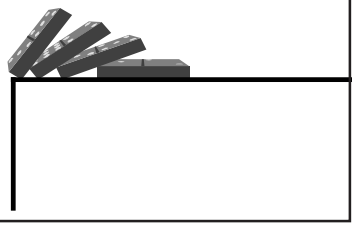
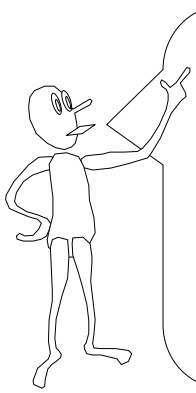
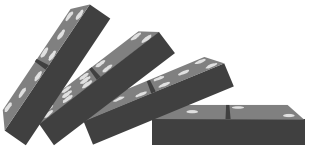
| Great Theoretical Ideas In Computer Science | | |
|---|---|---|
| Steven Rudich | | CS 15-251 Spring 2005 |
| Lecture 1 | Jan 11, 2005 | Carnegie Mellon University |

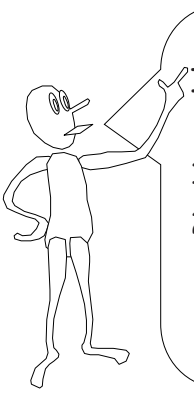
Induction: One Step At A Time





Today we will talk about INDUCTION





Induction is the primary way we:

1. Prove theorems
2. Construct and define objects

Let's start with dominoes

Domino Principle: Line up any number of dominos in a row; knock the first one over and they will all fall.

n dominoes numbered 1 to n

$F_k \equiv$ The $k^{th}$ domino falls

If we set them all up in a row then we know that each one is set up to knock over the next one:

For all $1 \leq k < n$:
$F_k \Rightarrow F_{k+1}$

## n dominoes numbered 1 to n

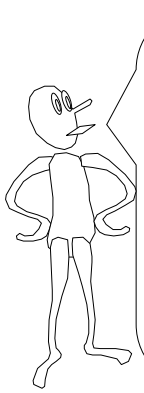
$F_k \equiv$ The $k^{th}$ domino falls

For all $1 \leq k < n$:

$F_k \Rightarrow F_{k+1}$

$F_1 \Rightarrow F_2 \Rightarrow F_3 \Rightarrow \ldots$

$F_1 \Rightarrow$ All Dominoes Fall



Computer Scientists don't start numbering things at 1, they start at 0.

YOU will spend a career doing this, so GET USED TO IT NOW.

## n dominoes numbered 0 to n-1

$F_k \equiv$ The $k^{th}$ domino falls

For all $0 \leq k < n-1$:

$F_k \Rightarrow F_{k+1}$

$F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow \ldots$

$F_0 \Rightarrow$ All Dominoes Fall

Standard Notation/Abbreviation
"for all" is written "$\forall$"

Example:

For all $k>0$, P(k)
is equivalent to
$\forall k>0$, P(k)

## n dominoes numbered 0 to n-1

$F_k \equiv$ The $k^{th}$ domino falls

$\forall k$, $0 \leq k < n-1$:

$F_k \Rightarrow F_{k+1}$

$F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow \ldots$

$F_0 \Rightarrow$ All Dominoes Fall

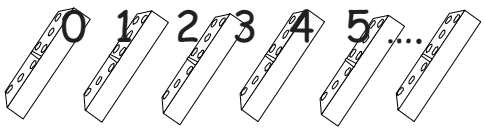
## The Natural Numbers

$\mathbb{N} = \{ 0, 1, 2, 3, \ldots \}$

## The Natural Numbers

$\mathbb{N} = \{ 0, 1, 2, 3, \ldots \}$

One domino for each natural number:




## The Infinite Domino Principle
$F_k \equiv$ The $k^{th}$ domino falls

Suppose $F_0$
Suppose for each natural number k,
$F_k \Rightarrow F_{k+1}$
Then All Dominoes Fall!

$F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow \ldots$

## The Infinite Domino Principle
$F_k \equiv$ The $k^{th}$ domino falls

Suppose $F_0$
Suppose for each natural number k,
$F_k \Rightarrow F_{k+1}$
Then All Dominoes Fall!

