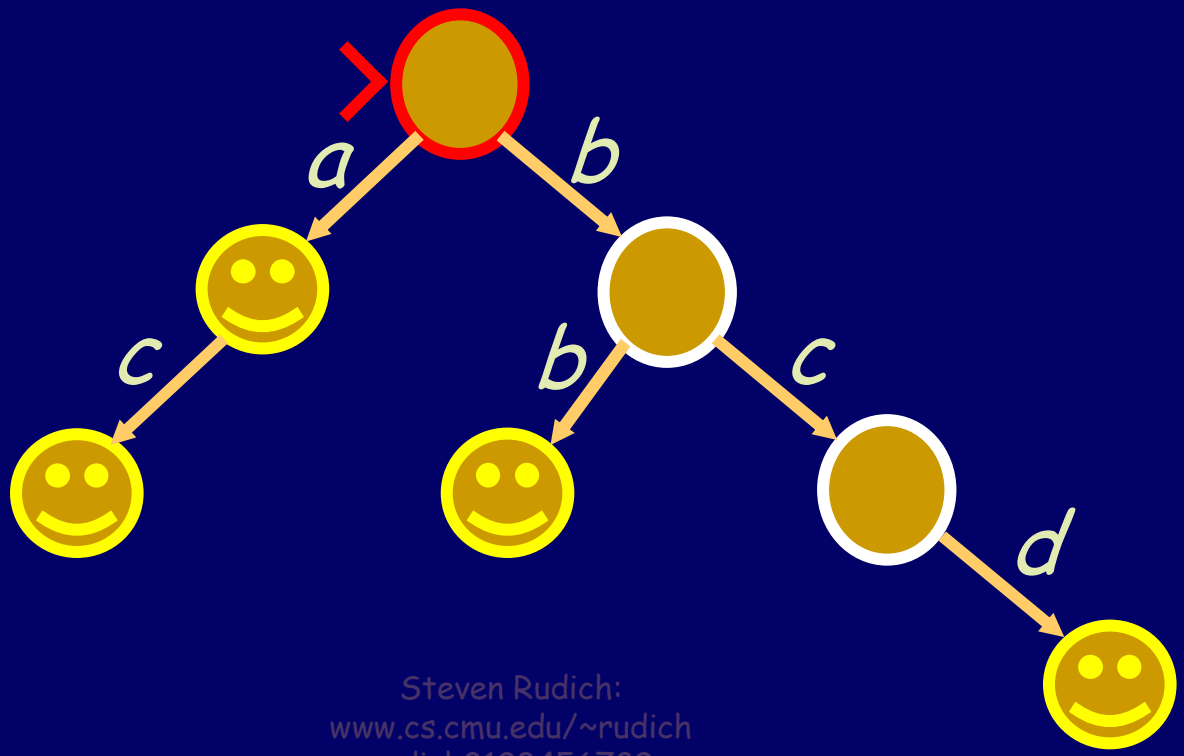
Theorem: Any finite langage is regular.

Proof: Make a machine with a "path" for each string in the language.

Example: $L = \{a, bcd, ac, bb\}$



Are all languages regular?

Consider the language

$$a^n b^n = \{\varepsilon, ab, aabb, aaabbb, \ldots\}$$

i.e., a bunch of *a*'s
followed by an equal
number of *b*'s

No finite automaton accepts this language.

Can you prove this?

$a^n b^n$ is not regular. No machine has enough states to keep track of the number of a's it might encounter.

