Great Theoretical Ideas In Computer Science

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CS 15-251 Spring 2005

Lecture 6 Jan 27, 2005

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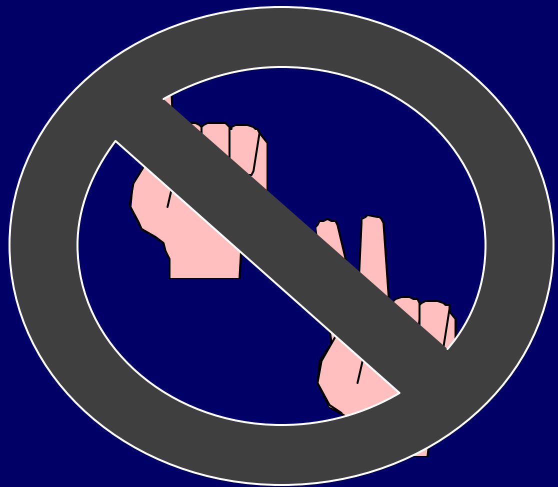
Counting I: One To One Correspondence and Choice Trees

How many seats in this auditorium?

Hint:
Count without counting!



If I have 14 teeth on the top and 12 teeth on the bottom, how many teeth do I have in all?

Addition Rule

Let A and B be two <u>disjoint</u> finite sets.

The size of $A \cup B$ is the sum of
the size of A and the size of B.

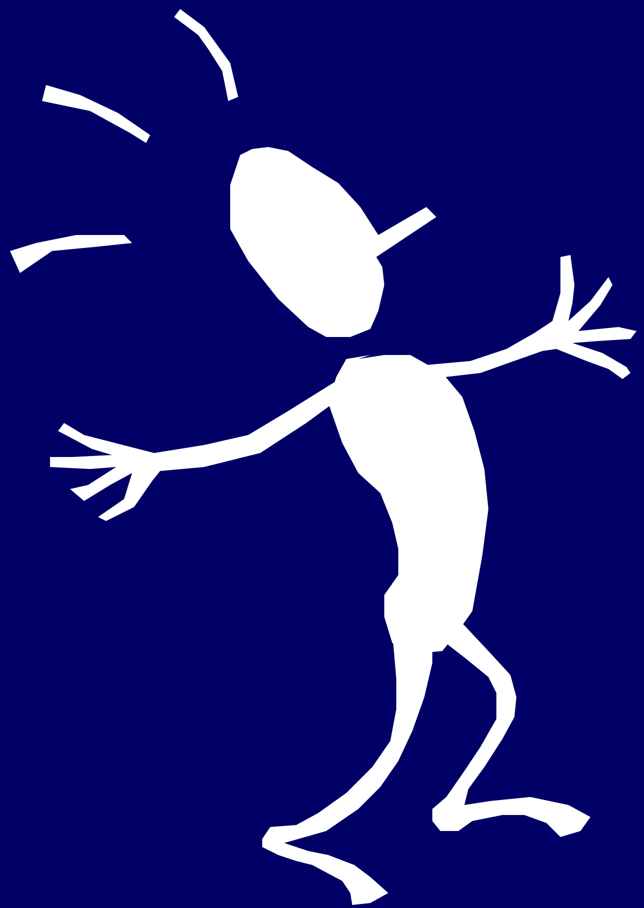
$$|A \cup B| = |A| + |B|$$

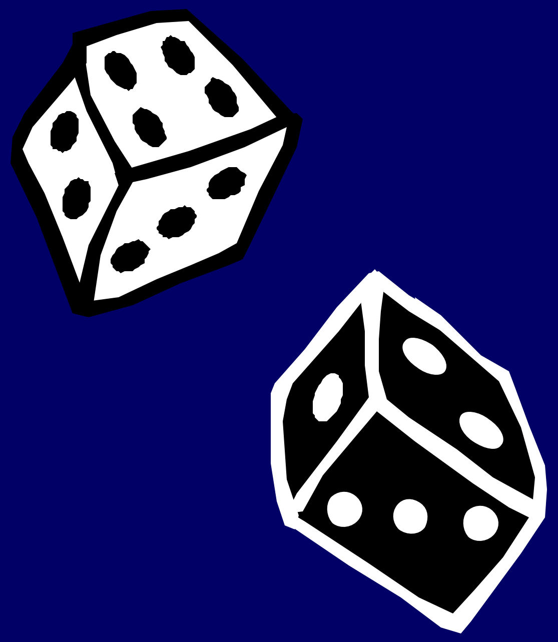
Corollary (by induction)

Let A_1, A_2, A_3, …, A_n be disjoint, finite sets.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Suppose I roll a white die and a black die.





S ≡ Set of all outcomes where the dice show different values.

$|S| = ?$

A_i ≡ set of outcomes where the black die says i and the white die says something else.

$$|S| = \left|\bigcup_{i=1}^{6} A_i\right| = \sum_{i=1}^{6} |A_i| = \sum_{i=1}^{6} 5 = 30$$

$S \equiv$ Set of all outcomes where the dice show different values.

$|S| = ?$

$T \equiv$ set of outcomes where dice agree.

$$|S \cup T| = \#\text{ of outcomes} = 36$$

$$|S| + |T| = 36 \qquad |T| = 6$$

$$|S| = 36 - 6 = 30$$

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$A_i \equiv$ set of outcomes where the black die says i and the white die says something larger.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

$$|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15$$

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$L \equiv$ set of all outcomes where the black die shows a larger number than the white die.

$$|S| + |L| = 30$$

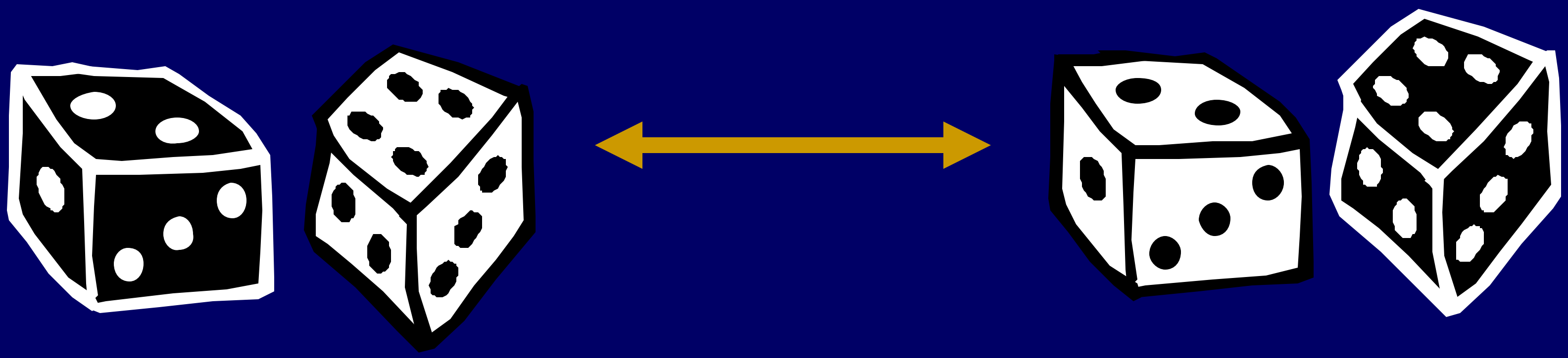
It is clear by symmetry that $|S| = |L|$.

Therefore $|S| = 15$

It is clear by symmetry that $|S| = |L|$.

Pinning down the idea of symmetry by exhibiting a correspondence.

Let's put each outcome in S in correspondence with an outcome in L by swapping the color of the dice.



S L

Pinning down the idea of symmetry by exhibiting a correspondence.

Let's put each outcome in S in correspondence with an outcome in L by swapping the color of the dice.

Each outcome in S gets matched with exactly one outcome in L, with none left over.

Thus: $|S| = |L|$.

