Great Theoretical Ideas In Computer Science

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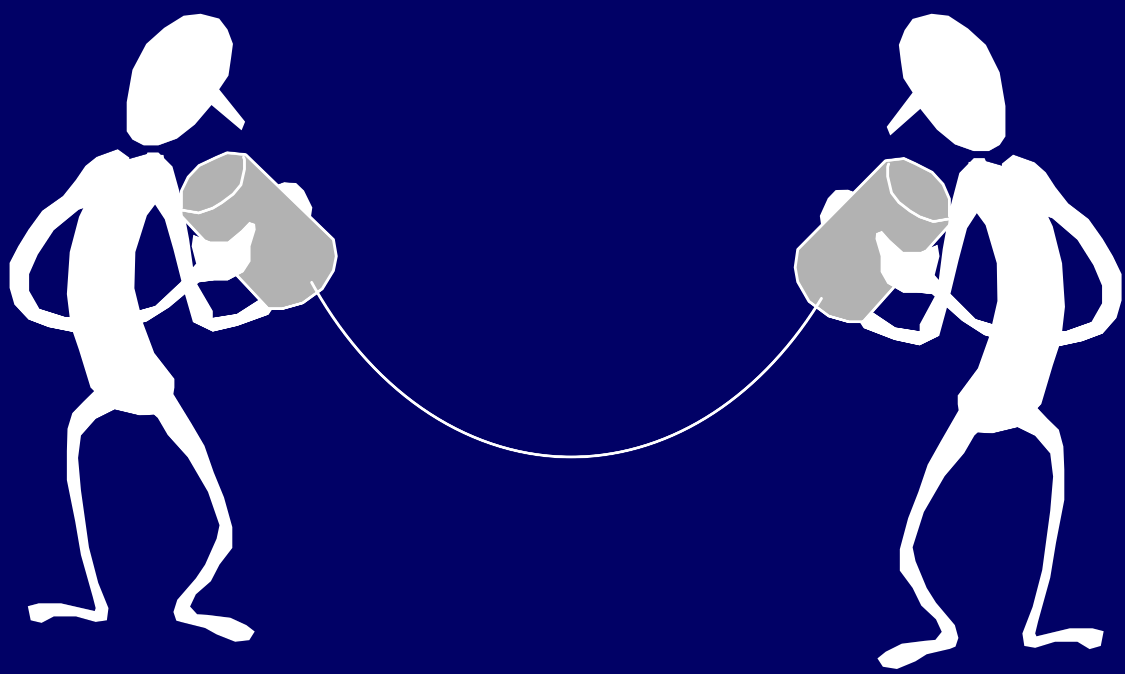
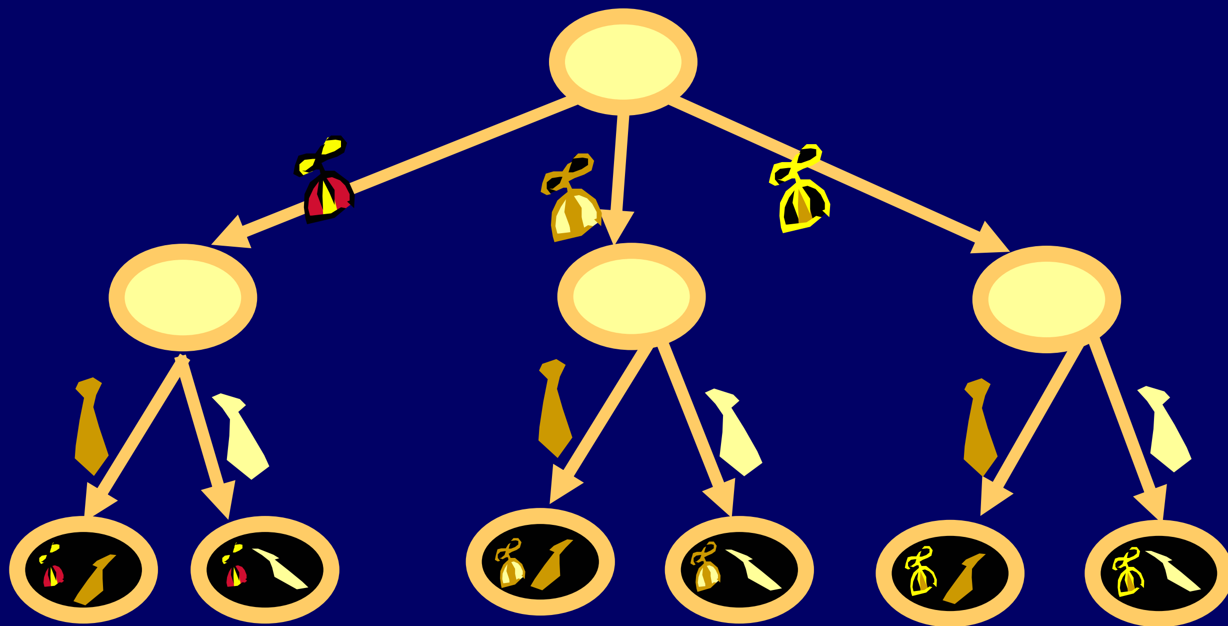
Lecture 20 March 25, 2004 Carnegie Mellon University

Decision Trees and Information: A Question of Bits

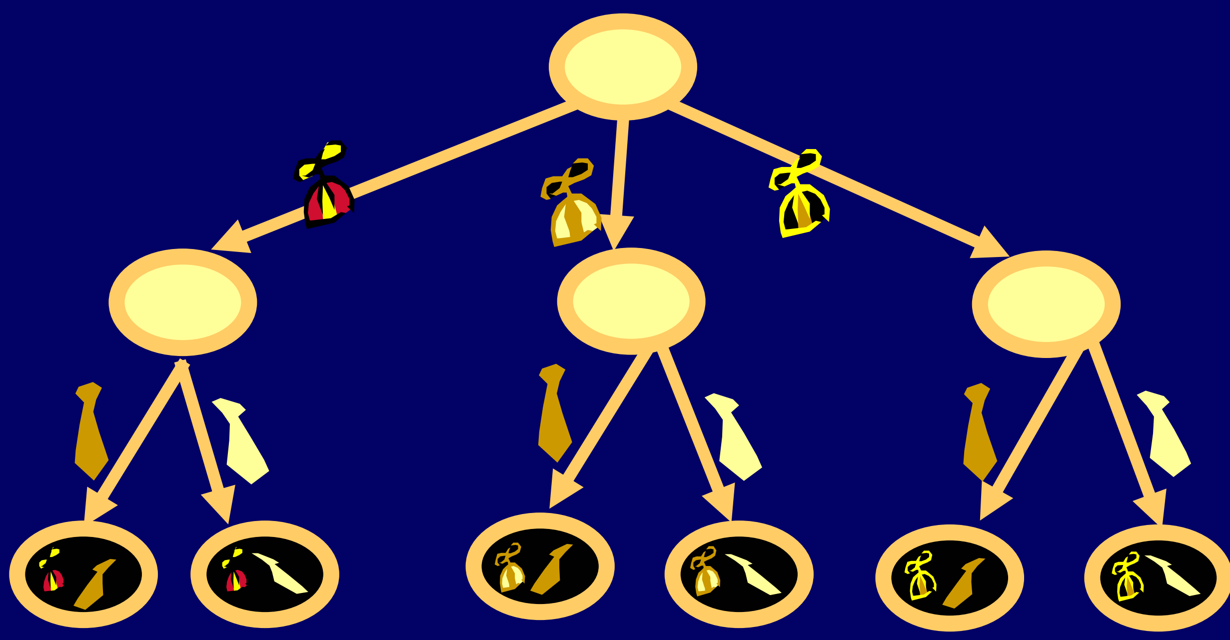


Choice Tree



A <u>choice tree</u> is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf.

