Great Theoretical Ideas In Computer Science		
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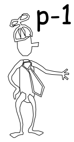
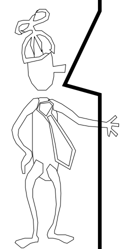
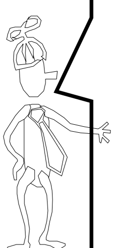
Lecture 8	Feb 3, 2005	Carnegie Mellon University

Modular Arithmetic and the RSA Cryptosystem

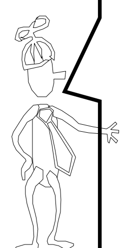

$^{p-1} \equiv_p 1$


MAX(a,b) + MIN(a,b) = a+b


n|m means that m is a an integer multiple of n.

We say that “n divides m”.

True: 5|25 2|-66 7|35,
False: 4|5 8|2

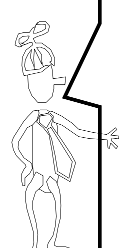

Greatest Common Divisor:

GCD(x,y) = greatest $k \geq 1$ s.t. k|x and k|y.

GCD: Greatest Common Divisor

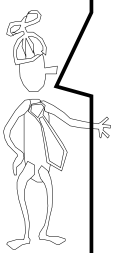
What is the GCD of 12 and 18?
$12 = 2^2 * 3$ $16 = 2*3^2$

Common factors: 2^1 and 3^1
Answer: 6


Least Common Multiple:

LCM(x,y) = smallest $k \geq 1$ s.t. x|k and y|k..

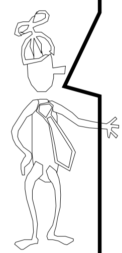
Prop:
GCD(x,y) = xy/LCM(x,y)
LCM(x,y) = xy/GCD(x,y)



$GCD(x,y) = xy/LCM(x,y)$
$LCM(x,y) = xy/GCD(x,y)$

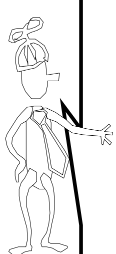
$x = 2^2*3 = 12$; $y = 3^2*5 = 45$

$GCD(12,45) = 3$
$LCM(12,45) = 2^2*3^2*5 = 180$
$x*y = 540$



$GCD(x,y) * LCM(x,y) = xy$

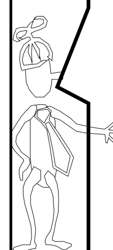
$MAX(a,b) + MIN(a,b) = a+b$



(a mod n) means the remainder when a is divided by n.

If $ad + r = n$, $0 \leq r < n$

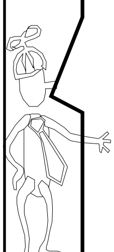
Then r = (a mod n) and d = (a div n)



Modular equivalence of integers a and b:

$a \equiv b$ [mod n]
$a \equiv_n b$
"a and b are equivalent modulo n"

iff (a mod n) = (b mod n)
iff $n|(a-b)$



31 equals 81 modulo 2

$31 \equiv 81$ [mod 2]

$31 \equiv_2 81$

(31 mod 2) = 1 = (81 mod 2)

$2|(31-81)$

$\equiv_n$ is an equivalence relation

In other words,

Reflexive:
$a \equiv_n a$

Symmetric:
$(a \equiv_n b) \Rightarrow (b \equiv_n a)$

Transitive:
$(a \equiv_n b$ and $b \equiv_n c) \Rightarrow (a \equiv_n c)$

$a \equiv_n b \leftrightarrow n|(a-b)$

"a and b are equivalent modulo n"

$\equiv_n$ induces a natural partition of the integers into n classes:

a and b are said to be in the same "residue class" or "congruence class" exactly when $a \equiv_n b$.

$a \equiv_n b \leftrightarrow n|(a-b)$

"a and b are equivalent modulo n"

Define the residue class [i] to be the set of all integers that are congruent to i modulo n.

Residue Classes Mod 3:

[0] = { …, -6, -3, 0, 3, 6, ..}
[1] = { …, -5, -2, 1, 4, 7, ..}
[2] = { …, -4, -1, 2, 5, 8, ..}
[-6] = { …, -6, -3, 0, 3, 6, ..}
[7] = { …, -5, -2, 1, 4, 7, ..}
[-1] = { …, -4, -1, 2, 5, 8, ..}

Equivalence mod n implies equivalence mod any divisor of n.

If $(x \equiv_n y)$ and $(k|n)$
Then: $x \equiv_k y$

Example: $10 \equiv_6 16 \Rightarrow 10 \equiv_3 16$

If $(x \equiv_n y)$ and $(k|n)$
Then: $x \equiv_k y$

Proof:
Recall, $x \equiv_n y \Leftrightarrow n|(x-y)$
$k|n$ and $n|(x-y)$
Hence, $k|(x-y)$
Of course, $k|(x-y) \Rightarrow x \equiv_k y$

Fundamental lemma of plus, minus, and times modulo n:

If $(x \equiv_n y)$ and $(a \equiv_n b)$
Then: 1) $x+a \equiv_n y+b$
2) $x-a \equiv_n y-b$
3) $xa \equiv_n yb$

Equivalently,

If n|(x-y) and n|(a-b) Then:

1) n|(x-y + a-b)
2) n | (x-y – [a-b])
3) n|(xa-yb)

Proof of 3:
xa-yb = a(x-y) – y(b-a)
n|a(x-y) and n|y(b-a)

Fundamental lemma of plus minus, and times modulo n:

When doing plus, minus, and time modulo n, I can at any time in the calculation replace a number with a number in the same residue class modulo n

Please calculate in your head:

329 * 666 mod 331

-2 * 4 = -8 = 323

A Unique Representation System Modulo n:

We pick exactly one representative from each residue class. We do all our calculations using the representatives.