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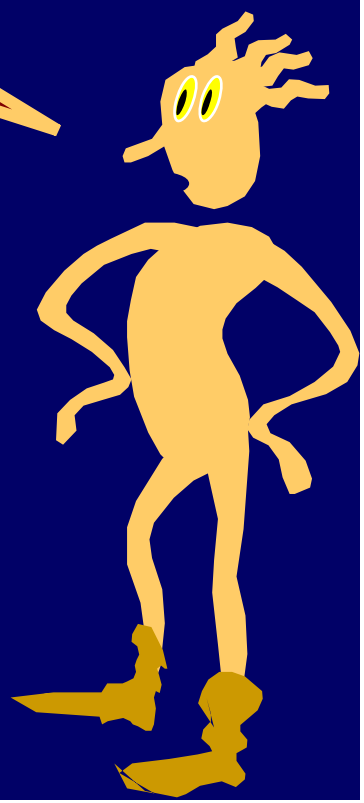
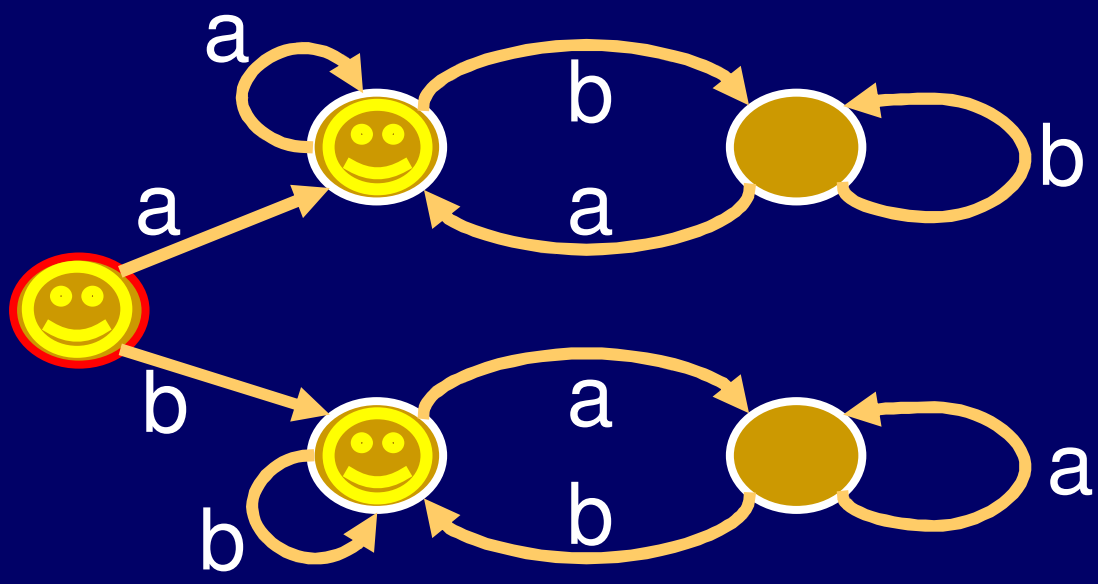
That is a fairly weak argument. Consider the following example…



L = strings where the # of occurrences of the pattern *ab* is equal to the number of occurrences of the pattern *ba*

Can't be regular. No machine has enough states to keep track of the number of occurrences of *ab*.

Remember "ABA"?



ABA accepts only the strings with an equal number of *ab*'s and *ba*'s!

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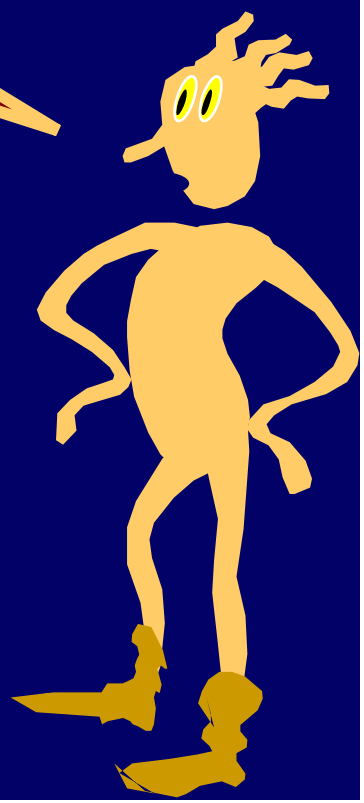
Let me show you a professional strength proof that $a^n b^n$ is not regular....



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Professional Strength Proof

Theorem: $a^n b^n$ is not regular.

Proof: Assume that it is. Then $\exists$ M with k states that accepts it.

For each $0 \leq i \leq k$, let S_i be the state M is in after reading a^i.

$\exists i, j \leq k$ s.t. $S_i = S_j$, but $i \neq j$

M will do the same thing on $a^i b^i$ and $a^j b^i$.

But a valid M must reject $a^j b^i$ and accept $a^i b^i$.

$\Rightarrow\Leftarrow$

MORAL:

Finite automata can't count.

Advertisement

You can learn much more about these creatures in the FLAC course.

Formal Languages, Automata, and Computation

- There is a unique smallest automaton for any regular language

- It can be found by a fast algorithm.

Cellular Automata

- Line up a bunch of ***identical*** finite automata in a straight line.

- Transitions are based on the states of the machine's two neighbors or an indicator that a neighbor is missing. (There is no other input.)