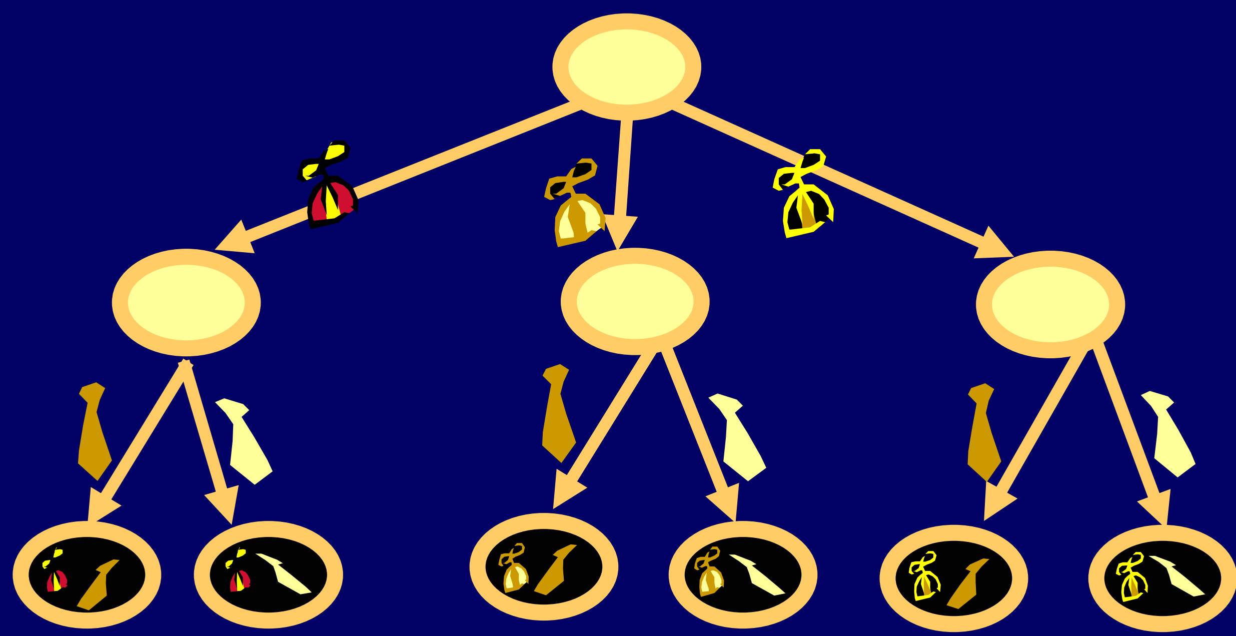
Choice Tree Representation of S



We satisfy these two conditions:

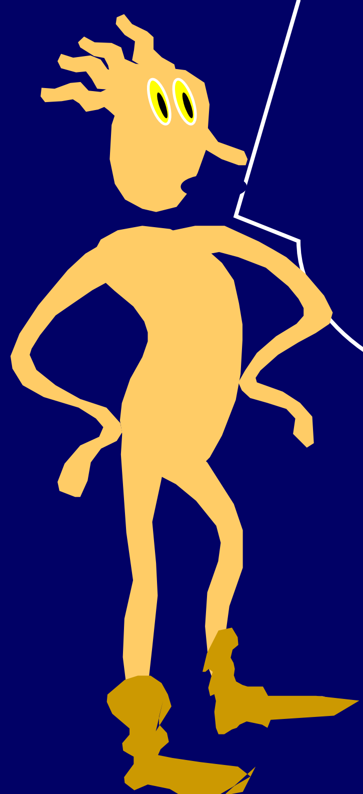
- Each leaf label is in S
- Each element from S on exactly one leaf.

Question Tree Representation of S



I am thinking of an outfit.
Ask me questions until you know which one.

What color is the beanie?
What color is the tie?



When a question tree has
at most 2 choices at each node,
we will call it a decision tree,
or a decision strategy.

Note: Nodes with one choices
represent stupid questions, but
we allow stupid questions.

20 Questions

S = set of all English nouns

Game:

I am thinking of an element of S.

You may ask up to 20 YES/NO questions.

What is a question strategy for this game?

20 Questions

Suppose $S = \{a_0, a_1, a_2, \ldots, a_k\}$

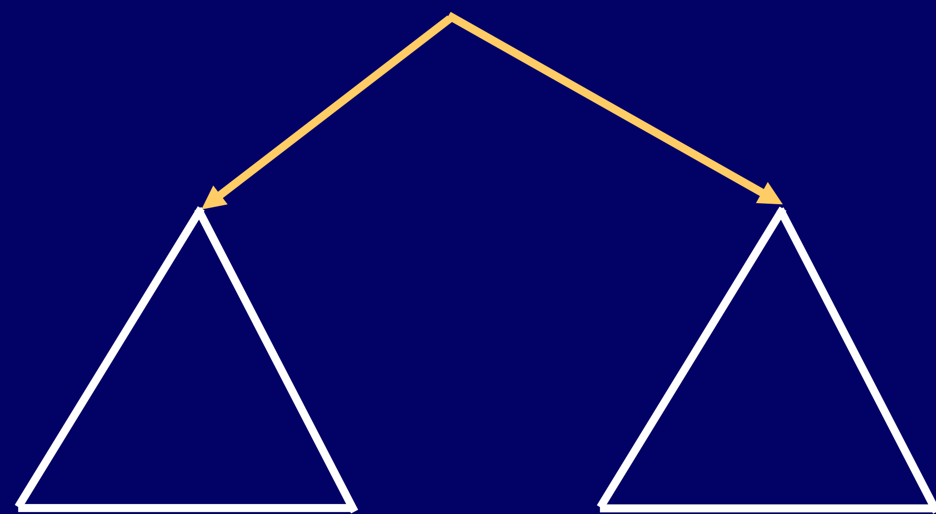
Binary search on S.

First question will be:

“Is the word in $\{a_0, a_1, a_2, \ldots, a_{(k-1)/2}\}$?”

20 Questions
Decision Tree Representation

A decision tree with depth at most 20, which has the elements of S on the leaves.



Decision tree for $\{a_0, a_1, a_2, \ldots, a_{(k-1)/2}\}$

Decision tree for $\{a_{(k+1)/2}, \ldots, a_{k-1}, a_k\}$

Decision Tree Representation

Theorem:

The binary-search decision tree for S with $k+1$ elements $\{ a_0, a_1, a_2, \ldots, a_k \}$ has depth

$$\lceil \log (k+1) \rceil$$

$$= \lfloor \log k \rfloor + 1$$

$$= |k|$$

Another way to look at it

If you are thinking of the noun a_m in S
We are asking about each bit of index m
Is the leftmost bit of m 0?
Is the next bit of m 0?

…

Theorem: The binary-search decision-tree for $S = \{ a_0, a_1, a_2, \ldots, a_k \}$ has depth

$$|k| = \lfloor \log k \rfloor + 1$$

A lower bound

Theorem: No decision tree for S can have depth $d < \lfloor \log k \rfloor + 1$.

Proof:

Any depth d tree can have at most 2^d leaves.

But $d < \lfloor \log k \rfloor + 1 \Rightarrow$ number of leaves $2^d < (k+1)$

Hence some element of S is not a leaf.

Tight bounds!

The optimal-depth decision tree
for any set S with (k+1) elements has depth

$$\lfloor \log k \rfloor + 1 = |k|$$

Prefix-free Set

Let T be a subset of $\{0,1\}^*$.

Definition:

T is prefix-free if for any distinct $x,y \in T$,

if $|x| < |y|$, then x is not a prefix of y

Example:

{000, 001, 1, 01} is prefix-free

{0, 01, 10, 11, 101} is not.

Prefix-free Code for S

Let S be any set.

Definition: A prefix-free code for S is a prefix-free set T and a 1-1 "encoding" function f: S -> T.