Unique representation system modulo 3

Finite set S = {0, 1, 2}

+ and * defined on S:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Unique representation system modulo 3

Finite set S = {0, 1, -1}

+ and * defined on S:

+	0	1	-1
0	0	1	-1
1	1	-1	0
-1	-1	0	1

*	0	1	-1
0	0	0	0
1	0	1	-1
-1	0	-1	1

The reduced system modulo n:

$Z_n = \{0, 1, 2, \ldots, n-1\}$

Define $+_n$ and $*_n$:

$a +_n b = (a+b \bmod n)$

$a *_n b = (a*b \bmod n)$

$Z_n = \{0, 1, 2, \ldots, n-1\}$

$a +_n b = (a+b \bmod n)$ $a *_n b = (a*b \bmod n)$

$+_n$ and $*_n$ are associative binary operators from $Z_n \times Z_n \rightarrow Z_n$:

When $\heartsuit = +_n$ or $*_n$:

[Closure] $x,y \in Z_n \Rightarrow x \heartsuit y \in Z_n$

[Associativity]
$x,y,z \in Z_n \Rightarrow (x \heartsuit y) \heartsuit z = x \heartsuit (y \heartsuit z)$

$Z_n = \{0, 1, 2, \ldots, n-1\}$

$a +_n b = (a+b \bmod n)$ $a *_n b = (a*b \bmod n)$

$+_n$ and $*_n$ are commutative, associative binary operators from $Z_n \times Z_n \rightarrow Z_n$:

[Commutativity]
$x,y \in Z_n \Rightarrow x \heartsuit y = y \heartsuit x$

The reduced system modulo 3

$Z_3 = \{0, 1, 2\}$

Two binary, associative operators on Z_3:

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$*_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

The reduced system modulo 2

$Z_2 = \{0, 1\}$

Two binary, associative operators on Z_2:

$+_2$	0	1
0	0	1
1	1	0

$*_2$	0	1
0	0	0
1	0	1

The Boolean interpretation of $Z_2 = \{0, 1\}$

0 means FALSE   1 means TRUE

$+_2$ XOR	0	1
0	0	1
1	1	0

$*_2$ AND	0	1
0	0	0
1	0	1

The reduced system
$Z_4 = \{0, 1,2,3\}$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

The reduced system
$Z_5 = \{0, 1,2,3,4\}$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

The reduced system
$Z_6 = \{0, 1,2,3,4,5\}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

The reduced system
$Z_6 = \{0, 1,2,3,4,5\}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

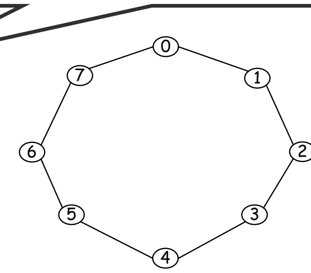
An operator has the permutation property if each row and each column has a permutation of the elements.

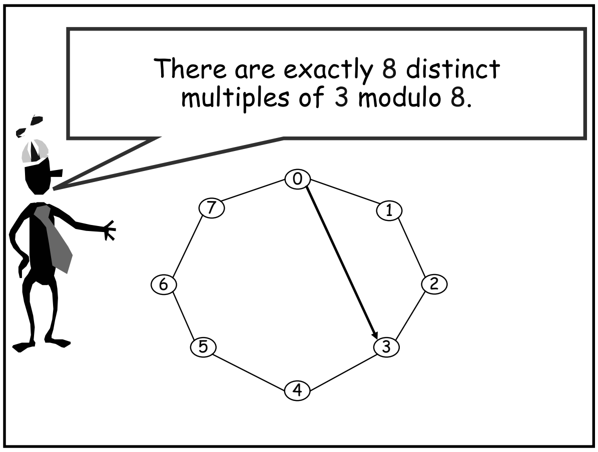
For every n, $+_n$ on Z_n has the permutation property

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

An operator has the permutation property if each row and each column has a permutation of the elements.

There are exactly 8 distinct multiples of 3 modulo 8.