$$Q \times Q \times Q \to Q$$

$$(q_k, q_i, q_j) = q_l$$

- All cells move to their next states at the same time: synchronous transition

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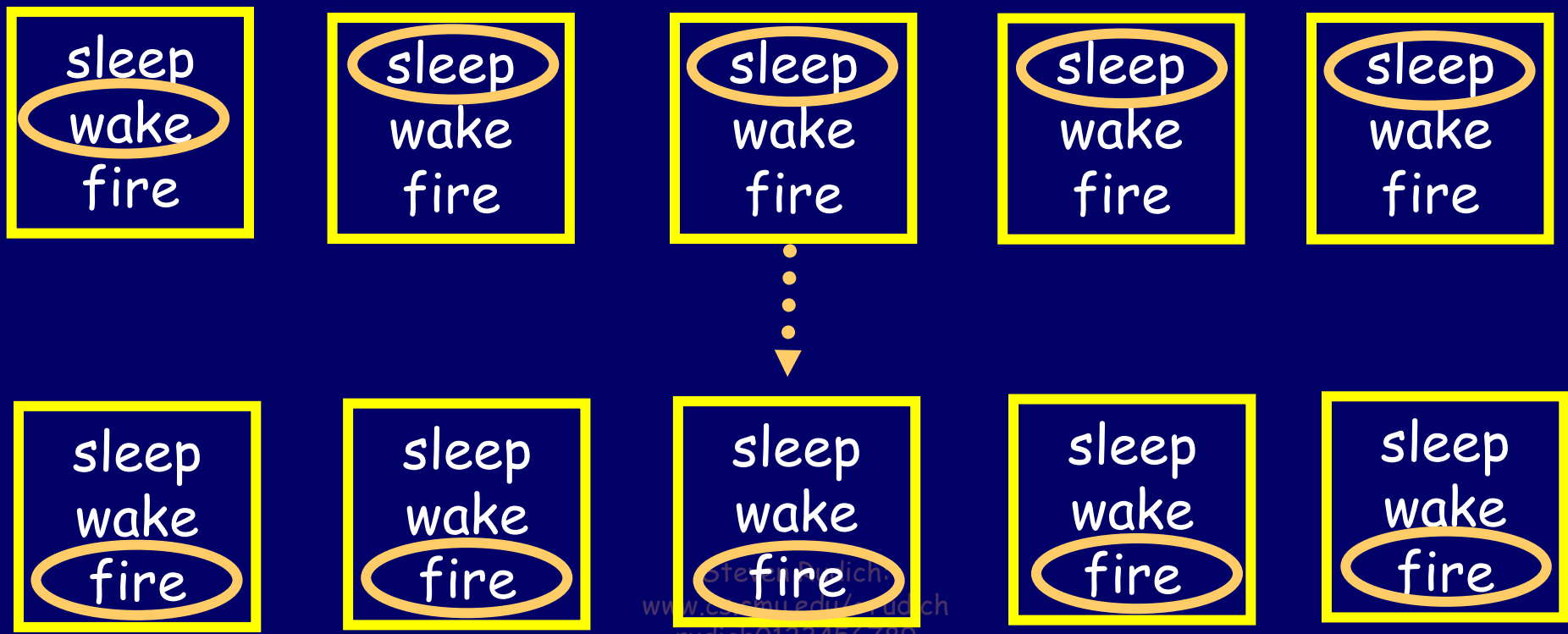
The Firing Squad Problem

- Five "soldiers" all start in the sleep state. You change the one on the left to the wake state.
- All five must get to the fire state at the same time (for the first time).



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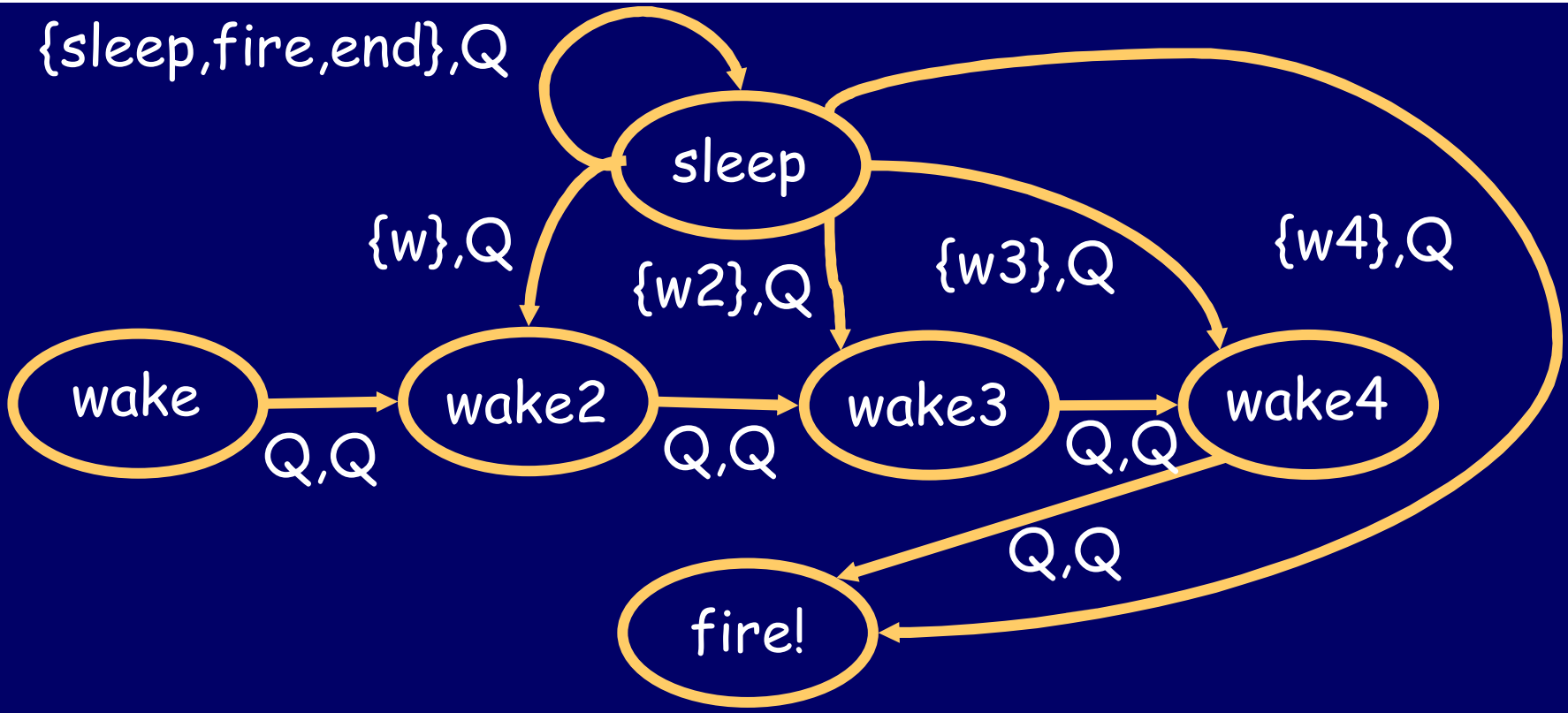
Shorthand