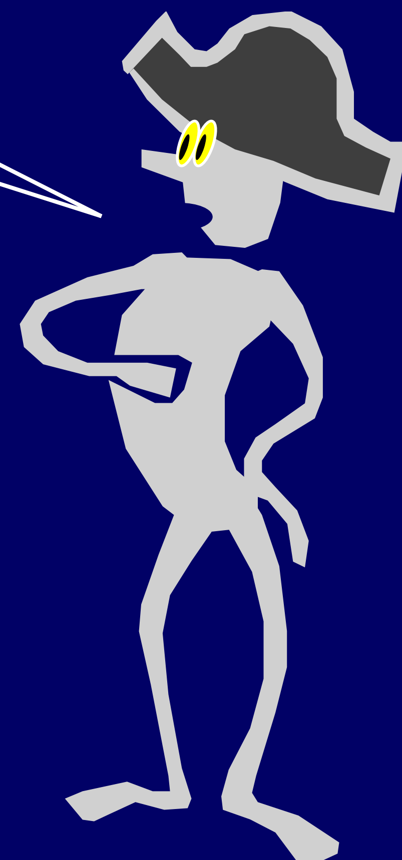
The inverse function f^{-1} is called the "decoding function".

Example: S = {apple, orange, mango}.

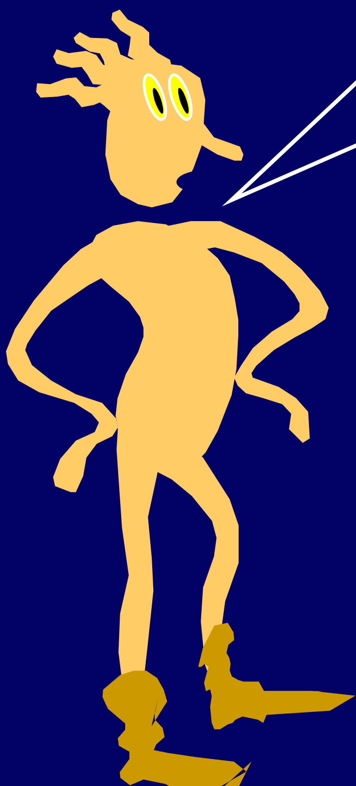
T = {0, 110, 1111}.

f(apple) = 0, f(orange) = 1111, f(mango) = 110.

What is so cool about prefix-free codes?



Sending sequences of elements of S over a communications channel



Let T be prefix-free and f be an encoding function. Wish to send $\langle x_1, x_2, x_3, \ldots \rangle$

Sender: sends $f(x_1)\, f(x_2)\, f(x_3)\ldots$

Receiver: breaks bit stream into elements of T and decodes using f^{-1}

Sending info on a channel

Example: S = {apple, orange, mango}.
T = {0, 110, 1111}.
f(apple) = 0, f(orange) = 1111, f(mango) = 110.

If we see

00011011111100…

we know it must be

0 0 0 110 1111 110 0 …

and hence

apple apple apple mango orange mango apple …

Morse Code is not Prefix-free!

SOS encodes as …---…

A .-	F ..-.	K -.-	P .--.	U ..-	Z --..
B -...	G --.	L .-..	Q --.-	V ...-	
C -.-.	H	M --	R .-.	W .--	
D -..	I ..	N -.	S ...	X -..-	
E .	J .---	O ---	T -	Y -.--	

Morse Code is not Prefix-free!

SOS encodes as ...---...

Could decode as: IAMIE

A .-	F ..-.	K -.-	P .--.	U ..-	Z --..
B -...	G --.	L .-..	Q --.-	V ...-	
C -.-.	H	M --	R .-.	W .--	
D -..	I ..	N -.	S ...	X -..-	
E .	J .---	O ---	T -	Y -.--	

Unless you use pauses

SOS encodes as ... --- ...

A .-	F ..-.	K -.-	P .--.	U ..-	Z --..
B -...	G --.	L .-..	Q --.-	V ...-	
C -.-.	H	M --	R .-.	W .--	
D -..	I ..	N -.	S ...	X -..-	
E .	J .---	O ---	T -	Y -.--	


Prefix-free codes are also called “self-delimiting” codes.