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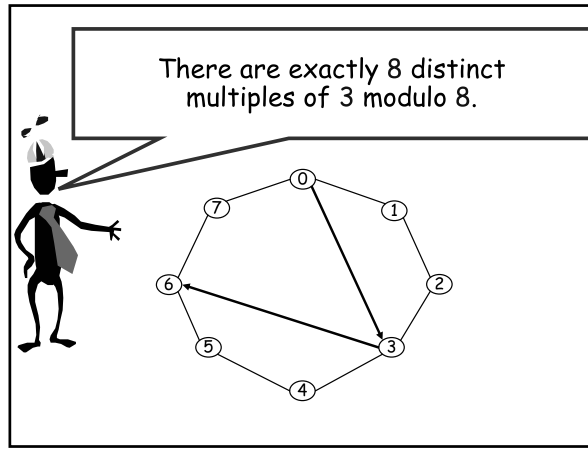
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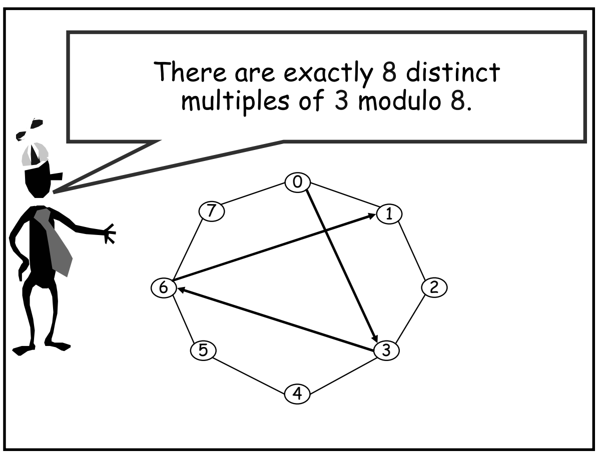
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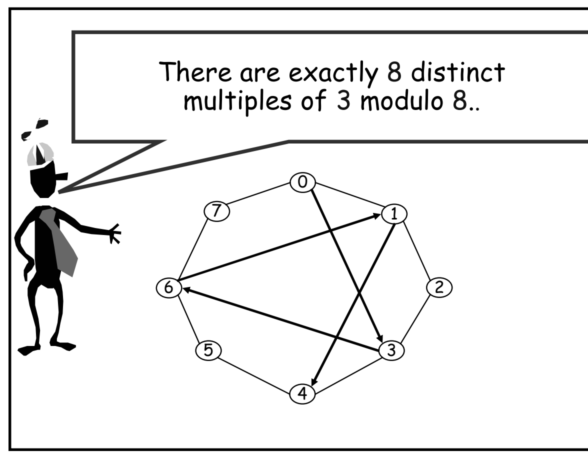
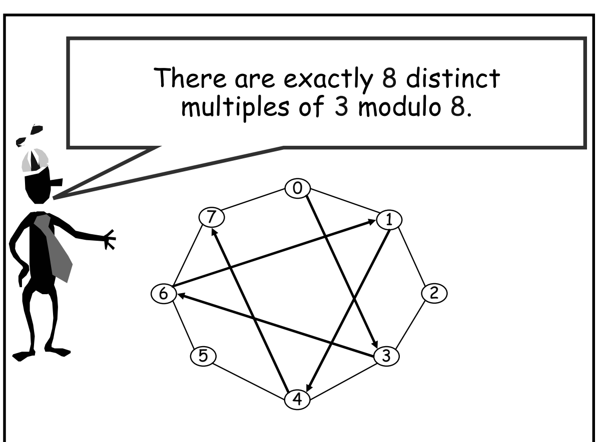

There are exactly 8 distinct
multiples of 3 modulo 8..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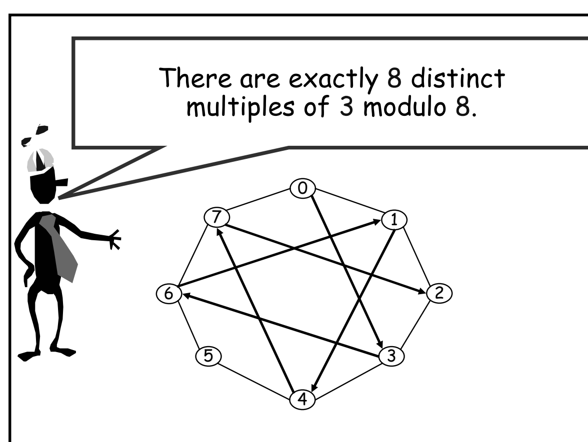
There are exactly 8 distinct
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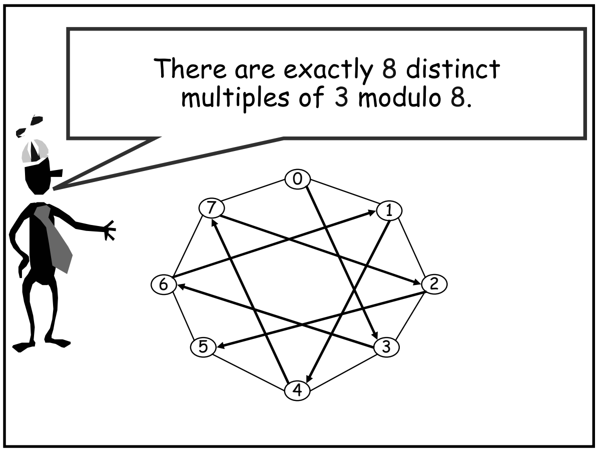
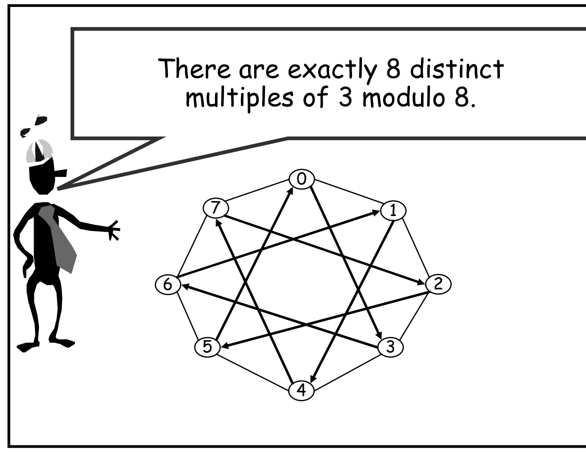