Proof: If they do not all fall, there must be a least numbered domino $d>0$ that did not fall. Hence, $F_{d-1}$ and not $F_d$. $F_{d-1} \Rightarrow F_d$. Hence, domino d fell and did not fall. Contradiction.

## Mathematical Induction: statements proved instead of dominoes fallen

Infinite sequence of dominoes.
$F_k \equiv$ domino k falls

Infinite sequence of statements: $S_0, S_1, \ldots$
$F_k \equiv S_k$ proved

Establish 1) $F_0$
2) $\forall$ k, $F_k \Rightarrow F_{k+1}$

Conclude that $F_k$ is true for all k

## Inductive Proof / Reasoning To Prove $\forall k, S_k$

Establish "Base Case": $S_0$
Establish "Domino Property": $\forall k, S_k \Rightarrow S_{k+1}$

$\forall k, S_k \Rightarrow S_{k+1}$ {
- Assume hypothetically that $S_k$ for *any* particular k;
- Conclude that $S_{k+1}$

## Inductive Proof / Reasoning To Prove $\forall k, S_k$

Establish "Base Case": $S_0$
Establish "Domino Property": $\forall k, S_k \Rightarrow S_{k+1}$

$\forall k, S_k \Rightarrow S_{k+1}$ {
- "Induction Hypothesis" $S_k$
- Use I.H. to show $S_{k+1}$

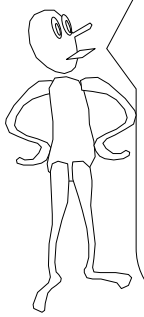
## Inductive Proof / Reasoning To Prove $\forall k \geq b$, $S_k$

Establish "Base Case": $S_b$
Establish "Domino Property": $\forall k \geq b$, $S_k \Rightarrow S_{k+1}$

Assume $k \geq b$

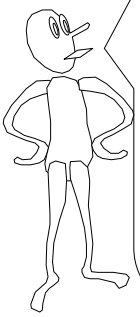
Assume "Inductive Hypothesis": $S_k$

Prove that $S_{k+1}$ follows



Theorem?

The sum of the first n odd numbers is $n^2$.

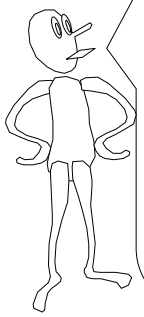


Theorem?

The sum of the first n odd numbers is $n^2$.

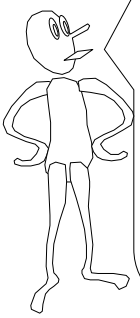
CHECK IT OUT ON SMALL VALUES:

| | |
|---|---|
| 1 | = 1 |
| 1+3 | = 4 |
| 1+3+5 | = 9 |
| 1+3+5+7 | = 16 |



Theorem: The sum of the first n odd numbers is $n^2$.

The $k^{th}$ odd number is expressed by the formula $(2k - 1)$, when $k>0$.



$S_n \equiv$
"The sum of the first n odd numbers is $n^2$."

Equivalently, $S_n$ is the statement that:
$\Sigma_{1 \leq k \leq n} (2k-1)$
$=1 + 3 + 5 + (2k-1) + . . +(2n-1)$
$= n^2$

$S_n \equiv$ "The sum of the first n odd numbers is $n^2$."
"$1 + 3 + 5 + (2k-1) + . . +(2n-1)= n^2$"

Trying to establish that: $\forall n \geq 1$ $S_n$

Base case: $S_1$ is true

The sum of the first 1 odd numbers is 1.

$S_n \equiv$ "The sum of the first n odd numbers is $n^2$."
"$1 + 3 + 5 + (2k-1) + \ldots + (2n-1) = n^2$"

Trying to establish that: $\forall n \geq 1\ S_n$

Assume "Induction Hypothesis": $S_k$
(for any particular $k \geq 1$)

$1+3+5+\ldots+(2k-1) = k^2$

$S_n \equiv$ "The sum of the first n odd numbers is $n^2$."
"$1 + 3 + 5 + (2k-1) + \ldots + (2n-1) = n^2$"

Trying to establish that: $\forall n \geq 1\ S_n$

Assume "Induction Hypothesis": $S_k$
(for any particular $k \geq 1$)

$1+3+5+\ldots+(2k-1) = k^2$

Add $(2k+1)$ to both sides.