Means use this transition when your left neighbor is in any state at all and your right neighbor is in state *a*,*b*, or *d*.



	1	2	3	4	5
	sleep	sleep	sleep	sleep	sleep
	sleep	sleep	sleep	sleep	sleep
	sleep	sleep	sleep	sleep	sleep
	sleep	sleep	sleep	sleep	sleep
	sleep	sleep	sleep	sleep	sleep

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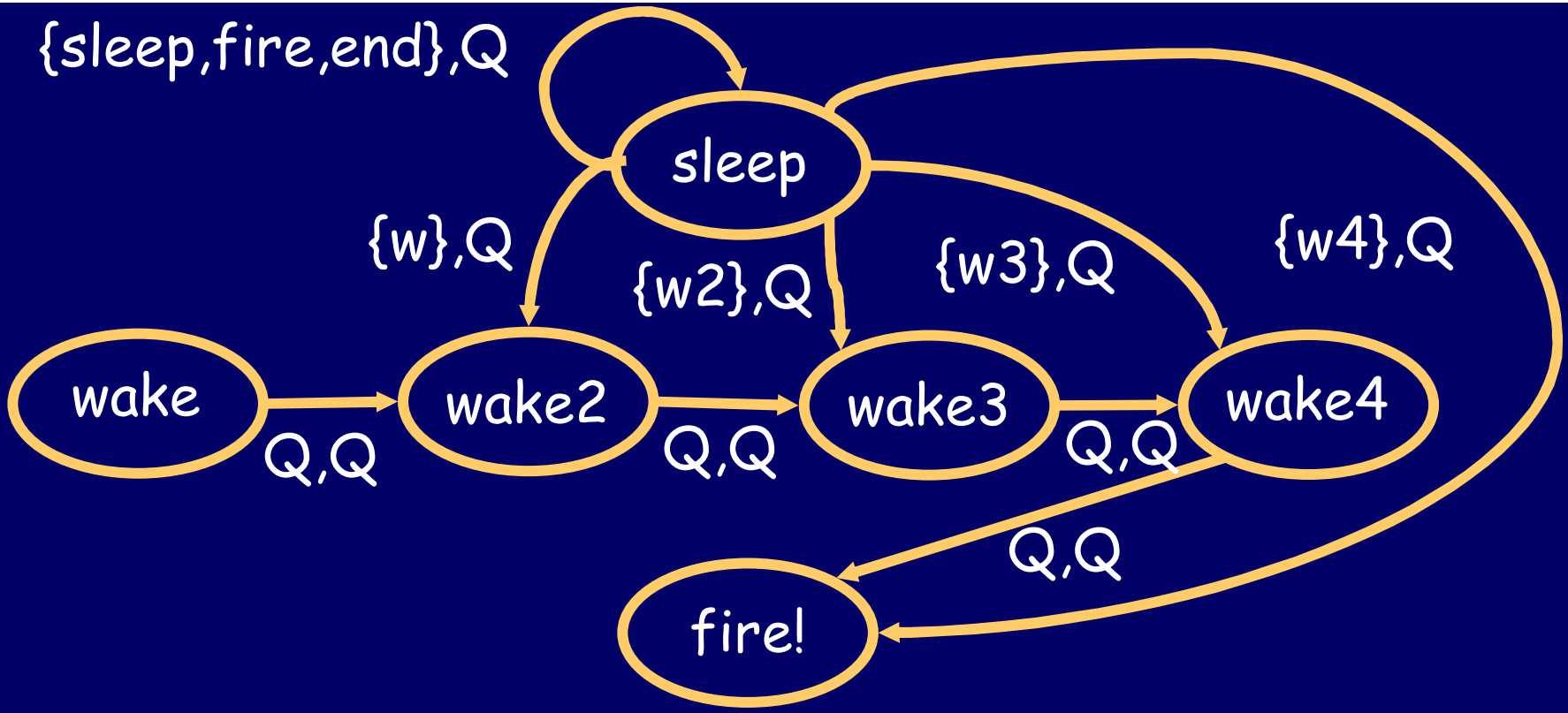