There are exactly 8 distinct multiples of 3 modulo 8.
0 1 2 3 4 5 6 7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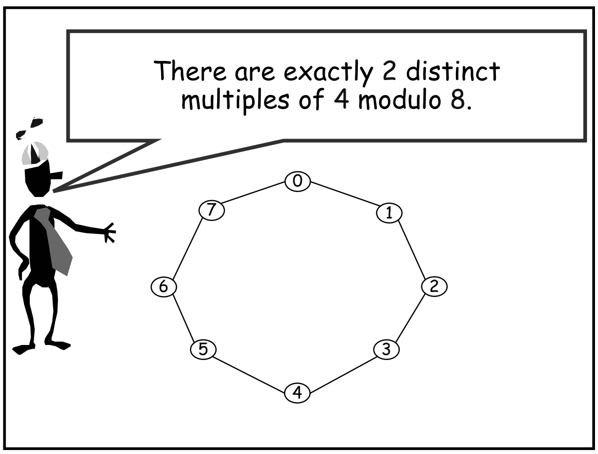
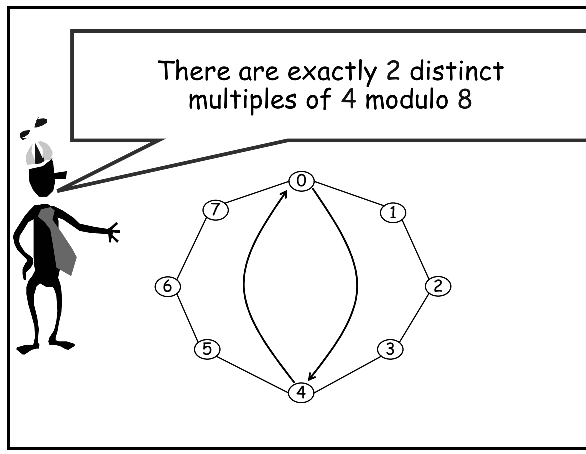
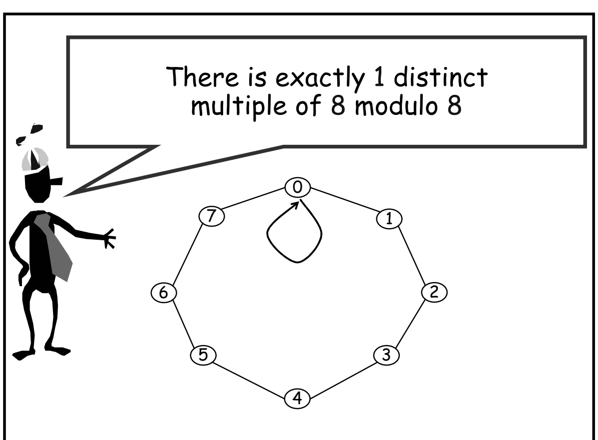
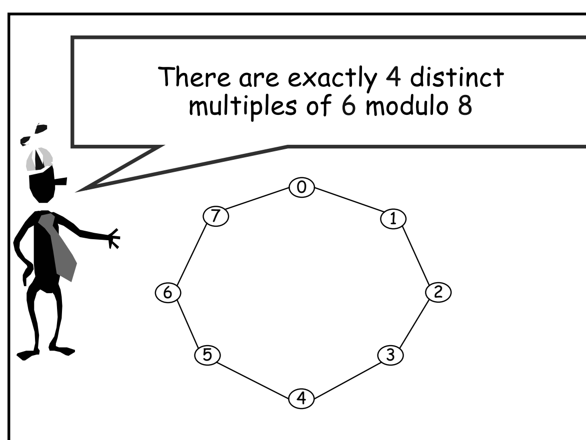
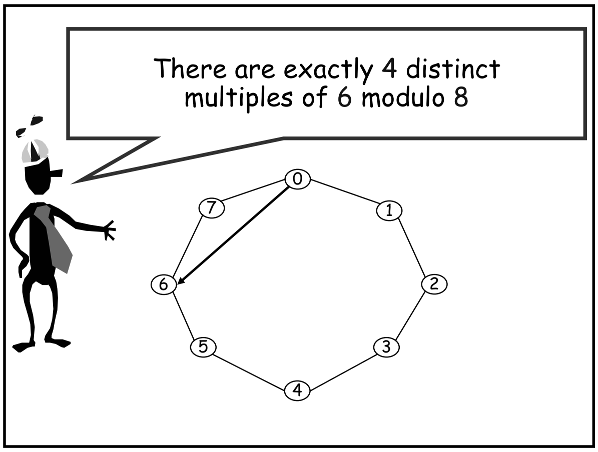