Representing prefix-free codes

A = 100

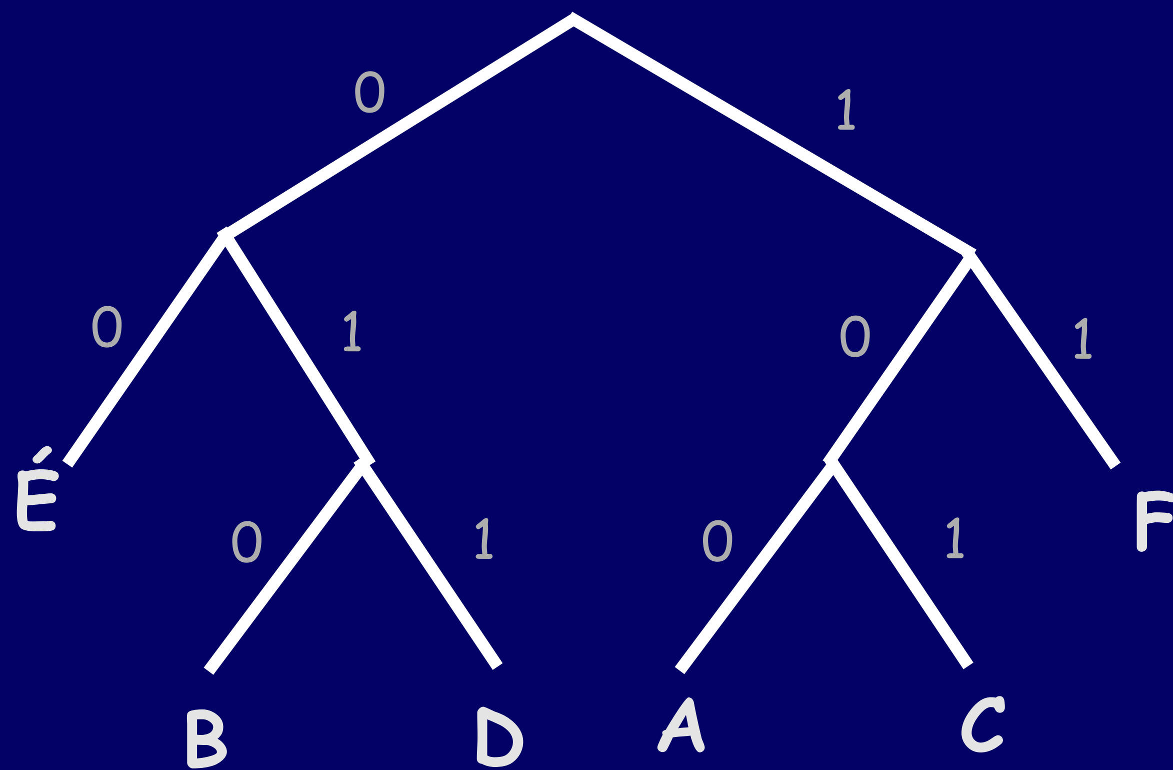
B = 010

C = 101

D = 011

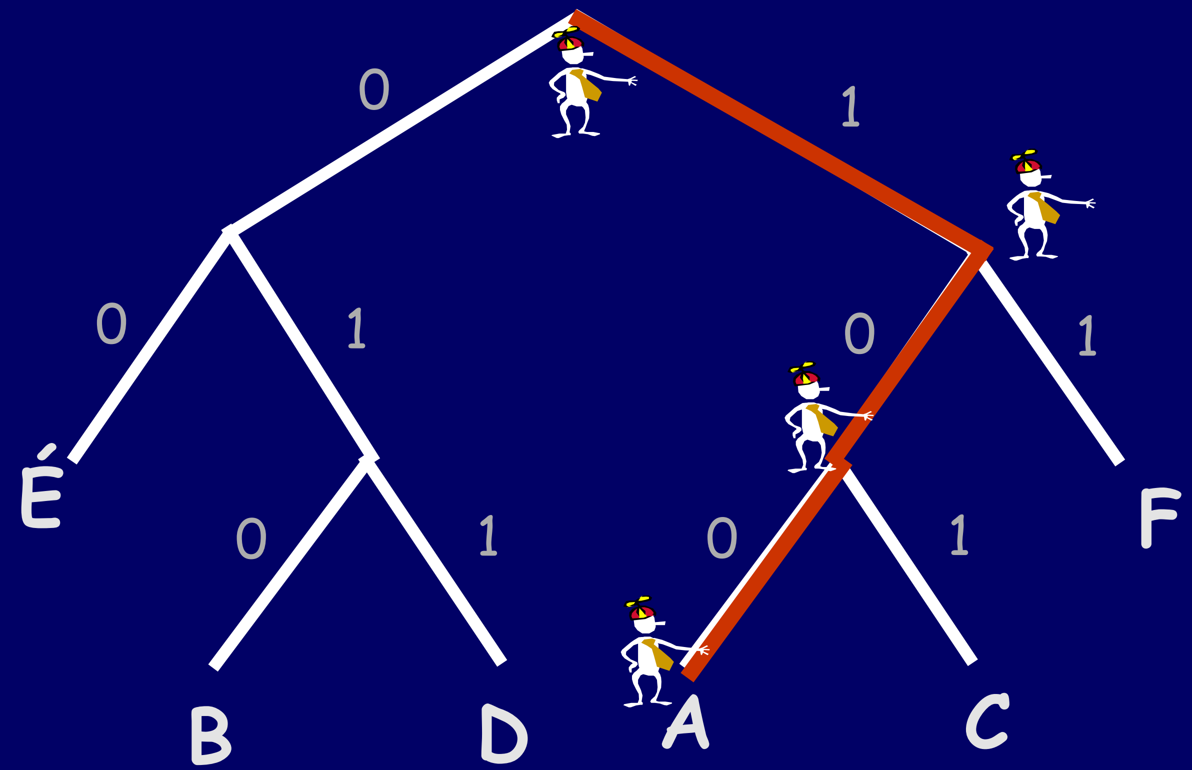
É = 00

F = 11



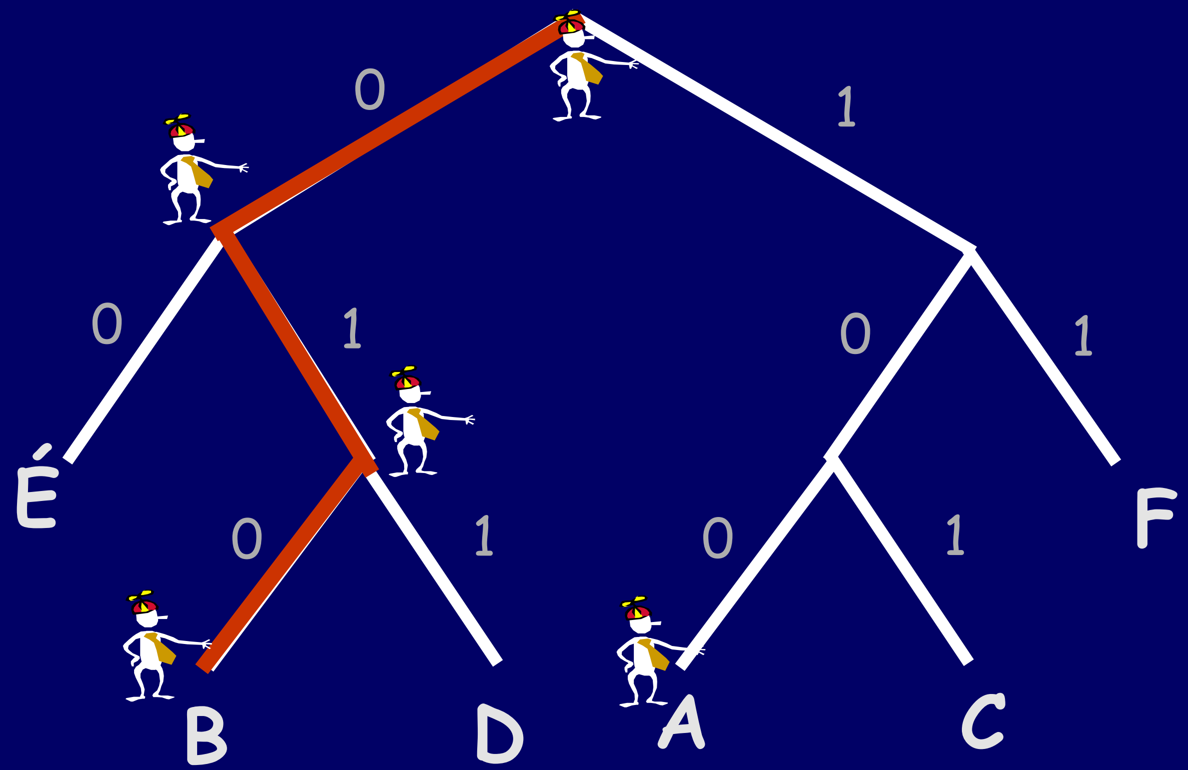
"CAFÉ" would encode as 1011001100

How do we decode 1011001100 (fast)?



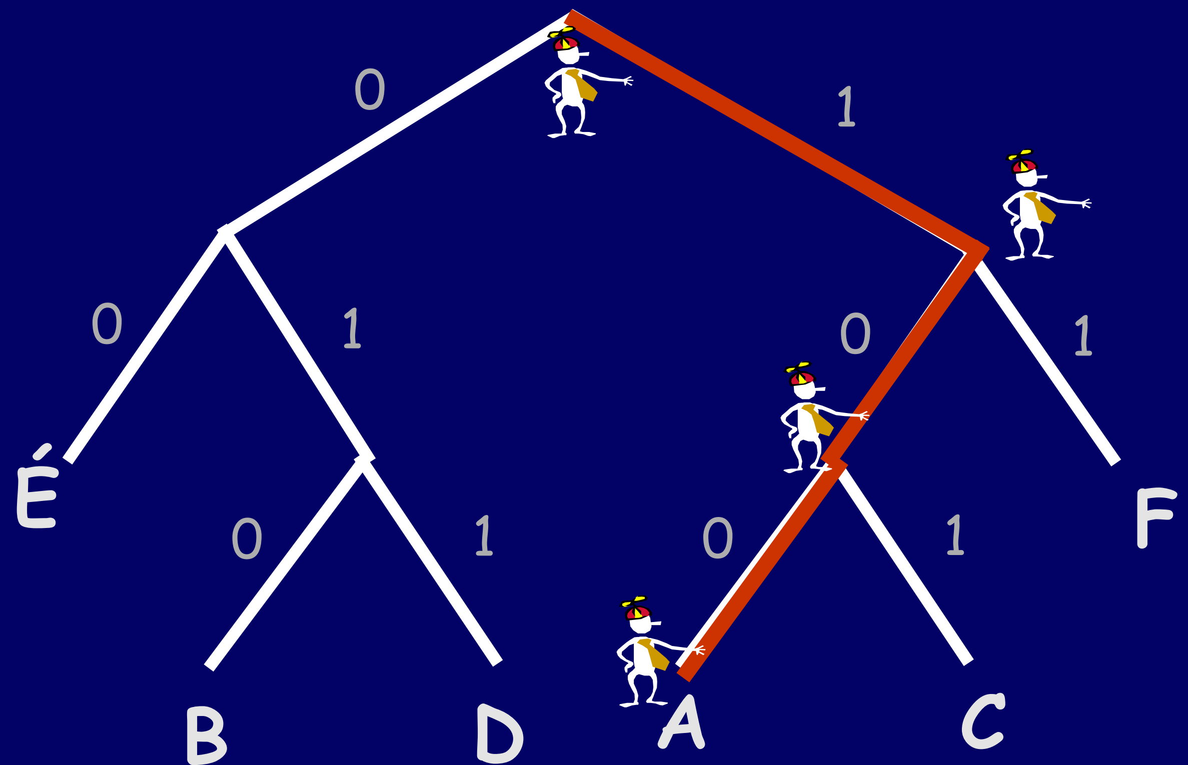
If you see: 1000101000111011001100

can decode as:



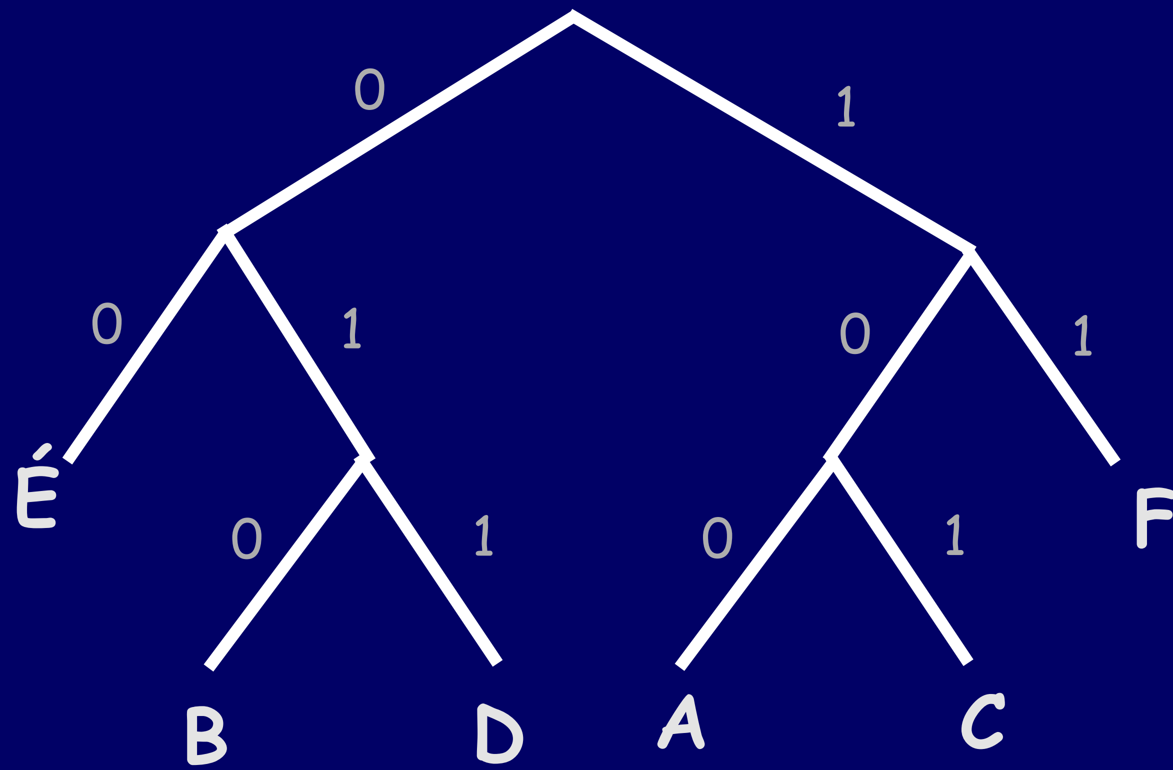
If you see: 1000101000111011001100

can decode as: A



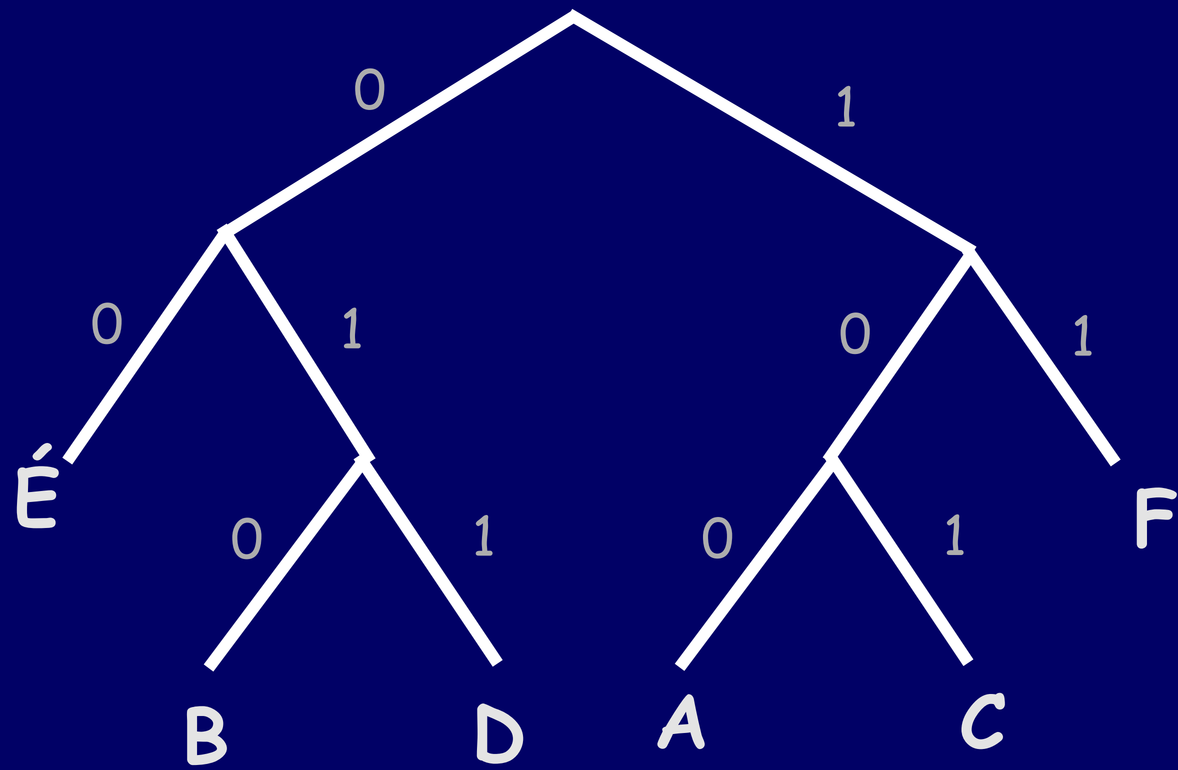
If you see: 100010100011101100110 0

can decode as: AB



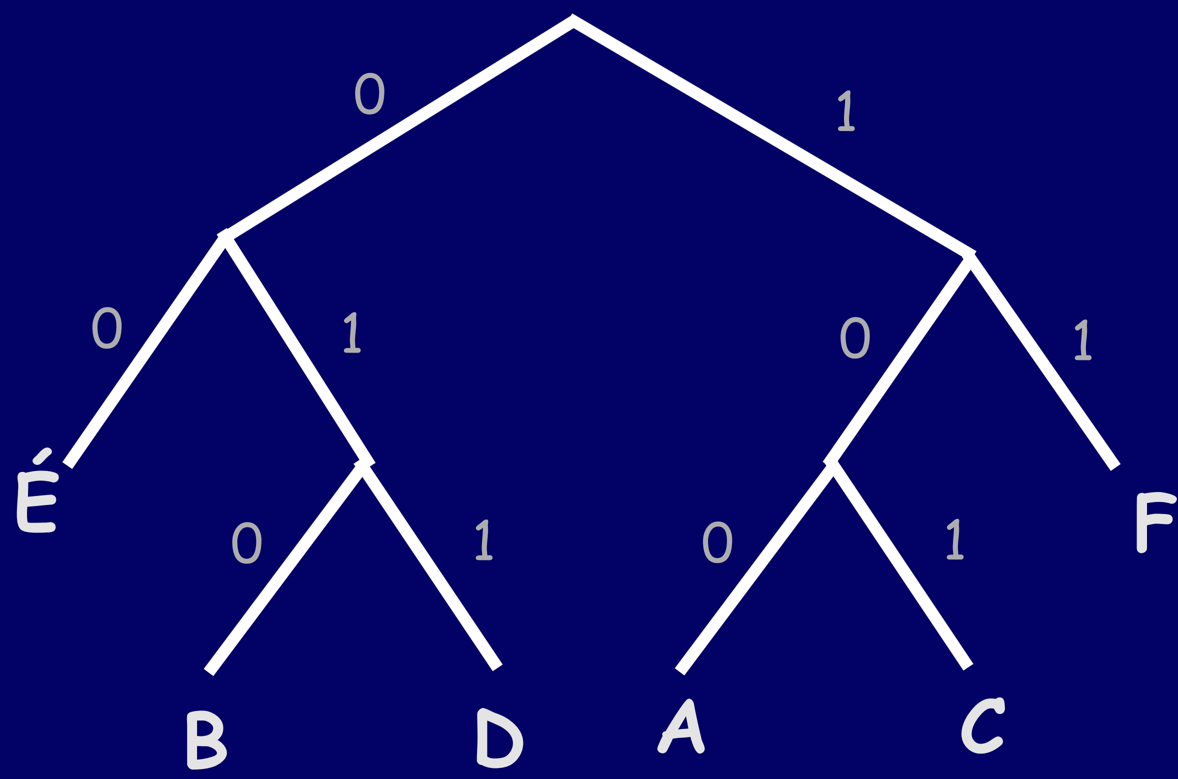
If you see: 1000101000111011001100

can decode as: ABA



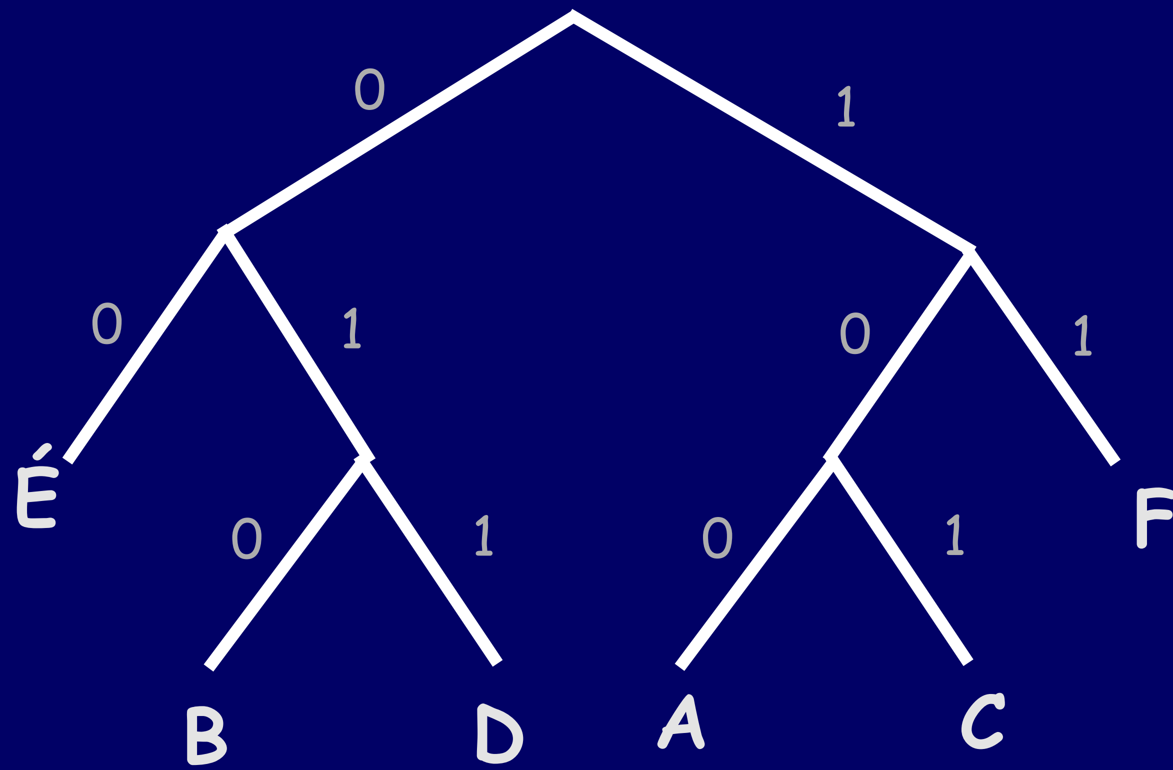
If you see: 1000101000111011001100

can decode as: ABAD



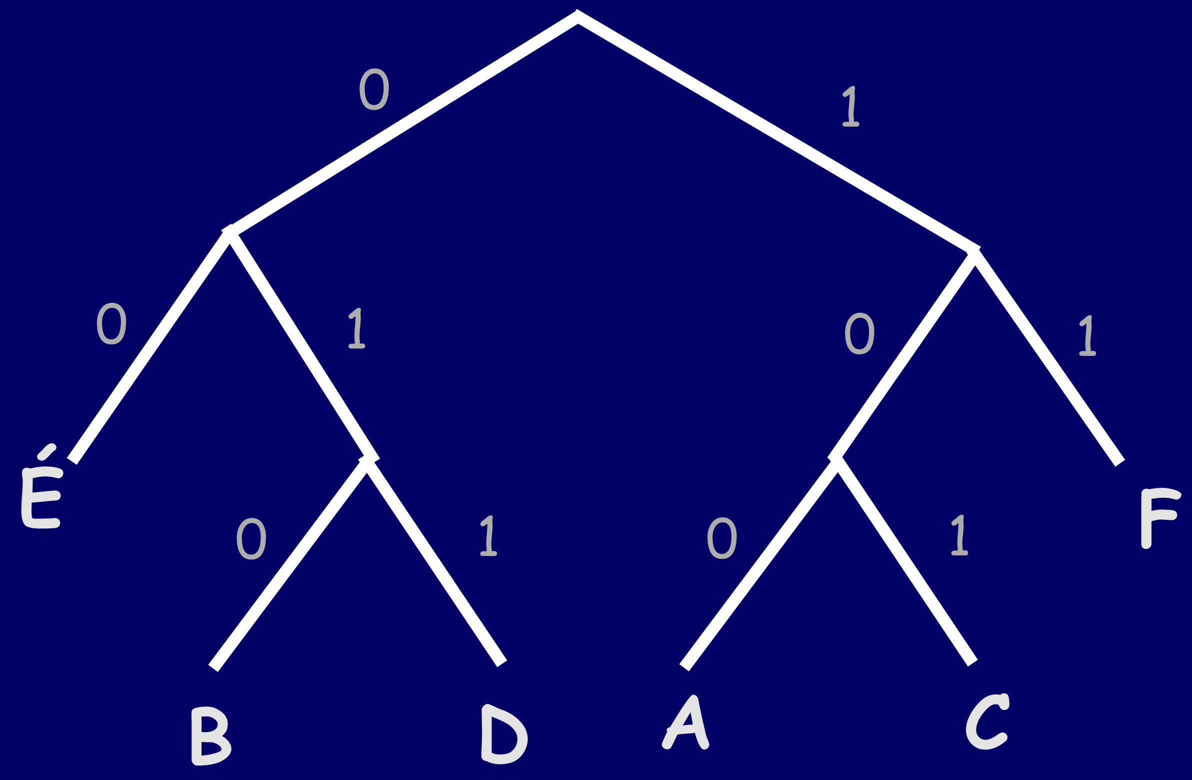
If you see: 1000101000111011001100

can decode as: ABADC



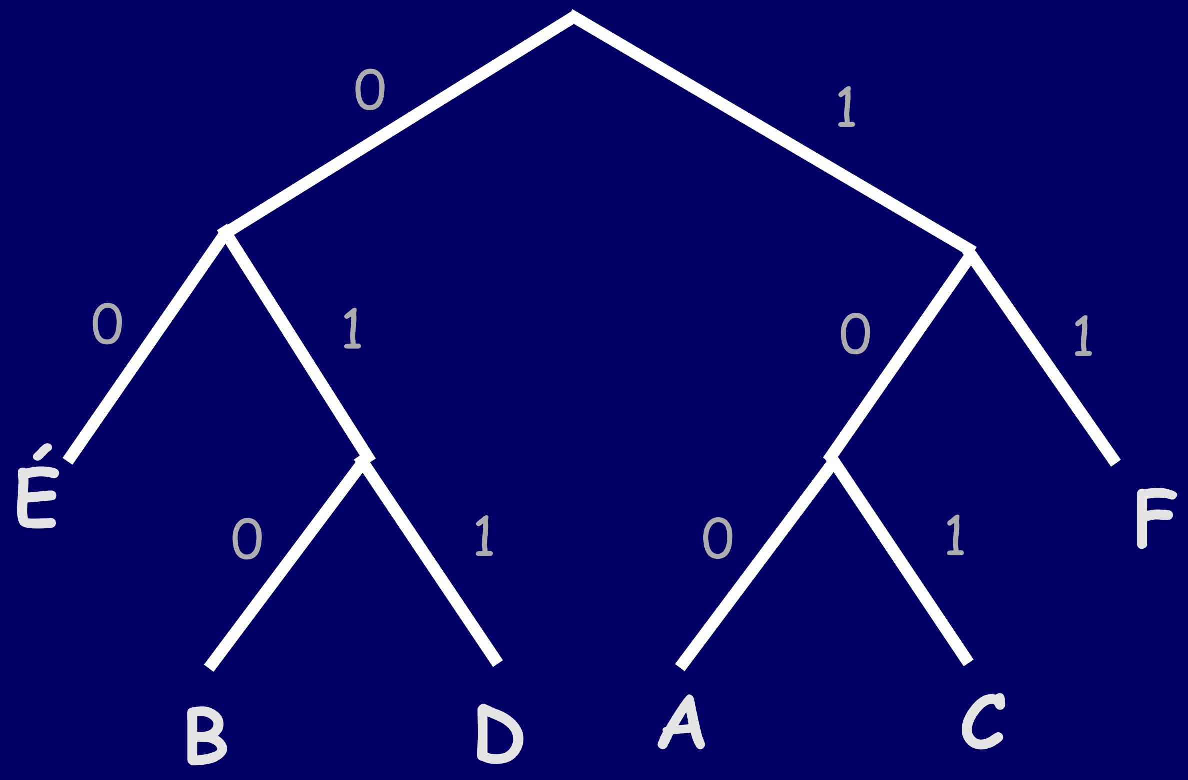