	1	2	3	4	5
	wake	sleep	sleep	sleep	sleep

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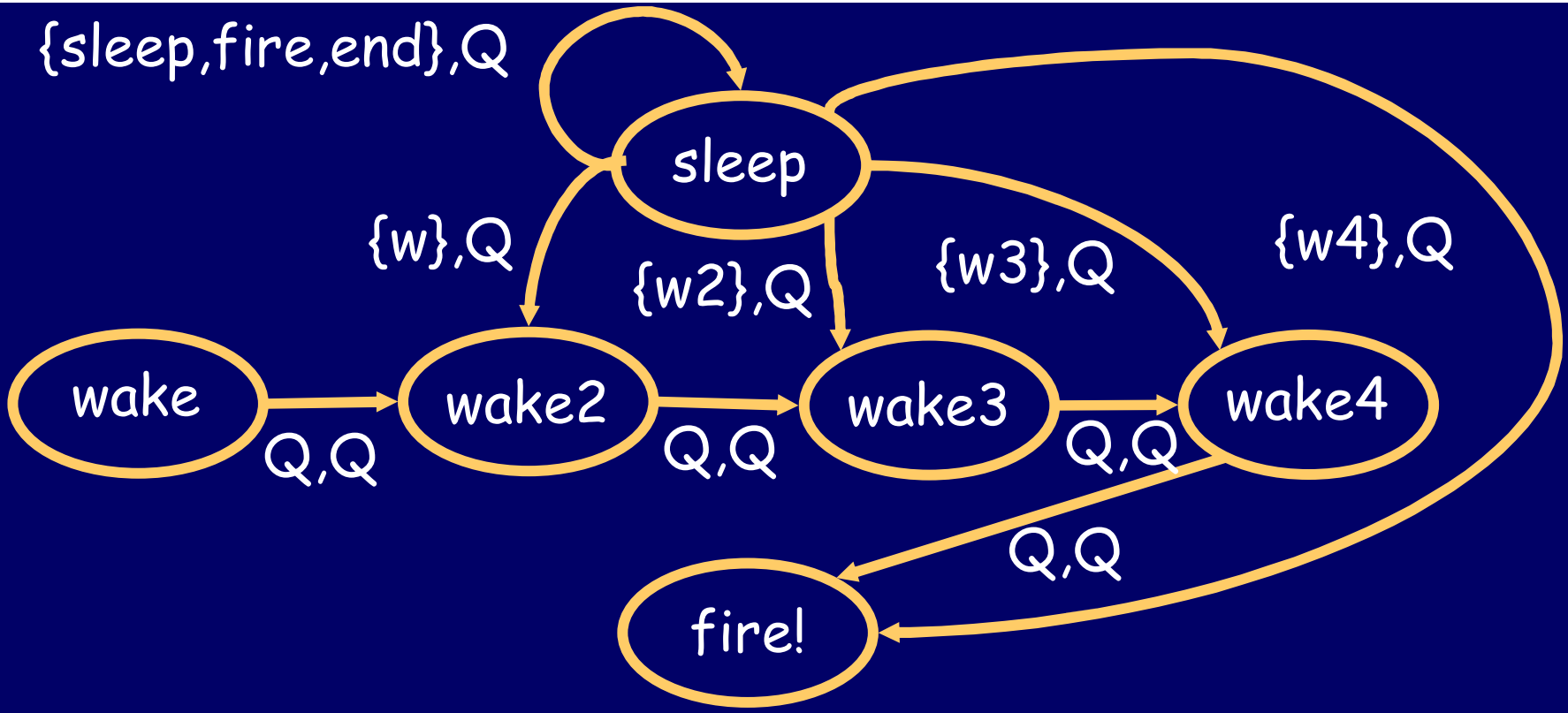