There are exactly 2 distinct multiples of 4 modulo 8.
0 1 2 3 4 5 6 7


There are exactly 2 distinct multiples of 4 modulo 8
0 1 2 3 4 5 6 7


There is exactly 1 distinct multiple of 8 modulo 8
0 1 2 3 4 5 6 7


There are exactly 4 distinct multiples of 6 modulo 8
0 1 2 3 4 5 6 7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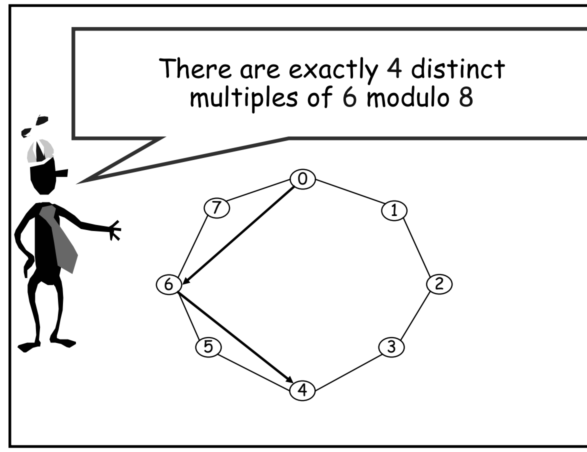
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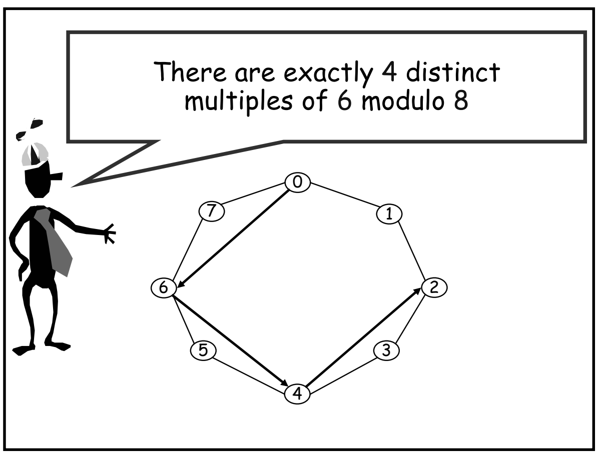
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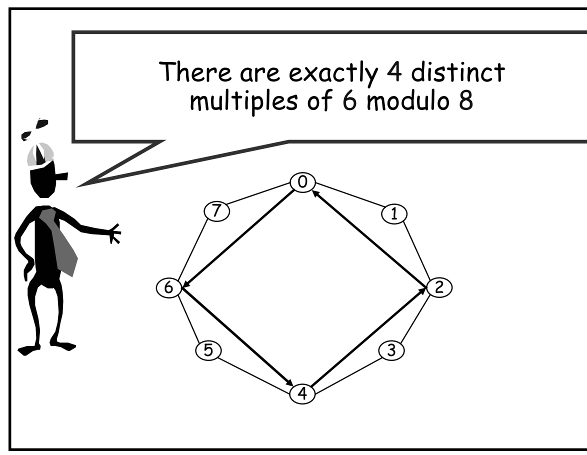
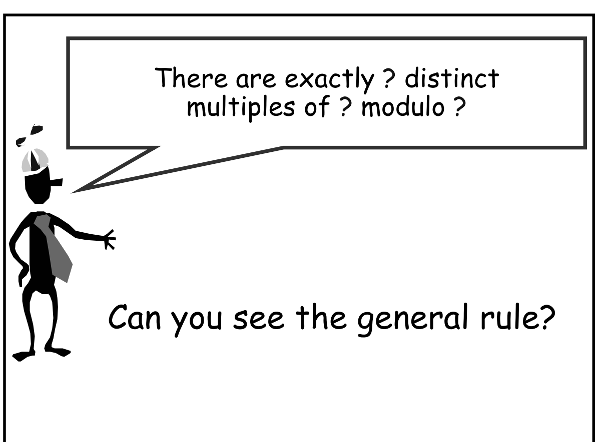
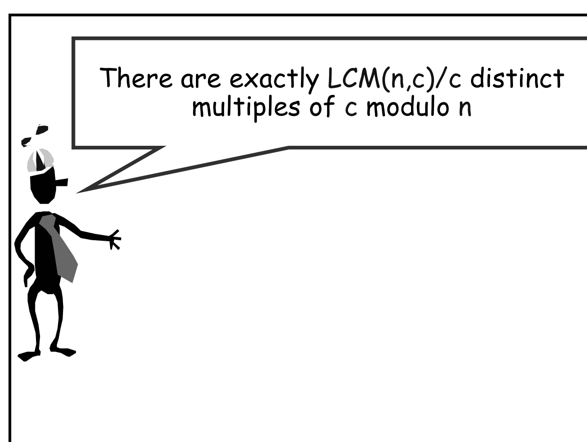
0 1 2 3 4 5 6 7


There are exactly 4 distinct multiples of 6 modulo 8
0 1 2 3 4 5 6 7


There are exactly ? distinct multiples of ? modulo ?
Can you see the general rule?