$1+3+5+\ldots+(2k-1)+(2k+1) = k^2+(2k+1)$

Sum of first k+1 odd numbers $= (k+1)^2$

CONCLUSE: $S_{k+1}$

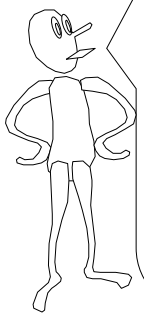
$S_n \equiv$ "The sum of the first n odd numbers is $n^2$."
"$1 + 3 + 5 + (2k-1) + \ldots + (2n-1) = n^2$"

Trying to establish that: $\forall n \geq 1\ S_n$

Established base case: $S_1$

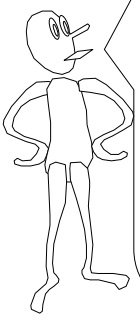
Established domino property: $\forall\ k \geq 1\ S_k \Rightarrow S_{k+1}$

By induction of n, we conclude that:
$\forall n \geq 1\ S_n$



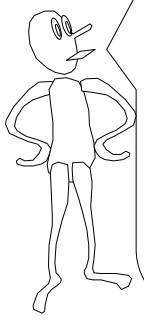
THEOREM:

The sum of the first n odd numbers is $n^2$.



Theorem?

The sum of the first n numbers is $\frac{1}{2}n(n+1)$.



Theorem? The sum of the first n numbers is $\frac{1}{2}n(n+1)$.

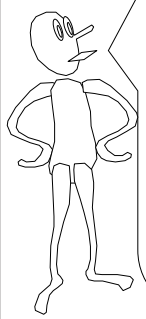
Try it out on small numbers!

$1 = 1 = \frac{1}{2}\ 1(1+1)$.

$1+2 = 3 = \frac{1}{2}\ 2(2+1)$.

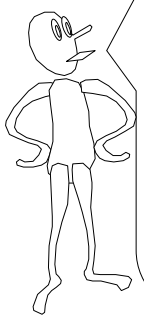
$1+2+3 = 6 = \frac{1}{2}\ 3(3+1)$.

$1+2+3+4 = 10 = \frac{1}{2}\ 4(4+1)$.



Theorem? The sum of the first n numbers is $\frac{1}{2}n(n+1)$.

= 0 = $=\frac{1}{2}$ 0(0+1).
1 = 1 = $=\frac{1}{2}$ 1(1+1).
1+2 = 3 = $\frac{1}{2}$ 2(2+1).
1+2+3 = 6 = $\frac{1}{2}$ 3(3+1).
1+2+3+4 = 10= $\frac{1}{2}$ 4(4+1).



Notation:

$\Delta_0 = 0$

$\Delta_n = 1 + 2 + 3 + \ldots + n-1 + n$

Let $S_n \equiv$ "$\Delta_n = n(n+1)/2$"

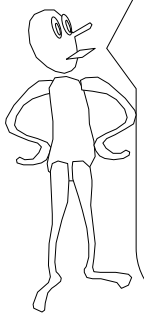
$S_n \equiv$ "$\Delta_n = n(n+1)/2$"
Use induction to prove $\forall k \geq 0$, $S_k$

Establish "Base Case": $S_0$. $\Delta_0$= The sum of the first 0 numbers = 0. Setting n=0 the formula gives 0(0+1)/2 = 0.