	1	2	3	4	5
	wake	sleep	sleep	sleep	sleep
	wake2	wake2	sleep	sleep	sleep

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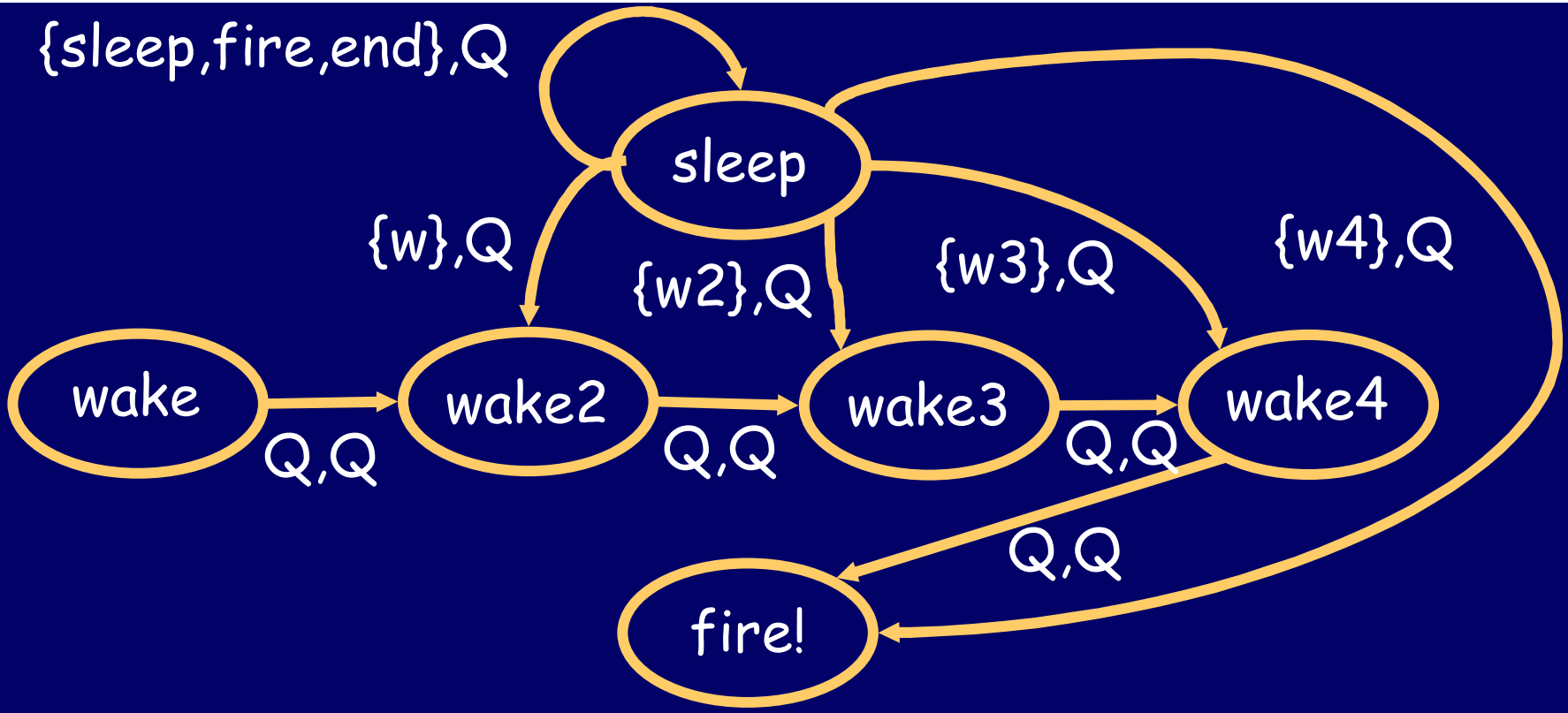

	1	2	3	4	5
	wake	sleep	sleep	sleep	sleep
	wake2	wake2	sleep	sleep	sleep
	wake3	wake3	wake3	sleep	sleep