Let f:A®B
be a function from a set A to a set B.

f is 1-1 if and only if

$$\forall x,y \in A,\ \ x \neq y \Rightarrow f(x) \neq f(y)$$

f is onto if and only if

$$\forall z \in B\ \ \exists x \in A\ \ f(x) = z$$

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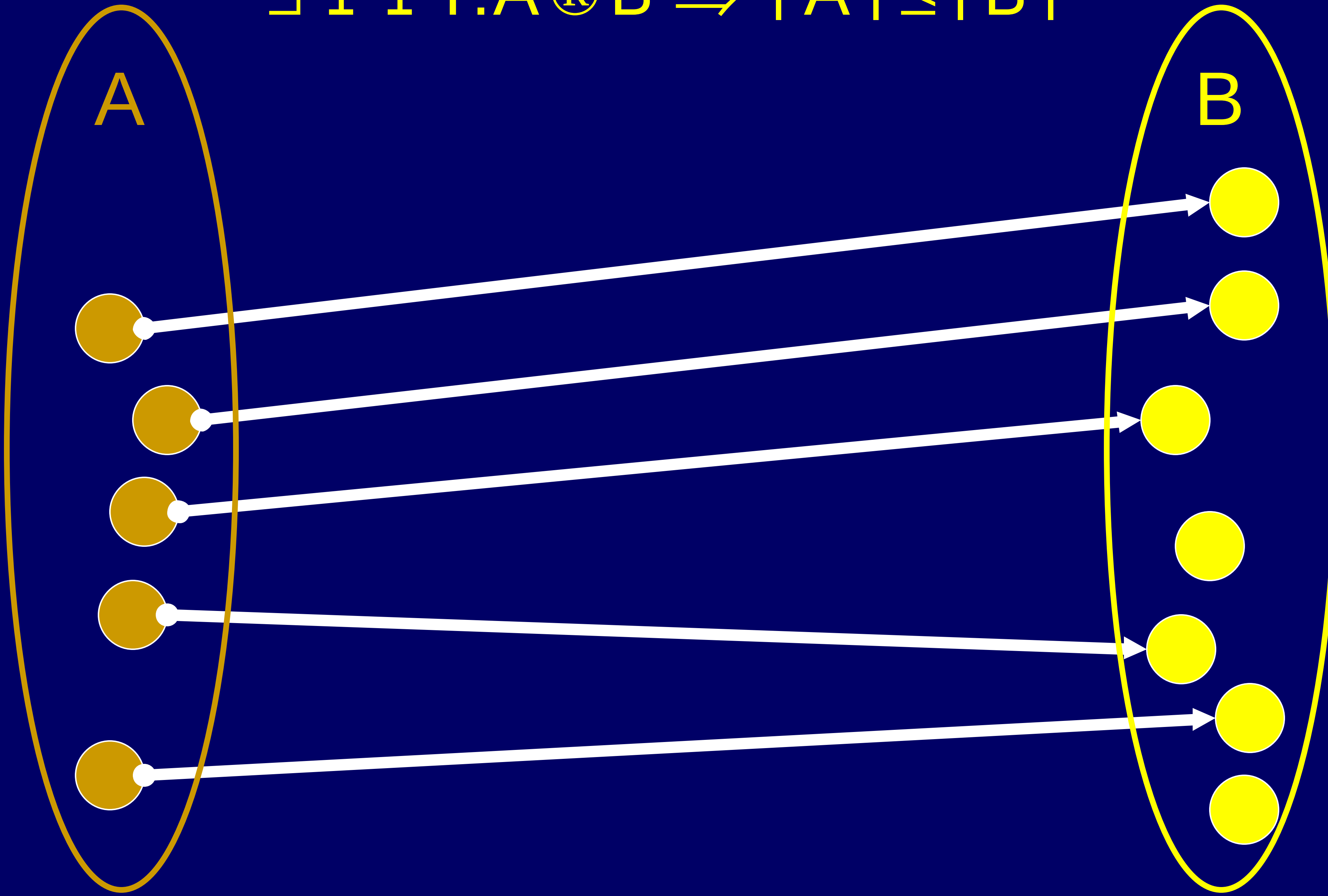
There Exists

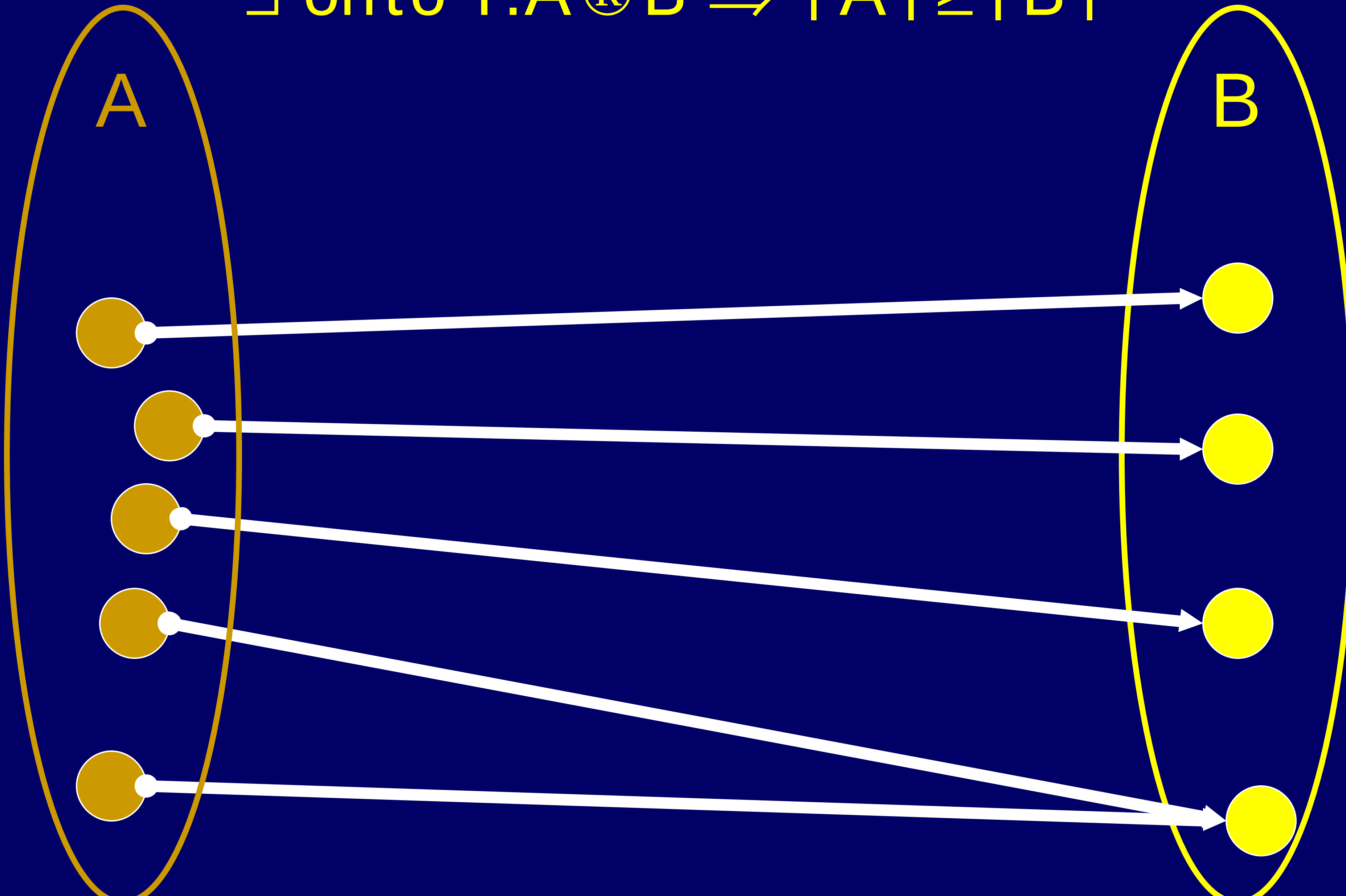
For Every

Let’s restrict our attention to finite sets.