Establish "Domino Property": $\forall k \geq 0$, $S_k \Rightarrow S_{k+1}$

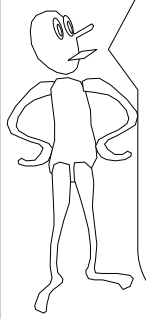
"Inductive Hypothesis" $S_k$: $\Delta_k = k(k+1)/2$

$\Delta_{k+1} = \Delta_k + (k+1)$
$= k(k+1)/2 + (k+1)$ [Using I.H.]
$= (k+1)(k+2)/2$ [which proves $S_{k+1}$]



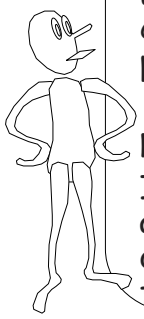
THEOREM:

The sum of the first n numbers is $\frac{1}{2}n(n+1)$.



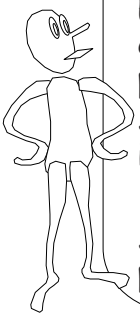
A natural number n>1 is prime if it has no divisors besides 1 and itself.

N.B.
1 is not considered prime.



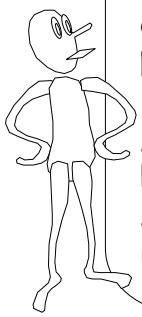
Easy theorem:
Every natural number>1 can be factored into primes.

N.B.:
It is much more subtle to argue for the existence of a unique prime factorization



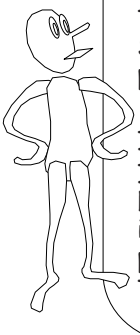
Easy theorem:
Every natural number>1 can be factored into primes.
$S_n \equiv$ "n can be factored into primes"

$S_2$ is true because 2 is prime.



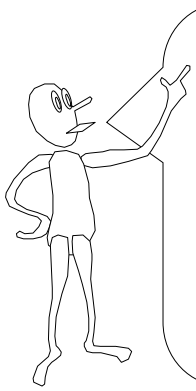
Every natural number>1 can be factored into primes. Base case: 2

Assume 2,3,……,k-1 all can be factored into primes.
Show k can be factored into primes.



Assume 2,3,……,k-1 all can be factored into primes.
Show k can be factored into primes.

If k is prime, we are done.
If not, k= ab where 1<a,b<k, hence a and b can be factored into primes. Thus, k is the product of the factors of a and the factors of b.



This illustrates a technical point about using and defining mathematical induction.

All Previous Induction
To Prove $\forall k, S_k$

Establish "Base Case": $S_0$

Establish that $\forall k, S_k \Rightarrow S_{k+1}$

Let k be any natural number.
Induction Hypothesis:
Assume $\forall j<k, S_j$
Derive $S_k$

"Strong" Induction
To Prove $\forall k, S_k$

Establish "Base Case": $S_0$

Establish that $\forall k, S_k \Rightarrow S_{k+1}$

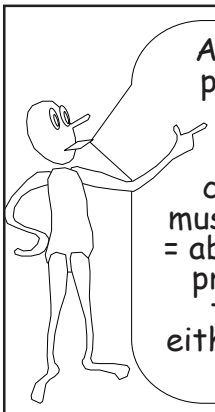
Let k be any natural number.
Assume $\forall j<k, S_j$
Prove $S_k$

## Least Counter-Example Induction to Prove $\forall k, S_k$

Establish "Base Case": $S_0$
Establish that $\forall k, S_k \Rightarrow S_{k+1}$

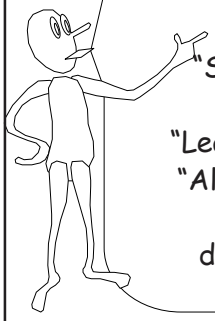
Assume that $S_k$ is the least counter-example.

Derive the existence of a smaller counter-example



All numbers > 1 has a prime factorization.

Let n be the least counter-example. n must not be prime – so n = ab. If both a and b had prime factorizations, then n would. Thus either a or b is a smaller counter-example.



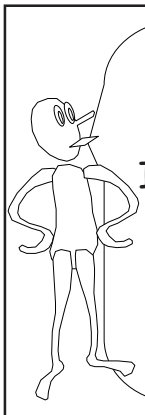
Inductive reasoning is the high level idea:

"Standard" Induction

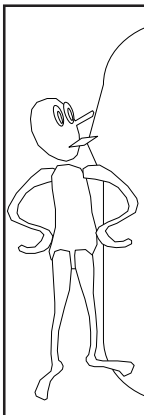
"Least Counter-example"
"All Previous" Induction
all just
different packaging.