If you see: 1000101000111011001100

can decode as: ABADCA



If you see: 1000101000111011001100

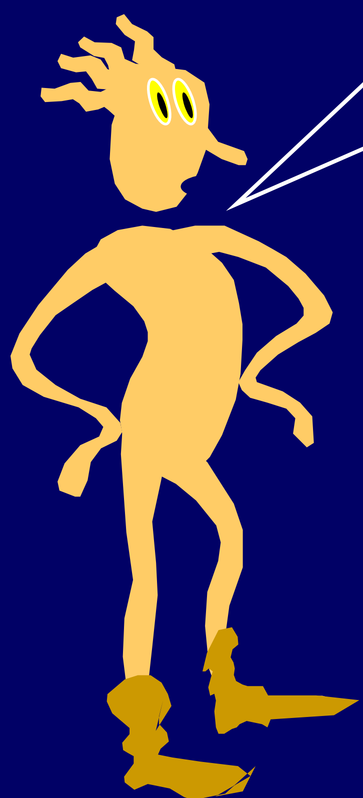
can decode as: ABADCAF



If you see: 1000101000111011001100

can decode as: ABADCAFÉ

Prefix-free codes are yet another representation of a decision tree.

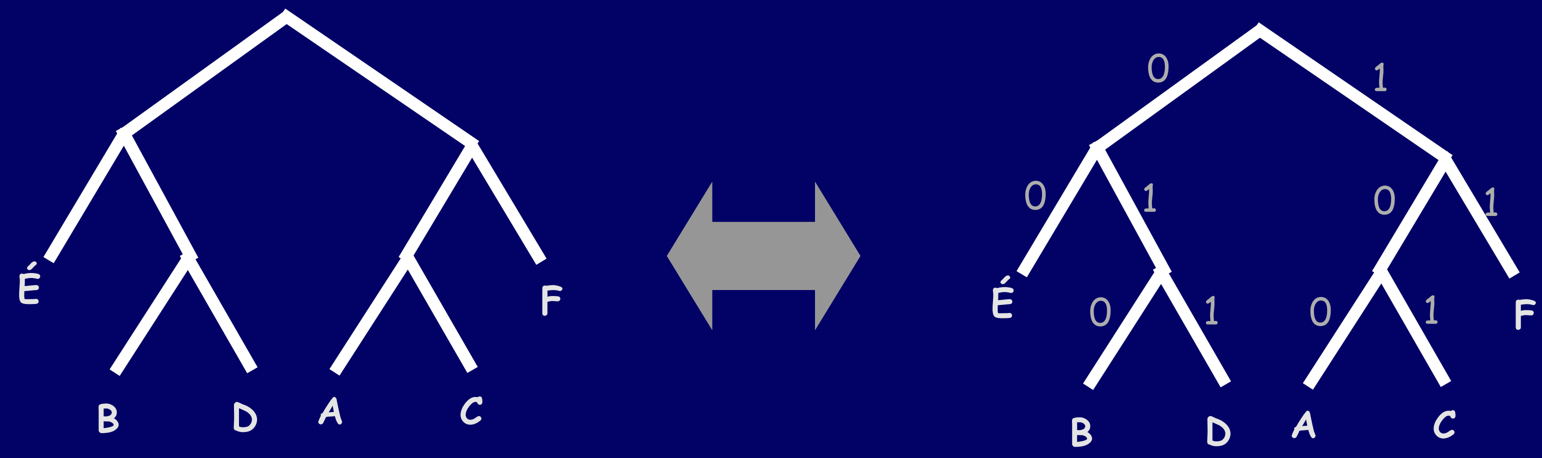


Theorem:

S has a decision tree of depth d

if and only if

S has a prefix-free code with all codewords bounded by length d



Theorem:

S has a decision tree of depth d