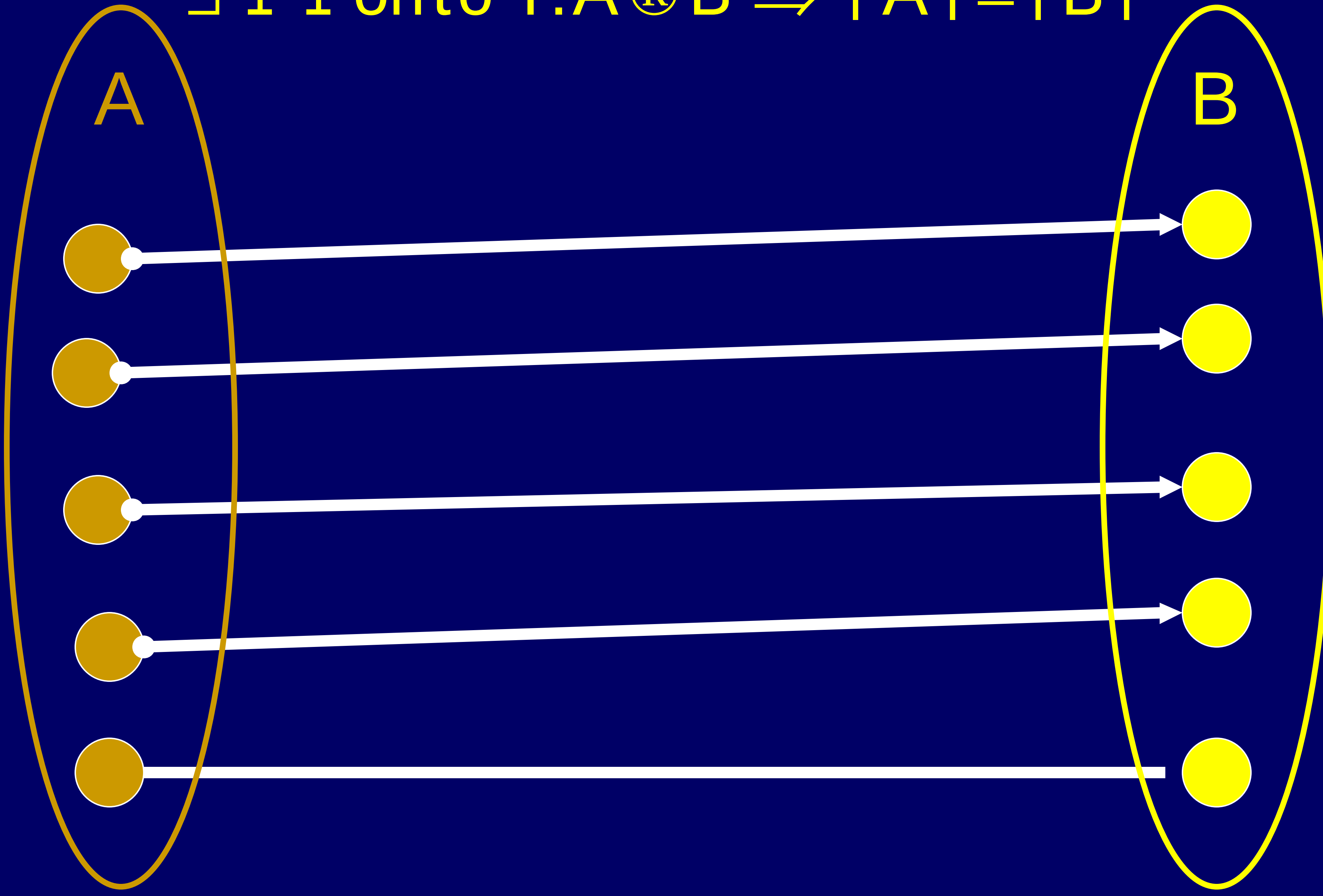
$\exists$ 1-1 f:A®B $\Rightarrow |A| \leq |B|$



$\exists$ onto $f{:}A \rightarrow B \Rightarrow |A| \geq |B|$



$\exists$ 1-1 onto f:A®B $\Rightarrow |A|=|B|$



1-1 Onto Correspondence
(just “correspondence” for short)

Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

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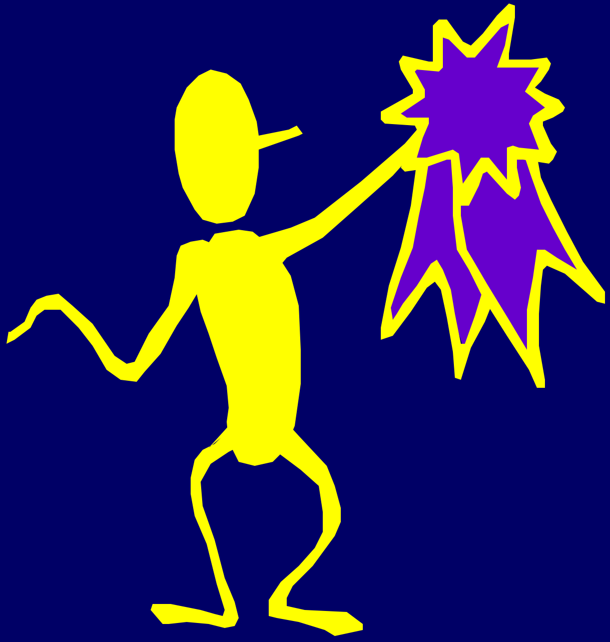


It's one of the most important mathematical ideas of all time!

Question: How many n-bit sequences are there?

000000	$\leftrightarrow$	0
000001	$\leftrightarrow$	1
000010	$\leftrightarrow$	2
000011	$\leftrightarrow$	3
	...	
1...11111	$\leftrightarrow$	2^n-1

2^n sequences

S = {a,b,c,d,e} has many subsets.

{a}, {a,b}, {a,d,e}, {a,b,c,d,e}, {e}, Ø, ...

The empty set is a set with all the rights and privileges pertaining thereto.

Question: How many subsets can be formed from the elements of a 5-element set?

a	b	c	d	e
0	1	1	0	1

{b c e}

1 means “TAKE IT”

0 means “LEAVE IT”

Question: How many subsets can be formed from the elements of a 5-element set?

a	b	c	d	e
0	1	1	0	1

Each subset corresponds to a 5-bit sequence (using the "take it or leave it" code)

$$S = \{a_1, a_2, a_3, \ldots, a_n\}$$

$$b = b_1 b_2 b_3 \ldots b_n$$

a_1	a_2	a_3	...	a_n
b_1	b_2	b_3	...	b_n

$$f(b) = \{a_i \mid b_i = 1\}$$

a_1	a_2	a_3	...	a_n
b_1	b_2	b_3	...	b_n

$$f(b) = \{ a_i \mid b_i = 1 \}$$

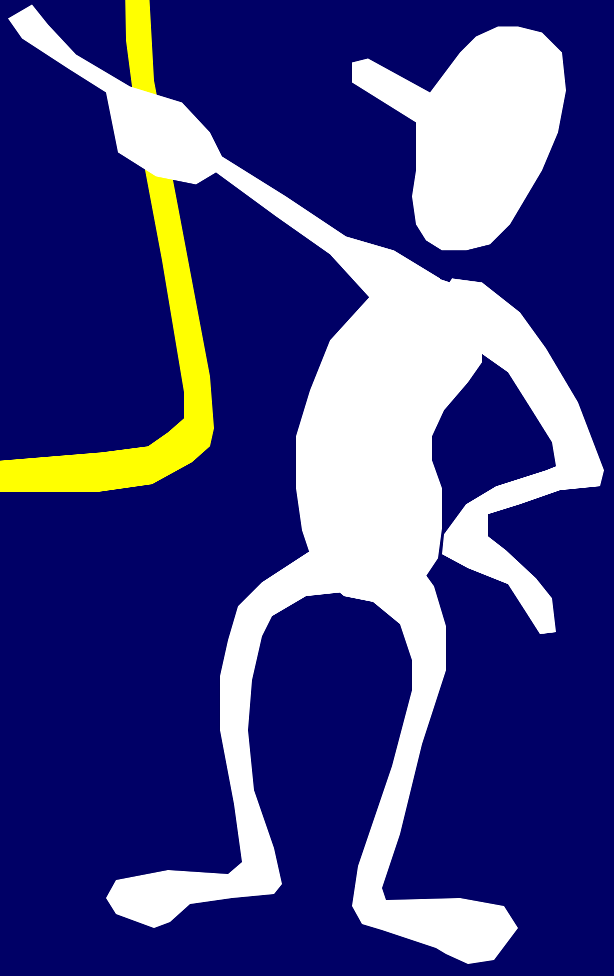
f is 1-1: Any two distinct binary sequences b and b' have a position i at which they differ. Hence, f(b) is not equal to f(b') because they disagree on element a_i.

a_1	a_2	a_3	...	a_n
b_1	b_2	b_3	...	b_n

$$f(b) = \{ a_i \mid b_i = 1 \}$$

f is onto: Let S be a subset of $\{a_1,\ldots,a_n\}$. Let $b_k = 1$ if a_k in S; $b_k = 0$ otherwise. $f(b_1 b_2 \ldots b_n) = S$.