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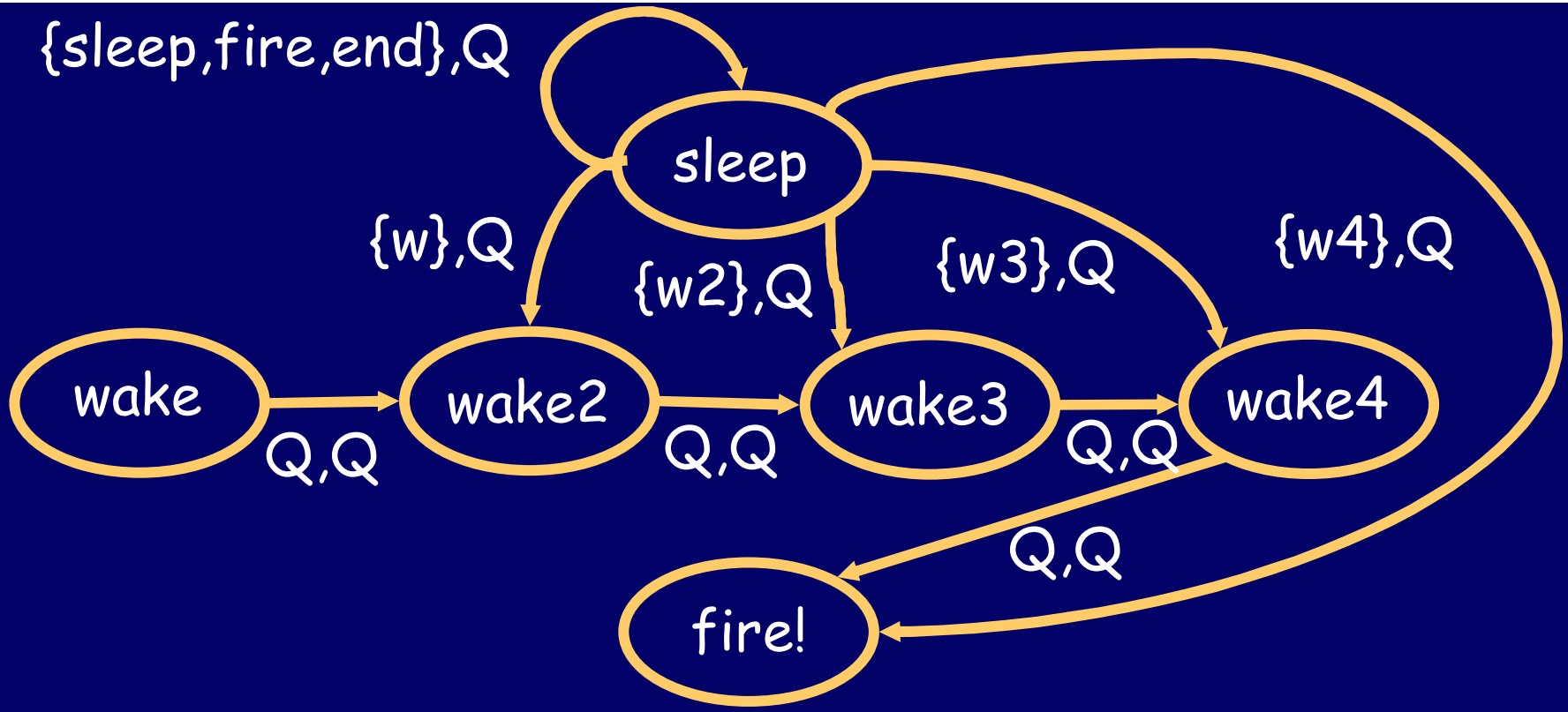

	1	2	3	4	5
	wake	sleep	sleep	sleep	sleep
	wake2	wake2	sleep	sleep	sleep
	wake3	wake3	wake3	sleep	sleep
	wake4	wake4	wake4	wake4	sleep

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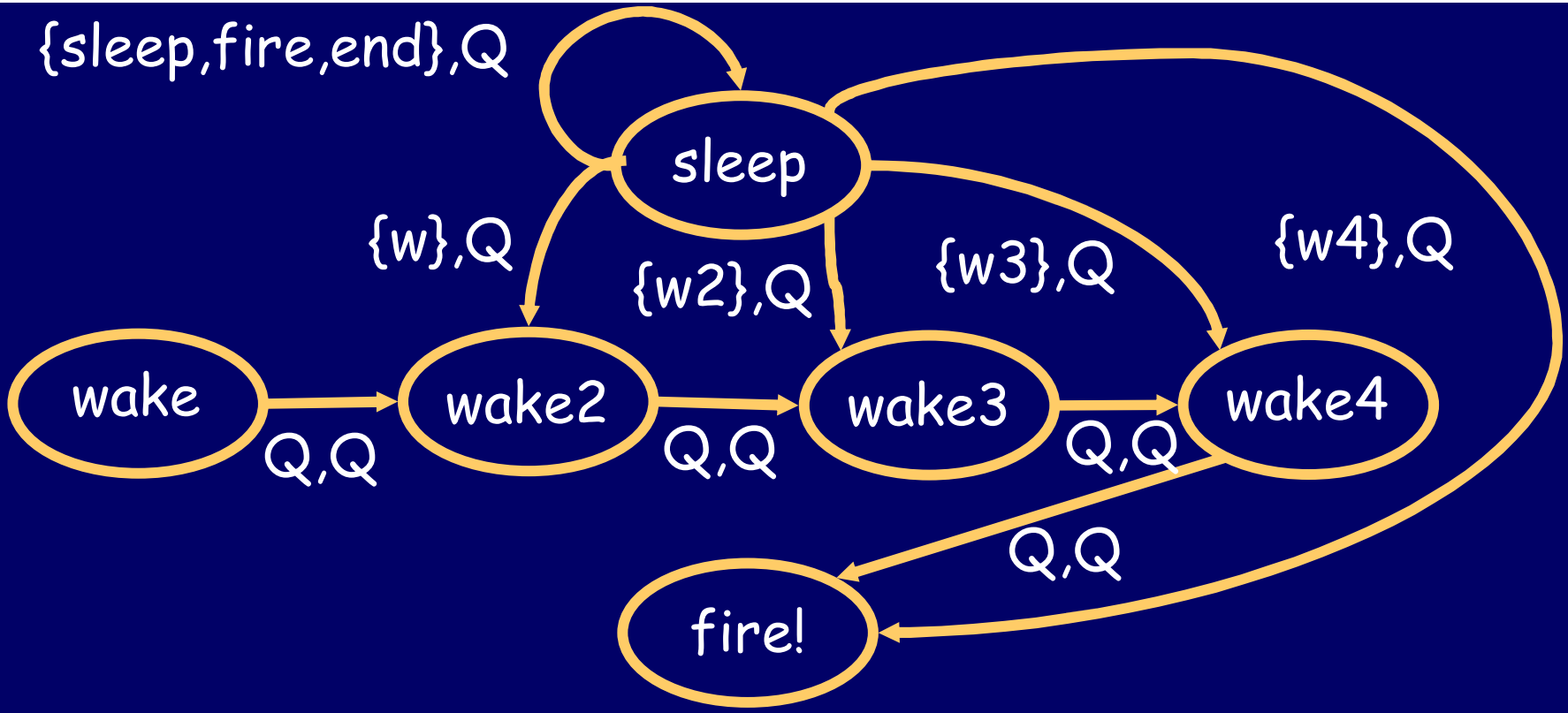