There are exactly LCM(n,c)/c distinct multiples of c modulo n

There are exactly LCM(n,c)/c distinct multiples of c modulo n

There are exactly n/(nc/LCM(n,c)) distinct multiples of c modulo n

There are exactly n/GCD(c,n) distinct multiples of c modulo n

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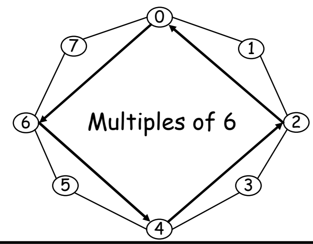
The <u>multiples of c modulo n</u> is the set:
$\{0, c, c +_n c, c +_n c +_n c, \ldots\}$
$= \{kc \bmod n \mid 0 \leq k \leq n-1\}$



Theorem: There are exactly
k= n/GCD(c.n) = LCM(c,n)/c
distinct multiples of c modulo n:
$\{ c*i \bmod n \mid 0 \leq i < k \}$

Clearly, $c/GCD(c,n) \geq 1$ is a whole number
$ck = n\,[c/GCD(c,n)] \equiv_n 0$
There are $\leq k$ distinct multiples of c mod n:
$c*0, c*1, c*2, \ldots, c*(k-1)$
k is all the factors of n missing from c
$cx \equiv_n cy \leftrightarrow n|c(x-y) \Rightarrow k|(x-y) \Rightarrow x-y \geq k$
There are $\geq k$ multiples of c

Is there a fundamental lemma of division modulo n?

$cx \equiv_n cy \Rightarrow x \equiv_n y$?

Is there a fundamental lemma of division modulo n?

$cx \equiv_n cy \Rightarrow x \equiv_n y$? NO!

If c=0 [mod n], $cx \equiv_n cy$ for any x and y. Canceling the c is like dividing by zero.

Repaired fundamental lemma of division modulo n?

$c \neq 0 \pmod n$, $cx \equiv_n cy \Rightarrow x \equiv_n y$?

$2*2 \equiv_6 2*5$, but not $2 \equiv_6 5$.
$6*3 \equiv_{10} 6*8$, but not $3 \equiv_{10} 8$.

When can I divide by c?

Theorem: There are exactly n/GCD(c.n) distinct multiples of c modulo n.

Corollary: If GCD(c,n) > 1, then the number of multiples of c is less than n.

Corollary: If GCD(c,n)>1 then you can't always divide by c.

Proof: There must exist distinct x,y<n such that c*x=c*y (but $x \neq y$)

Fundamental lemma of division modulo n.

$$GCD(c,n)=1,\ ca \equiv_n cb \Rightarrow a \equiv_n b$$

$$ab = ac \bmod n$$
$$n \mid (ab - ac)$$
$$n \mid a(b-c)$$
$$n \mid b-c \qquad \text{since } (a,n)=1$$
$$b = c \bmod n$$

Corollary for general c:

$$cx \equiv_n cy \Rightarrow x \equiv_{n/GCD(c,n)} y$$

$cx \equiv_n cy$

$\Rightarrow cx \equiv_{n/(c,n)} cy$ and (c, n/$_{GCD}$(c,n))=1

$\Rightarrow x \equiv_{n/(c,n)} y$

Fundamental lemma of division modulo n.

$$GCD(c,n)=1,\ ca \equiv_n cb \Rightarrow a \equiv_n b$$

$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$

Multiplication over Z_n^* will have the cancellation property.

$Z_6 = \{0, 1,2,3,4,5\}$

$Z_6^* = \{1,5\}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Suppose GCD(x,n) =1 and GCD(y,n) =1

Let z = xy and z' = (xy mod n)

It is obvious that GCD(z,n) =1

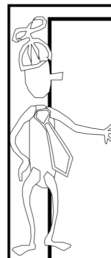
It requires a moment to convince ourselves that GCD(z',n) =1

$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$

$*_n$ is an associative, binary operator. In particular, Z_n^* is closed under $*_n$:

$$x,y \in Z_n^* \Rightarrow x *_n y \in Z_n^*.$$

Proof: Let z = xy. Let z' = z mod n. z = z' + kn.
Suppose there exists a prime p>1 p|z' and p|n.
z is the sum of two multiples of p, so p|z.
p|z ⇒ that p|x or p|y. Contradiction of $x,y \in Z_n^*$

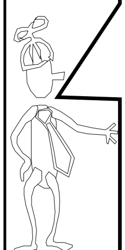


$Z_{12}^* = \{1,5,7,11\}$

$*_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Z_{15}^*

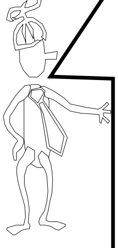
*	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1



The column permutation property is equivalent to the right cancellation property:

$[b * a = c * a] \Rightarrow b=c$

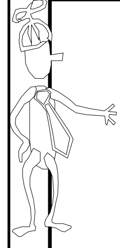
*	1	2	a	4
b	1	2	3	4
2	2	4	1	3
c	3	1	4	2
4	4	3	2	1



The row permutation property is equivalent to the left cancellation property:

$[a * b = a * c] \Rightarrow b=c$

*	b	2	c	4
1	1	2	3	4
2	2	4	1	3
a	3	1	4	2
4	4	3	2	1



$Z_5^* = \{1,2,3,4\}$

$*_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Euler Phi Function

$\Phi(n)$ = size of z_n^*

= number of $1<=k<n$ that are relatively prime to n.

p prime $\Rightarrow Z_p^* = \{1,2,3,\ldots,p-1\}$
$\Rightarrow \Phi(p) = p-1$

$Z_{12}^* = \{1,5,7,11\}$
$\phi(12) = 4$

$*_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

$\phi(pq) = (p-1)(q-1)$ if p,q distinct primes

pq = # of numbers from 1 to pq
p = # of multiples of q up to pq
q = # of multiples of p up to pq
1 = # of multiple of both p and q up to pq

$\phi(pq) = pq - p - q + 1 = (p-1)(q-1)$

Let's consider how we do arithmetic in Z_n and in Z_n^*

The additive inverse of $a \in Z_n$ is the unique $b \in Z_n$ such that $a +_n b \equiv_n 0$. We denote this inverse by "$-a$".