The number of subsets of an n-element set is 2^n.



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$$\forall z \in B\ \ \exists x \in A\ \ f(x) = z$$

Let f:A®B

be a function from a set A to a set B.

f is a 1 to 1 correspondence iff

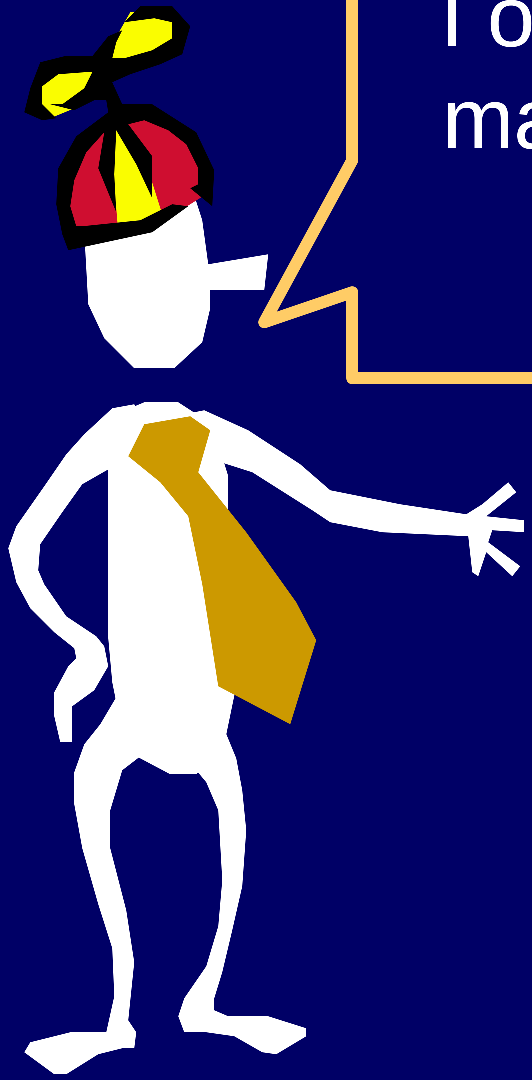
∀z∈B ∃ exactly one x2A s.t. f(x)=z

f is a k to 1 correspondence iff

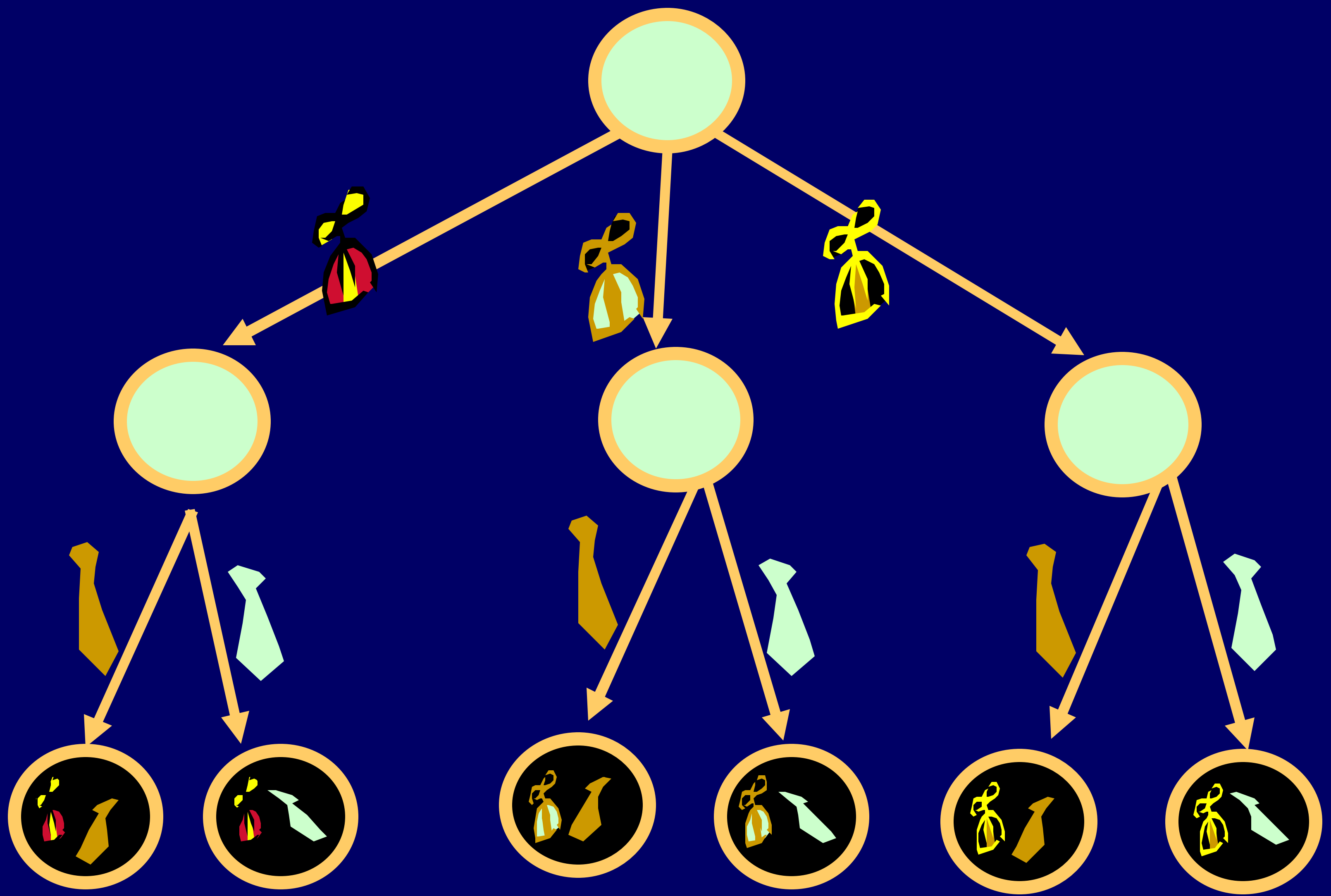
∀z∈B ∃ exactly k x2A s.t. f(x)=z

To count the number of horses in a barn, we count the number hoofs and then divide by 4.

If Finite set A has a k to 1 correspondence to finite set B, then #B = #A/k



I own 3 beanies and 2 ties. How many different ways can I dress up in a beanie and a tie?



A restaurant has a menu with 5 appetizers, 6 entrees, 3 salads, and 7 desserts.

How many items on the menu?

- 5 + 6 + 3 + 7 = 21

How many ways to choose a complete meal?

- 5 * 6 * 3 * 7 = 630

A restaurant has a menu with 5 appetizers, 6 entrees, 3 salads, and 7 desserts.

How many ways to order a meal if I might not have some of the courses?

- 6 * 7 * 4 * 8 = 1344

Hobson's restaurant has only 1 appetizer, 1 entree, 1 salad, and 1 dessert.

2^4 ways to order a meal if I might not have some of the courses.