if and only if

S has a prefix-free code with all codewords bounded by length d

Extends to infinite sets

Let S is a subset of Σ^*

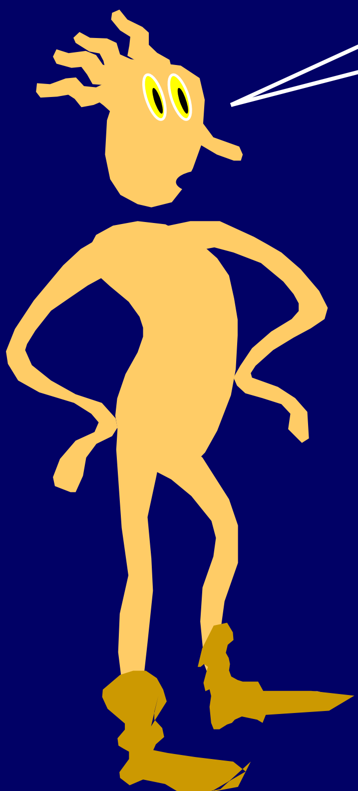
Theorem:

S has a decision tree where all length n elements of S have depth $\leq$ D(n)

if and only if

S has a prefix-free code where all length n strings in S have encodings of length $\leq$ D(n)

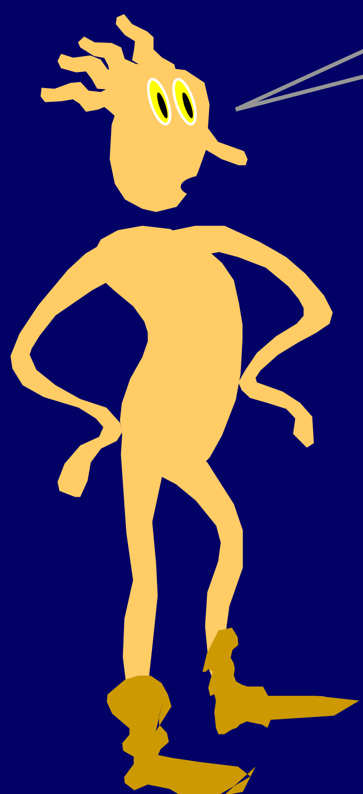
I am thinking of some natural number k -
ask me YES/NO questions in order to determine k.