## "All Previous" Induction Can Be Repackaged As Standard Induction

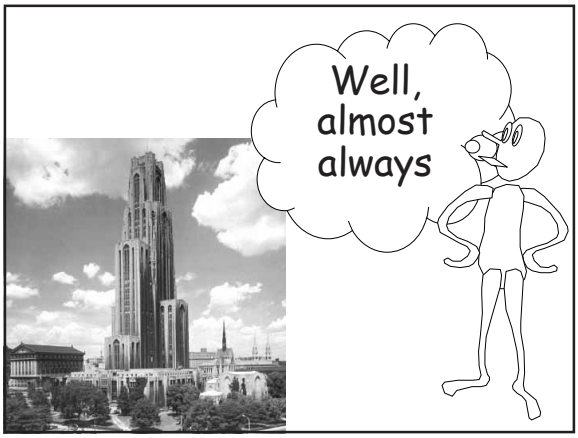
| Standard Induction | "All Previous" Induction |
|---|---|
| | Define $T_i = \forall j \leq i, S_j$ |
| Establish "Base Case": $S_0$ | Establish "Base Case": $T_0$ |
| Establish that $\forall k, S_k \Rightarrow S_{k+1}$ | Establish that $\forall k, T_k \Rightarrow T_{k+1}$ |
| Let k be any natural number. | Let k be any natural number. |
| Assume $\forall j<k, S_j$ | Assume $T_{k-1}$ |
| Prove $S_k$ | Prove $T_k$ |



Induction is also how we can define and construct our world.



So many things, from buildings to computers, are built up stage by stage, module by module, each depending on the previous stages.




## Inductive Definition Of Functions

Stage 0, Initial Condition, or Base Case:
Declare the value of the function on some subset of the domain.

Inductive Rules
Define new values of the function in terms of previously defined values of the function

F(x) is defined if and only if it is implied by finite iteration of the rules.

## Inductive Definition Of Functions

Stage 0, Initial Condition, or Base Case:
Declare the value of the function on some subset of the domain.

Inductive Rules
Define new values of the function in terms of previously defined values of the function

If there is an x such that F(x) has more than one value – then the whole inductive definition is said to be inconsistent.

## Inductive Definition Recurrence Relation for F(X)

Initial Condition, or Base Case:
F(0) = 1

Inductive Rule
For n>0, F(n) = F(n-1) + F(n-1)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| F(n) | 1 | | | | | | | |

## Inductive Definition Recurrence Relation for F(X)

Initial Condition, or Base Case:
F(0) = 1

Inductive Rule
For n>0, F(n) = F(n-1) + F(n-1)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| F(n) | 1 | 2 | | | | | | |

## Inductive Definition Recurrence Relation for F(X)

Initial Condition, or Base Case:
F(0) = 1

Inductive Rule
For n>0, F(n) = F(n-1) + F(n-1)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| F(n) | 1 | 2 | 4 | | | | | |

## Inductive Definition
## Recurrence Relation for F(X)

Initial Condition, or Base Case:
F(0) = 1

Inductive Rule
For n>0, F(n) = F(n-1) + F(n-1)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| F(n) | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 |

## Inductive Definition
## Recurrence Relation for $F(X) = 2^X$

Initial Condition, or Base Case:
F(0) = 1

Inductive Rule
For n>0, F(n) = F(n-1) + F(n-1)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| F(n) | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 |

## Inductive Definition
## Recurrence Relation

Initial Condition, or Base Case:
F(1) = 1

Inductive Rule
For n>1, F(n) = F(n/2) + F(n/2)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| F(n) | | 1 | | | | | | |

## Inductive Definition
## Recurrence Relation

Initial Condition, or Base Case:
F(1) = 1

Inductive Rule
For n>1, F(n) = F(n/2) + F(n/2)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| F(n) | | 1 | 2 | | | | | |