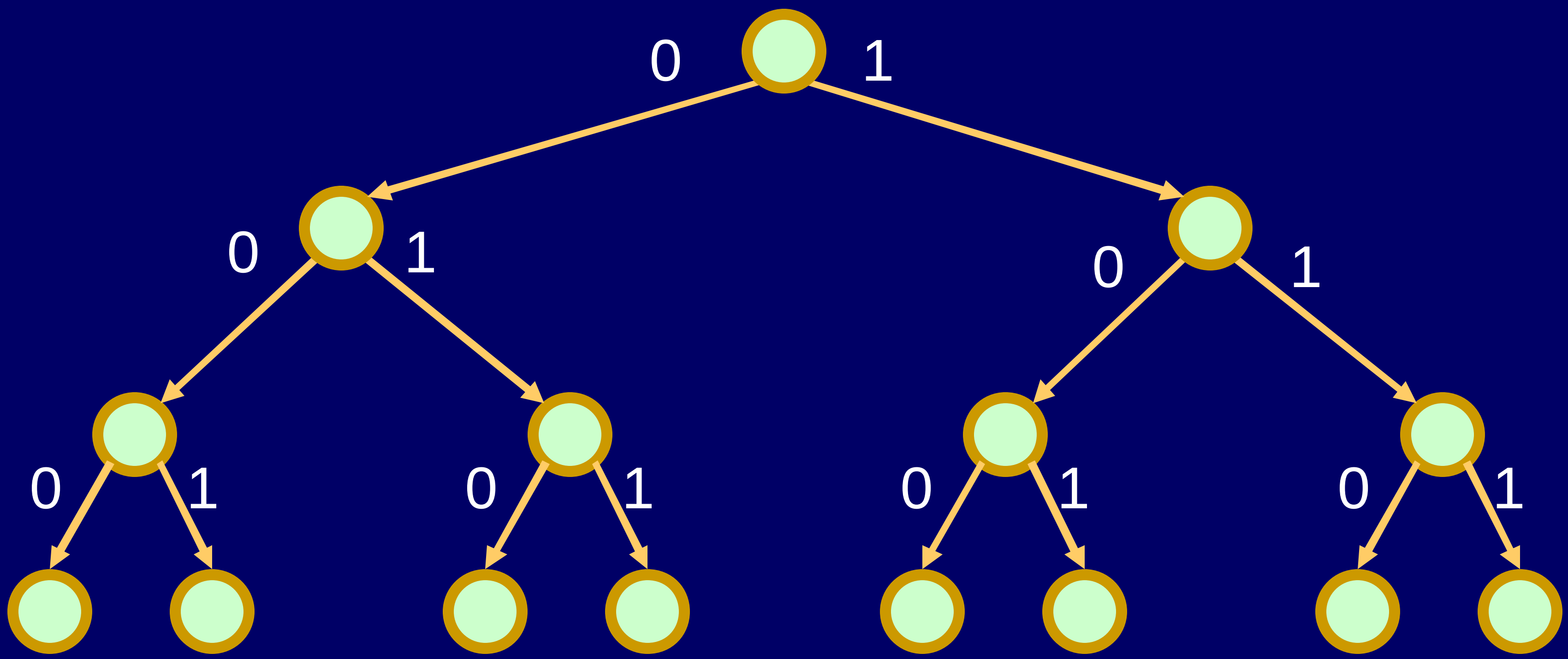
Same as number of subsets of the set {Appetizer, Entrée, Salad, Dessert}

Leaf Counting Lemma

Let T be a depth n tree when each node at depth $0 \leq i \leq n-1$ has P_{i+1} children. The number of leaves of T is given by:

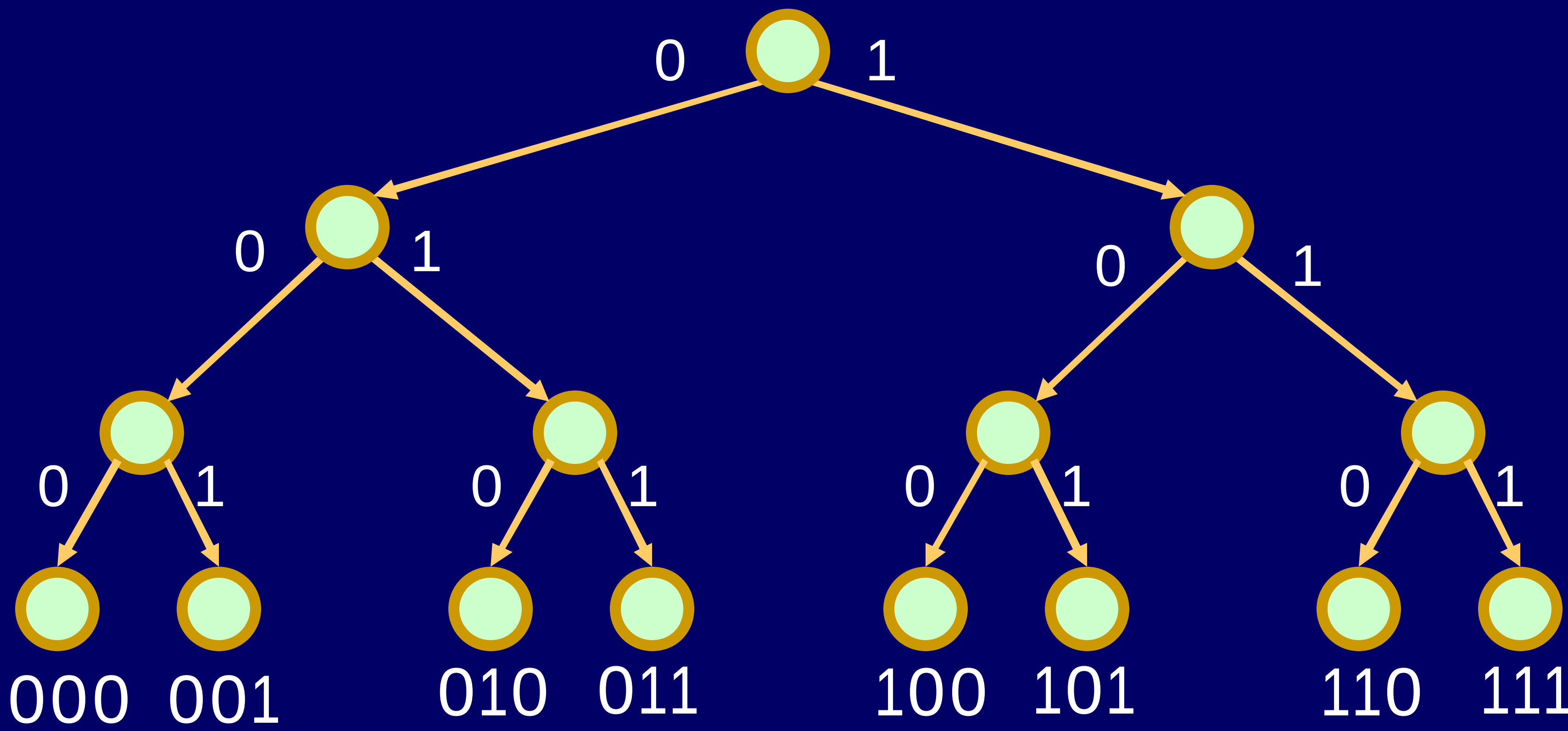
$$P_1 P_1 P_2 \ldots P_n$$

Choice Tree
for 2^n n-bit sequences

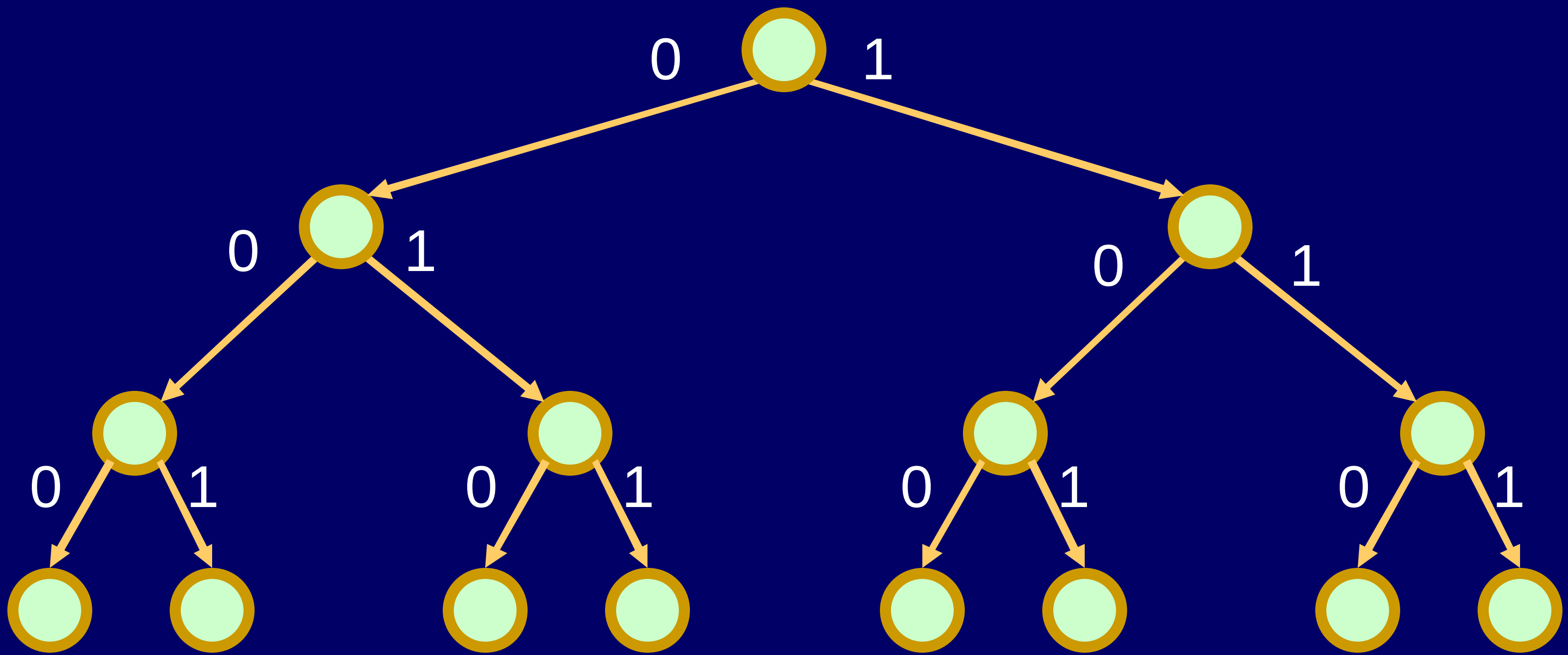


We can use a "choice tree" to represent the construction of objects of the desired type.

2^n n-bit sequences



Label each leaf with the object constructed by the choices along the path to the leaf.