It is trivial to calculate: "$-a$" = $(n-a)$.

The multiplicative inverse of $a \in Z_n^*$ is the unique $b \in Z_n^*$ such that $a *_n b \equiv_n 1$. We denote this inverse by "a^{-1}" or "$1/a$".

The unique inverse of a must exist because the a row contains a permutation of the elements and hence contains a unique 1.

*	1	b	3	4
1	1	2	3	4
2	2	4	1	3
a	3	1	4	2
4	4	3	2	1

$Z_n = \{0, 1, 2, \ldots, n-1\}$
$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$

Define $+_n$ and $*_n$:
$a +_n b = (a+b \bmod n)$ $\quad a *_n b = (a*b \bmod n)$

$c *_n (a +_n b) \equiv_n (c *_n a) +_n (c *_n b)$

$\langle Z_n, +_n \rangle$	$\langle Z_n^*, *_n \rangle$
1. Closed	1. Closed
2. Associative	2. Associative
3. 0 is identity	3. 1 is identity
4. Additive Inverses	4. Multiplicative Inverses
5. Cancellation	5. Cancellation
6. Commutative	6. Commutative

The multiplicative inverse of $a \in Z_n^*$ is the unique $b \in Z_n^*$ such that $a *_n b \equiv_n 1$. We denote this inverse by "a^{-1}" or "$1/a$".

Efficient algorithm to compute a^{-1} from a and n.

Execute the Extended Euclid Algorithm on a and n (previous lecture). It will give two integers r and s such that:
$ra + sn = (a,n) = 1$

Taking both sides mod n, we obtain:
$rn \equiv_n 1$
Output r, which is the inverse of a

Fundamental lemma of powers?

If $(a \equiv_n b)$
Then $x^a \equiv_n x^b$?

If $(a \equiv_n b)$ Then $x^a \equiv_n x^b$?

NO!

$(16 \equiv_{15} 1)$, but it is not the case that: $2^1 \equiv_{15} 2^{16}$

Calculate a^b mod n:
Except for b, work in a reduced mod system to keep all intermediate results less than $\lfloor \log_2 (n) \rfloor + 1$ bits long.

Phase I (Repeated Multiplication)
For $\lfloor \log b \rfloor$ steps
multiply largest so far by a
$(a, a^2, a^4, \ldots)$

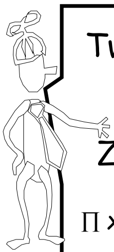
Phase II (Make a^b from bits and pieces)
Expand n in binary to see how n is the sum powers of 2. Assemble a^b by multiplying together appropriate powers of a.

Two names for the same set:

$Z_n^* = Z_n^a$

$Z_n^a = \{a *_n x \mid x \in Z_n^*\}, a \in Z_n^*$

*	b	2	c	4
1	1	2	3	4
2	2	4	1	3
a	3	1	4	2
4	4	3	2	1



Two products on the same set:

$Z_n^* = Z_n^a$

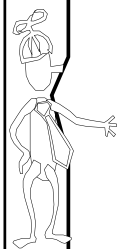
$Z_n^a = \{a *_n x \mid x \in Z_n^*\}$, $a \in Z_n^*$

$\Pi\, x \equiv_n \Pi\, ax$ [as x ranges over Z_n^*]

$\Pi x \equiv_n \Pi\, x\ (a^{\text{size of } Zn^*})$ [Commutativity]

$1 = a^{\text{size of } Zn^*}$ [Cancellation]

$a^{\Phi(n)} = 1$

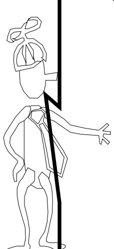


Euler's Theorem

$a \in Z_n^*$, $a^{\Phi(n)} \equiv_n 1$

Fermat's Little Theorem

p prime, $a \in Z_p^* \Rightarrow a^{p-1} \equiv_p 1$



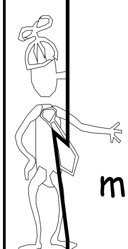
Fundamental lemma of powers.

Suppose $x \in Z_n^*$, and a,b,n are naturals.

If $a \equiv_{\Phi(n)} b$ Then $x^a \equiv_n x^b$

Equivalently,

$x^{a \bmod \Phi(n)} \equiv_n x^{b \bmod \Phi(n)}$

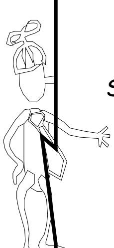


Defining negative powers.