## Inductive Definition
## Recurrence Relation

Initial Condition, or Base Case:
F(1) = 1

Inductive Rule
For n>1, F(n) = F(n/2) + F(n/2)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| F(n) | | 1 | 2 | | 4 | | | |

## Inductive Definition
## Recurrence Relation

Initial Condition, or Base Case:
F(1) = 1

Inductive Rule
For n>1, F(n) = F(n/2) + F(n/2)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| F(n) | % | 1 | 2 | % | 4 | % | % | % |

## Inductive Definition
## Recurrence Relation
F(X) = X for X a whole power of 2.

Initial Condition, or Base Case:
F(1) = 1

Inductive Rule
For n>1, F(n) = F(n/2) + F(n/2)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| F(n) | % | 1 | 2 | % | 4 | % | % | % |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | | | | | | | | |
| 3 | | | | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | | |
| 1 | 1 | | | | | | | |
| 2 | 2 | | | | | | | |
| 3 | 3 | | | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | | | | | | |
| 1 | 1 | 2 | | | | | | |
| 2 | 2 | 3 | | | | | | |
| 3 | 3 | 4 | | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | | | | | |
| 1 | 1 | 2 | 3 | | | | | |
| 2 | 2 | 3 | 4 | | | | | |
| 3 | 3 | 4 | 5 | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

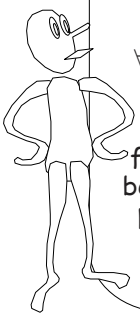
Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| X+Y | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |



Definition of P:

$\forall x \in \mathbb{N}$ P(X,0) = X
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

Any inductive definition with a finite number of base cases, can be translated into a program. The program simply calculates from the base cases on up.

Definition of P:

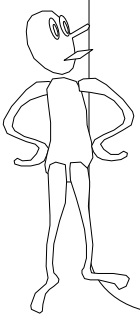
$\forall x \in \{0,1,2,3\}$ P(X,0) = X
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

What would be the bottom up implementation of P?

```
For k = 0 to 3
      P(k,0)=k
For j = 1 to 7
      For k = 0 to 3
            P(k,j) = P(k,j-1) + 1
```

Bottom-Up Program for P

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |



Suppose we wanted to know P(2,3) in particular, but we had not yet done any calculation.

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | | | | ? | | | | |
| 3 | | | | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | | | ? | ? | | | | |
| 3 | | | | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | | ? | ? | ? | | | | |
| 3 | | | | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | ? | ? | ? | ? | | | | |
| 3 | | | | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | 2 | ? | ? | ? | | | | |
| 3 | | | | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | 2 | 3 | ? | ? | | | | |
| 3 | | | | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | 2 | 3 | 4 | ? | | | | |
| 3 | | | | | | | | |

Base Case: $\forall x \in \mathbb{N}$ P(X,0) = X

Inductive Rule:
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | 2 | 3 | 4 | 5 | | | | |
| 3 | | | | | | | | |

Procedure P(x,y): Top Down
If y=0 return x
Otherwise return P(x,y-1)+1;

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | 2 | 3 | 4 | 5 | | | | |
| 3 | | | | | | | | |

Procedure P(x,y): Recursive Programming
If y=0 return x
Otherwise return P(x,y-1)+1;

| P(x,y) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | 2 | 3 | 4 | 5 | | | | |
| 3 | | | | | | | | |

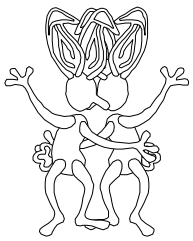
Inductive Definition:
$\forall x \in \mathbb{N}$ P(X,0) = X
$\forall x,y \in \mathbb{N}$, y>0, P(x,y) = P(x,y-1) + 1

Bottom-Up, Iterative Program:
For k = 0 to 3
P(k,0)=k
For j = 1 to 7
For k = 0 to 3
P(k,j) = P(k,j-1) + 1

Top-Down, Recursive Program:
Procedure P(x,y):
If y=0 return x
Otherwise return P(x,y-1)+1;

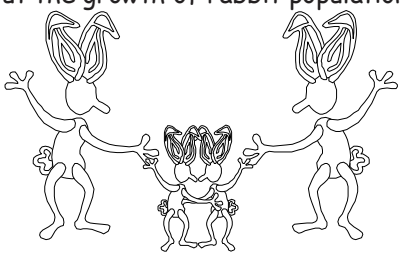
## Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.