2 choices for first bit

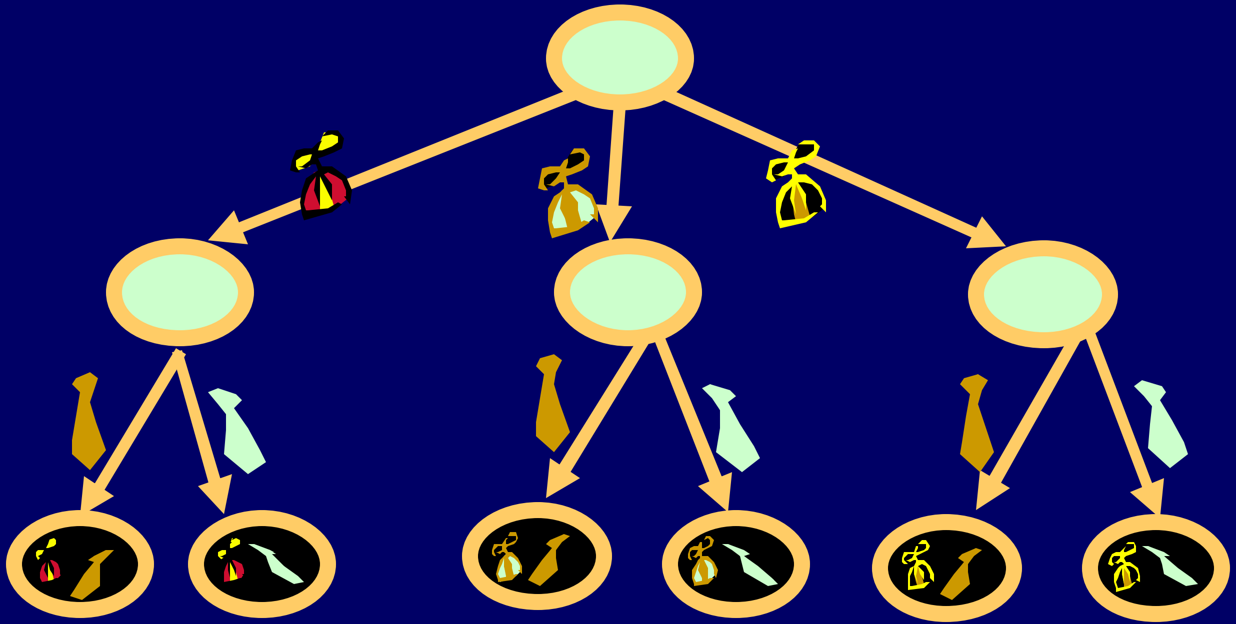
X 2 choices for second bit

X 2 choices for third bit

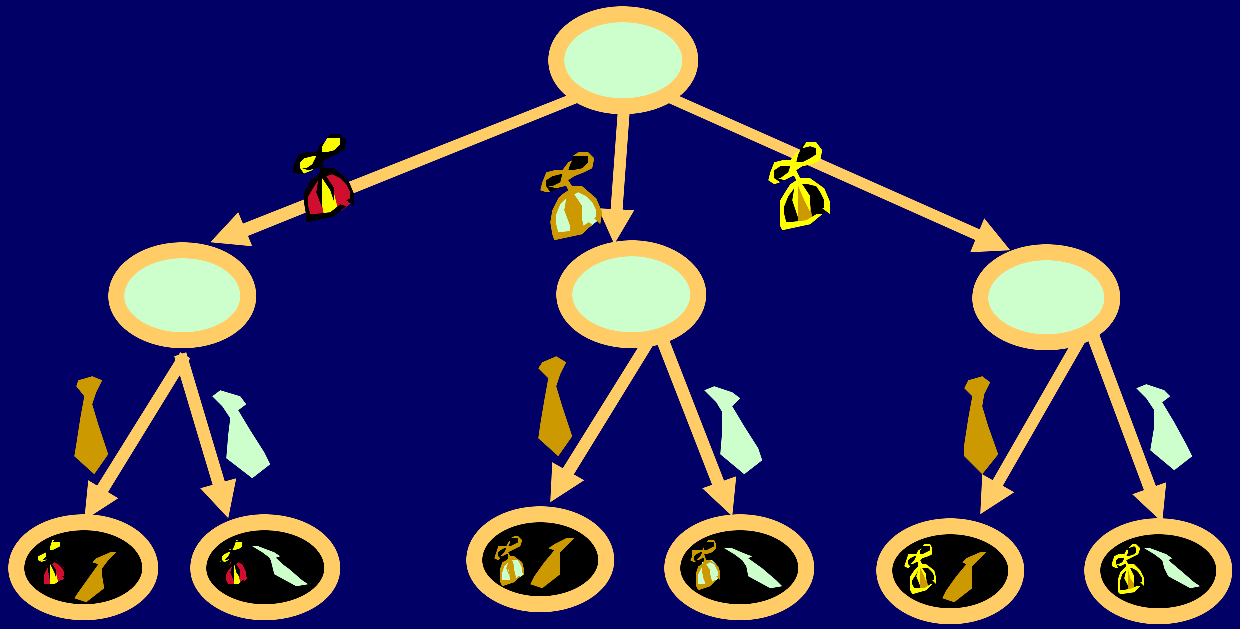
…

X 2 choices for the n^{th}

Choice Tree

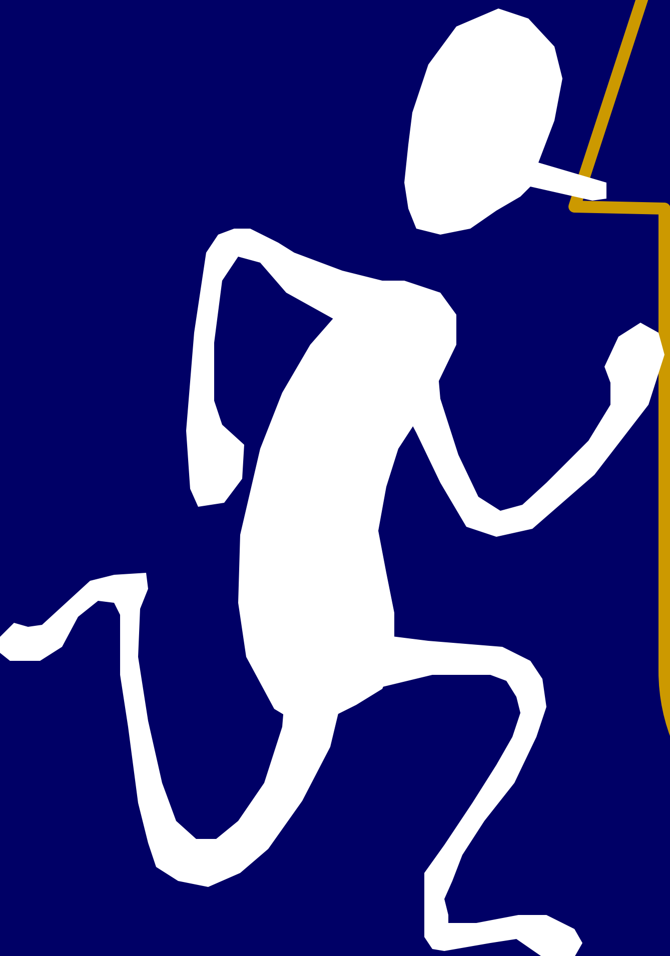


A <u>choice tree</u> is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf.



A choice tree provides a “choice tree representation” of a set S, if

1) Each leaf label is in S
2) No two leaf labels are the same



We will now combine the correspondence principle with the leaf counting lemma to make a powerful counting rule for choice tree representation.

Product Rule

IF S has a choice tree representation with
P_1 possibilities for the first choice,
P_2 for the second, and so on,

THEN

there are $P_1 P_2 P_3 \ldots P_n$ objects in S

Proof: The leaves of the choice tree are in 1-1 onto correspondence with the elements of S.

Product Rule

Suppose that all objects of a type S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF

1) Each sequence of choices constructs an object of type S

AND

2) No two different sequences create the same object

THEN

there are $P_1 P_2 P_3 \ldots P_n$ objects of type S.

How many different orderings of deck with 52 cards?

What type of object are we making?

- Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;

51 possible choices for the second card;

50 possible choices for the third card;

…

1 possible choice for the 52cond card.

How many different orderings of deck with 52 cards?

By the product rule:

52 * 51 * 50 * … * 3 * 2 * 1 = 52!

52 “factorial” orderings

A <u>permutation</u> or <u>arrangement</u> of n objects is an ordering of the objects.

The number of permutations of n distinct objects is n!

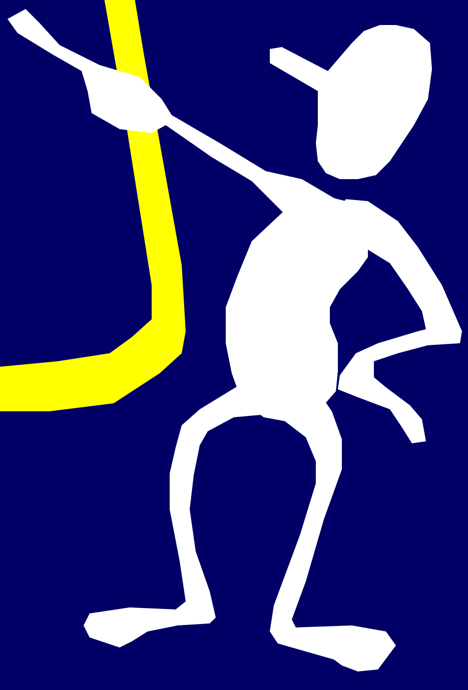
How many sequences of 7 letters are there?

26^7

How many sequences of 7 letters contain at least two of the same letter?

$26^7 - 26*25*24*23*22*21*20$

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.



A formalization

Let $S(x): \Sigma^* \rightarrow \{True, False\}$ be any predicate.

We can associate S with the set:

$OBJECTS_S = \{x \in \Sigma^* \mid S(x)\}$

the "object space" S (or objects of type S)

When $OBJECTS_S$ is finite, let us define

$\#OBJECTS_S$ = the size of $OBJECTS_S$

In fact, define #S as $\#OBJECTS_S$

Object property Q on object space S

Consider $Q(x)$: $\mathrm{OBJECTS}_S \rightarrow \{\mathrm{True}, \mathrm{False}\}$

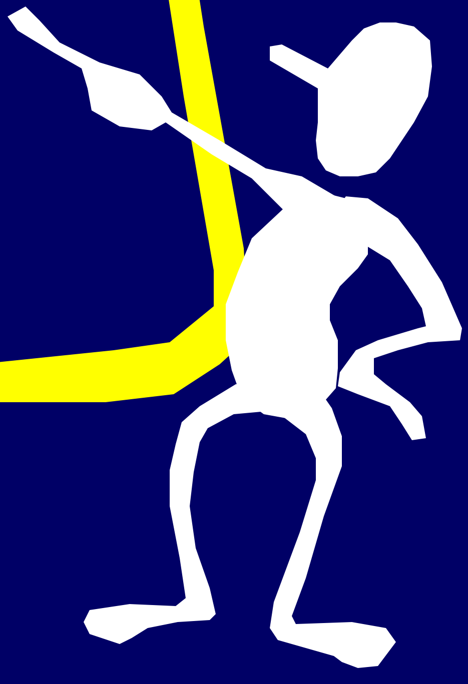
Define :Q(x): $\mathrm{OBJECTS}_S \rightarrow \{\mathrm{True}, \mathrm{False}\}$
As Input(x); return NOT Q(x)

Proposition: $\#Q = \#S - \#(:Q)$

How many of our objects have property Q in object space S?

$\#Q$

$= \#OBJECTS_S - \#(:Q)$



Helpful Advice:

In logic, it can be useful to represent a statement in the contrapositive.

In counting, it can be useful to represent a set in terms of its complement.

If 10 horses race, how many orderings of the top three finishers are there?