	1	2	3	4	5
	wake	sleep	sleep	sleep	sleep
	wake2	wake2	sleep	sleep	sleep
	wake3	wake3	wake3	sleep	sleep
	wake4	wake4	wake4	wake4	sleep
	fire	fire	fire	fire	fire

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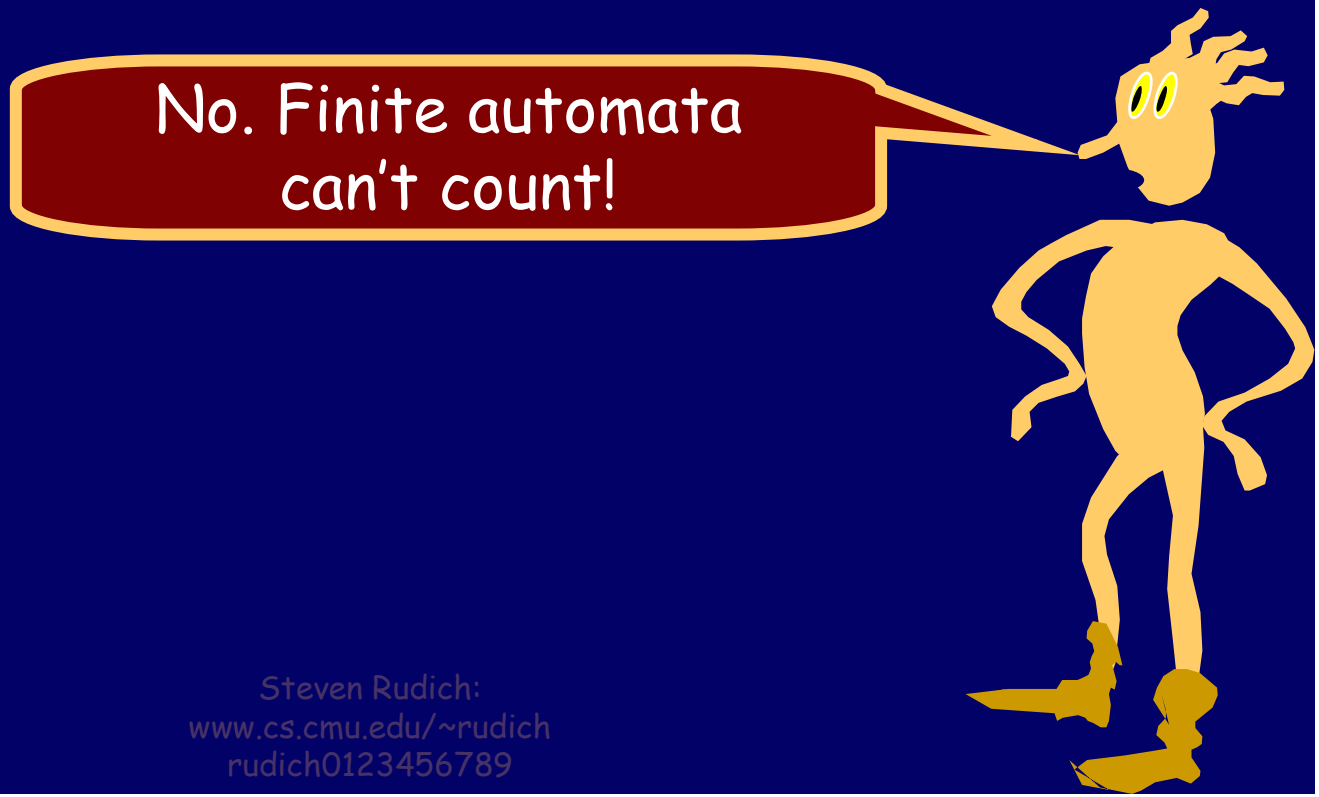
Question

Can you build the soldier's finite automaton brain before you know how many soldiers will be in the line?



Don't jump to conclusions!
It *is* possible to design a single cellular automaton that works for any number of soldiers!

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