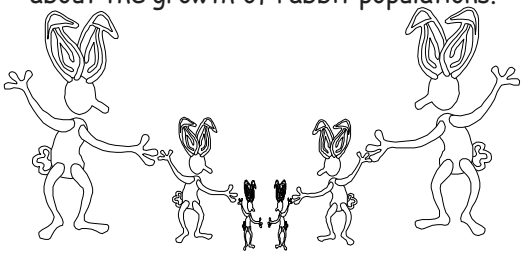
## Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.



## Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.



## The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

$F_n$= # of rabbit pairs at the beginning of the $n^{th}$ month

| month | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| rabbits | | | | | | | |

## The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

$F_n$= # of rabbit pairs at the beginning of the $n^{th}$ month

| month | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| rabbits | 1 | | | | | | |

## The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

$F_n$= # of rabbit pairs at the beginning of the $n^{th}$ month

| month | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| rabbits | 1 | 1 | | | | | |

## The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

$F_n$= # of rabbit pairs at the beginning of the $n^{th}$ month

| month | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| rabbits | 1 | 1 | 2 | | | | |

## The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

$F_n$= # of rabbit pairs at the beginning of the $n^{th}$ month

| month | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| rabbits | 1 | 1 | 2 | 3 | | | |

## The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

$F_n$= # of rabbit pairs at the beginning of the $n^{th}$ month

| month | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| rabbits | 1 | 1 | 2 | 3 | 5 | | |

## The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

$F_n$= # of rabbit pairs at the beginning of the $n^{th}$ month

| month | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| rabbits | 1 | 1 | 2 | 3 | 5 | 8 | 13 |

## Inductive Definition or Recurrence Relation for the Fibonacci Numbers

Stage 0, Initial Condition, or Base Case:
Fib(1) = 1; Fib (2) = 1

Inductive Rule
For n>3, Fib(n) = Fib(n-1) + Fib(n-2)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Fib(n) | % | 1 | 1 | 2 | 3 | 5 | 8 | 13 |

## Inductive Definition or Recurrence Relation for the Fibonacci Numbers

Stage 0, Initial Condition, or Base Case:
Fib(0) = 0; Fib (1) = 1

Inductive Rule
For n>1, Fib(n) = Fib(n-1) + Fib(n-2)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Fib(n) | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 |

Inductive Definition:
Fib(0)=0, Fib(1)=1, k>1, Fib(k)=Fib(k-1)+Fib(k-2)

Bottom-Up, Iterative Program:
Fib(0) = 0; Fib(1) =1;
Input x;
For k= 2 to x do Fib(x)=Fiib(x-1)+Fib(x-2);
Return Fib(k);

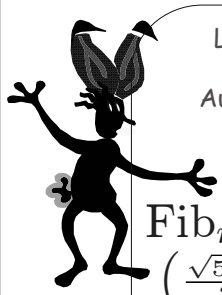
Top-Down, Recursive Program:
Procedure Fib(k)
If k=0 return 0
If k=1 return 1
Otherwise return Fib(k-1)+Fib(k-2);

## What is a closed form formula for Fib(n) ????

Stage 0, Initial Condition, or Base Case:
Fib(0) = 0; Fib (1) = 1

Inductive Rule
For n>1, Fib(n) = Fib(n-1) + Fib(n-2)

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Fib(n) | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 |



Leonhard Euler (1765)
J. P. M. Binet (1843)
August de Moivre (1730)

$$\mathrm{Fib}_n = \frac{\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}+1}{2}\right)^{-n}}{\sqrt{5}}$$

Study Bee

Inductive Proof
- Standard Form
- All Previous Form
- Least-Counter Example Form
- Invariant Form

Inductive Definition
- Bottom-Up Programming
- Top-Down Programming
- Recurrence Relations
- Solving a Recurrence