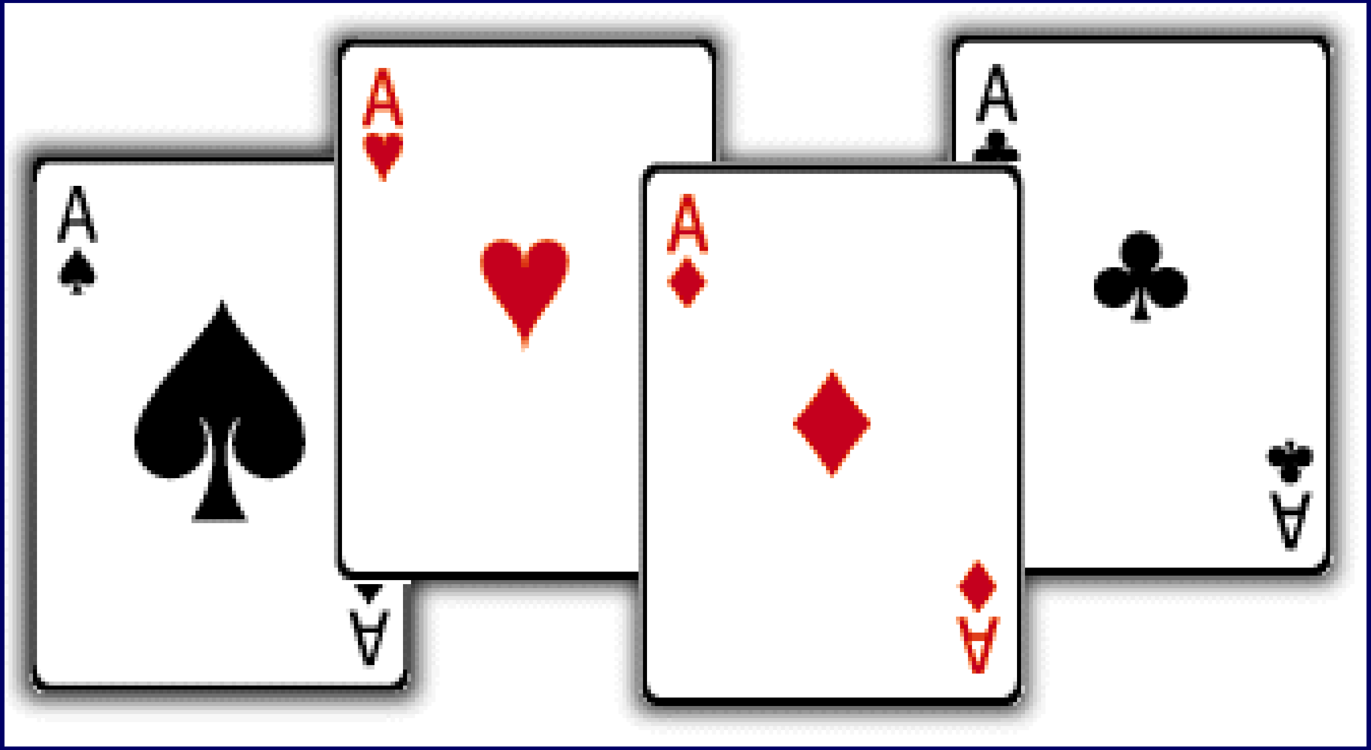
$$10 * 9 * 8 = 720$$

The number of ways of ordering, permuting, or arranging r out of n objects.

n choices for first place, n-1 choices for second place, . . .

$$n * (n-1) * (n-2) *...* (n-(r-1))$$

$$= \frac{n!}{(n-r)!}$$



Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

- 52 * 51

How many unordered pairs?

- 52*51 / 2 ← divide by overcount

  Each unordered pair is listed twice on a list of the ordered pairs, but we consider the ordered pairs to be the same.

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

- 52 * 51

How many unordered pairs?

- 52*51 / 2 ← divide by overcount

We have a 2 to 1 map from ordered pairs to unordered pairs. Hence: the #unordered pairs = (#ordered pairs)/2

Ordered Versus Unordered

From a deck of 52 cards how many ordered 5 card sequences can be formed?

- 52 * 51 * 50 * 49 * 48

How many orderings of 5 cards?

- 5!

How many unordered 5 card hands? pairs?

- 52*51*50*49*48 / 5! = 2,598,960

A <u>combination</u> or <u>choice</u> of r out of n objects is an (unordered) set of r of the n objects.

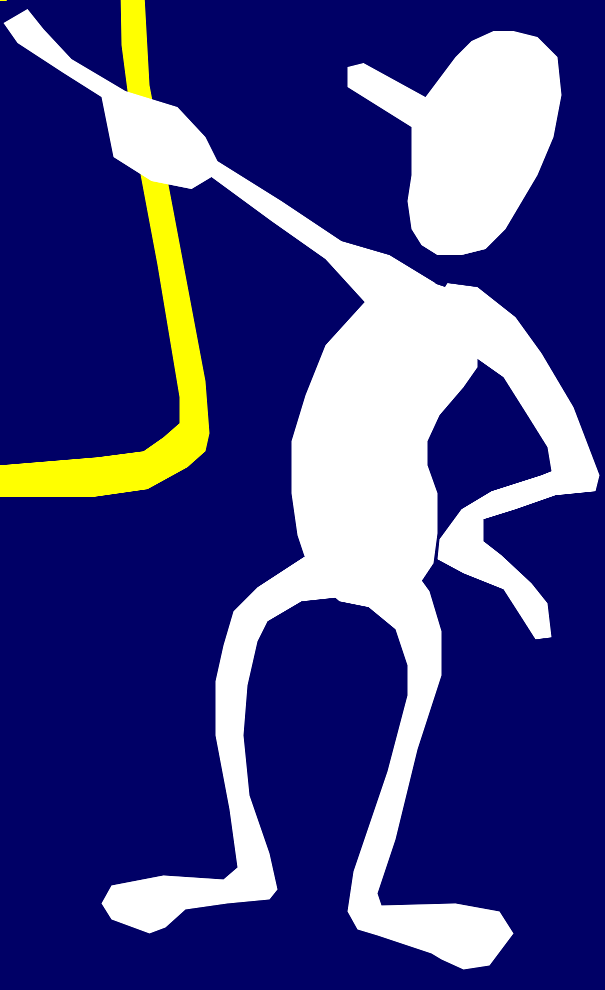
The number of r combinations of n objects:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

$\boxed{\text{n choose r}}$

The number of subsets of size r that can be formed from an n-element set is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$



How many 8 bit sequences have 2 0's and 6 1's?

Tempting, but incorrect:

8 ways to place first 0 times

7 ways to place second 0

Violates condition 2 of product rule!

Choosing position i for the first 0 and then position j for the second 0 gives the same sequence as choosing position j for the first 0 and position i for the second.

How many 8 bit sequences have 2 0's and 6 1's?

1) Choose the set of 2 positions to put the 0's. The 1's are forced.

$$\binom{8}{2} \times 1 = \binom{8}{2}$$

2) Choose the set of 6 positions to put the 1's. The 0's are forced.

$$\binom{8}{6} \times 1 = \binom{8}{6}$$

Symmetry in the formula:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$$

How many hands have at least 3 aces?

$$\binom{4}{3} = 4 \text{ ways of picking 3 of the 4 aces.}$$

$$\binom{49}{2} = 1176 \text{ ways of picking 2 cards from the remaining 49 cards.}$$

$$4 \times 1176 = 4704$$

How many hands have at least 3 aces?

How many hands have exactly 3 aces?

$\binom{4}{3} = 4$ ways of picking 3 of the 4 aces.

$\binom{48}{2} = 1128$ ways of picking 2 cards non – ace cards.

$4 \times 1128 = 4512$

How many hands have exactly 4 aces?

$\binom{4}{4} = 1$ way of picking 4 of the 4 aces.

48 ways of picking one of the remaining cards

4512 + 48 = 4560

4704 1 4560

At least one of the two counting arguments is not correct.

Four different sequences of choices produce the same hand

$\binom{4}{3}$ = 4 ways of picking 3 of the 4 aces.

$\binom{49}{2}$ = 1176 ways of picking 2 cards from the remaining 49 cards.

$4 \times 1176 = 4704$

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦

Is the other argument correct? How do I avoid fallacious reasoning?

The Sleuth's Criterion

Condition (2) of the product rule:

For any object it should be possible to reconstruct <u>the</u> sequence of choices which lead to it.

1) Choose 3 of 4 aces
2) Choose 2 of the remaining cards

A♣ A♦ A♥ A♠ K♦

Sleuth can’t determine which cards came from which choice.

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A ♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦

Is the other argument correct? How do I avoid fallacious reasoning?

1) Choose 3 of 4 aces
2) Choose 2 non-ace cards

A♣ Q♠ A♦ A♥ K♦

Sleuth reasons:

The aces came from the first choice and the non-aces came from the second choice.

1) Choose 4 of 4 aces
2) Choose 1 non-ace

A♣ A♠ A♦ A♥ K♦

Sleuth reasons:

The aces came from the first choice and the non-ace came from the second choice.