Suppose $x \in Z_n^*$, and a,n are naturals.

x^{-a} is defined to be the multiplicative inverse of X^a

$X^{-a} = (X^a)^{-1}$

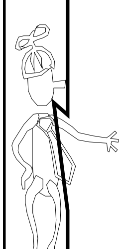


Rule of integer exponents

Suppose $x,y \in Z_n^*$, and a,b are integers.

$(xy)^{-1} \equiv_n x^{-1} y^{-1}$

$X^a X^b \equiv_n X^{a+b}$



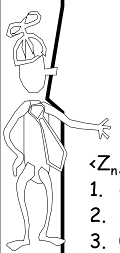
Lemma of integer powers.

Suppose $x \in Z_n^*$, and a,b are integers.

If $a \equiv_{\Phi(n)} b$ Then $x^a \equiv_n x^b$

Equivalently,

$x^{a \bmod \Phi(n)} \equiv_n x^{b \bmod \Phi(n)}$

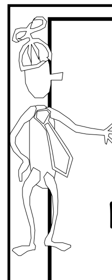


$Z_n = \{0, 1, 2, \ldots, n-1\}$

$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$

Quick raising to power.

$\langle Z_n, +_n \rangle$	$\langle Z_n^*, *_n \rangle$
1. Closed	1. Closed
2. Associative	2. Associative
3. 0 is identity	3. 1 is identity
4. Additive Inverses Fast + amd -	4. Multiplicative Inverses Fast * and /
5. Cancellation	5. Cancellation
6. Commutative	6. Commutative



Euler Phi Function

$\Phi(n)$ = size of z_n^*

p prime $\Rightarrow Z_p^* = \{1,2,3,\ldots,p-1\}$

$\Rightarrow \Phi(p) = p-1$

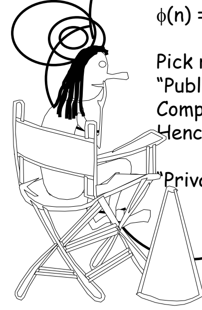
$\phi(pq) = (p-1)(q-1)$

if p,q distinct primes

The RSA Cryptosystem

Rivest, Shamir, and Adelman (1978)

RSA is one of the most used cryptographic protocols on the net. Your browser uses it to establish a secure session with a site.



Pick secret, random k-bit primes: p,q

"Publish": n = p*q

$\phi(n) = \phi(p)\ \phi(q) = (p-1)*(q-1)$

Pick random $e \in Z^*_{\phi(n)}$

"Publish": e

Compute d = inverse of e in $Z^*_{\phi(n)}$

Hence, $e*d = 1$ [mod $\phi(n)$]

"Private Key": d

p,q random primes, e random $\in Z^*_{\phi(n)}$

n = p*q

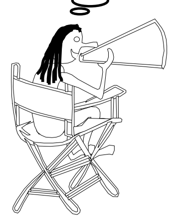
$e*d = 1$ [mod $\phi(n)$]

n,e is my public key. Use it to send a message to me.

p,q prime, e random $\in Z^*_{\phi(n)}$

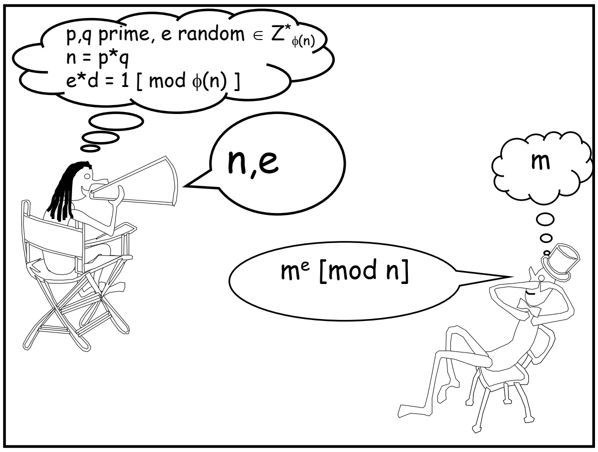
n = p*q

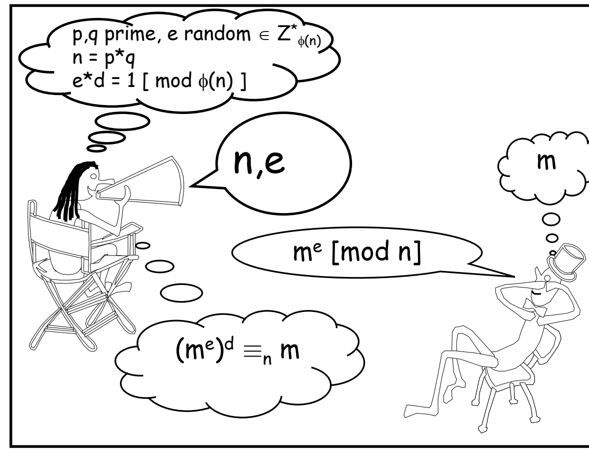
$e*d = 1$ [mod $\phi(n)$]



n,e

m


p,q prime, e random $\in Z^*_{\phi(n)}$
n = p*q
e*d = 1 [mod $\phi(n)$]
n,e
m
m^e [mod n]


p,q prime, e random $\in Z^*_{\phi(n)}$
n = p*q
e*d = 1 [mod $\phi(n)$]
n,e
m
m^e [mod n]
$(m^e)^d \equiv_n m$