Let d(k) be the number of questions that you ask when I am thinking of k.

Let D(n) = max { d(j) over n-bit numbers j}.

I am thinking of some natural number k - ask me YES/NO questions in order to determine k.



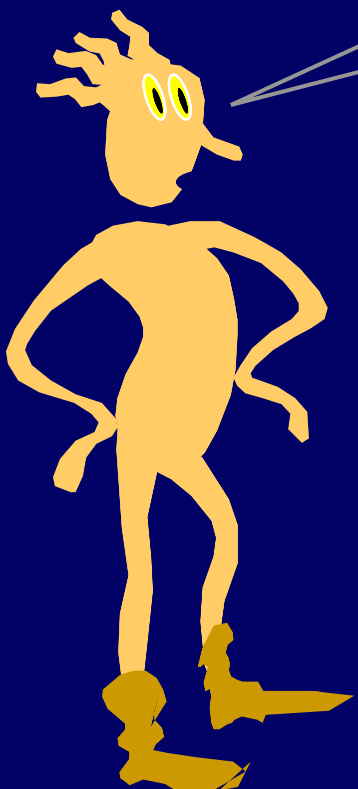
Lousy strategy: Is it 0? 1? 2? 3? ...

$d(k) = k+1$

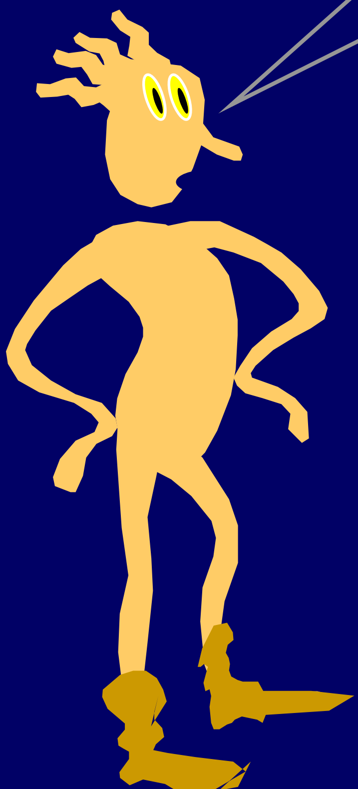
$D(n) = 2^{n+1}$ since $2^{n+1} -1$ uses only n bits.

Effort is exponential in length of k !!!

I am thinking of some natural number k -
ask me YES/NO questions in order to determine k.



What is an efficient question strategy?



Does k have length 1? NO

Does k have length 2? NO

Does k have length 3? NO

…

Does k have length n? YES

Do binary search on strings of length n.

$d(k) = |k| + |k|$

$= 2\,(\lfloor \log k \rfloor + 1)$

$D(n) = 2n$

Size First/ Binary Search

Does k have length 1? NO

Does k have length 2? NO

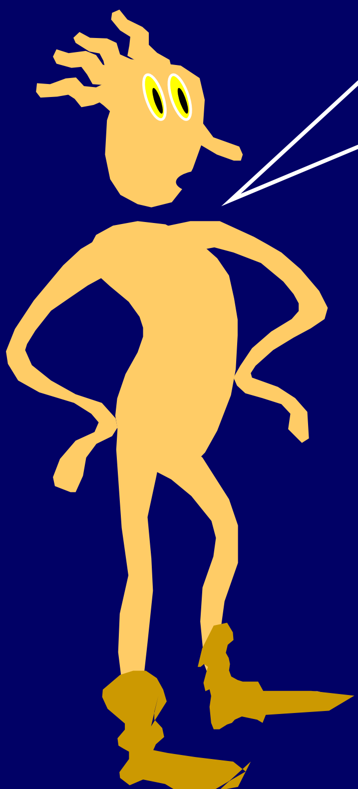
Does k have length 3? NO

...

Does k have length n? YES

Do binary search on strings of length n.

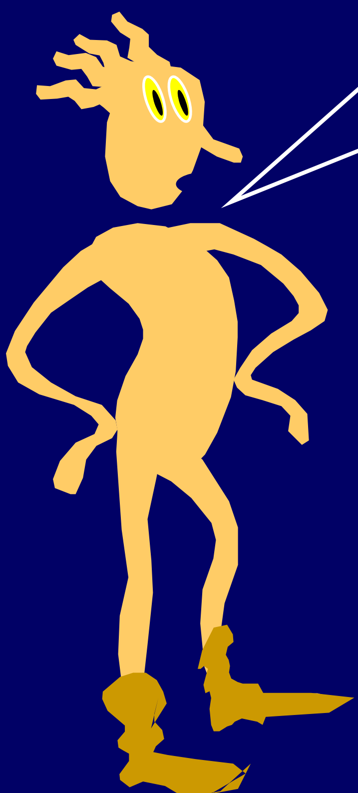
What prefix-free code corresponds to the Size First / Binary Search decision strategy?



f(k) = (|k| - 1) zeros, followed by 1, and then by the binary representation of k

|f(k)| = 2 |k|

What prefix-free code corresponds to the Size First / Binary Search decision strategy?



Or,

length of k in unary $\Rightarrow$ |k| bits

k in binary $\Rightarrow$ |k| bits

Another way to look at f

k = 27 = 11011, and hence |k| = 5

f(k) = 00001 11011

Another way to look at f

k = 27 = 11011, and hence |k| = 5

11011

f(k) = 00001 11011

0101000111

g(k) = 0101000111

Another way to look at the function *g*:

g(final 0) -> 10
g(final 1) -> 11

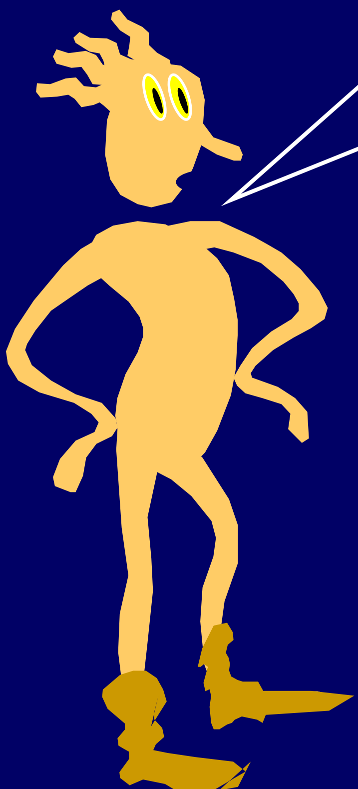
g(all other 0's) -> 00
g(all other 1's) -> 01

"Fat Binary" ⇔ Size First/Binary Search strategy

Is it possible to beat 2n questions to find a number of length n?

Look at the prefix-free code…

Any obvious improvement suggest itself here?



the fat-binary map f concatenates

length of k in ~~unary~~ $\Rightarrow$ |k| bits

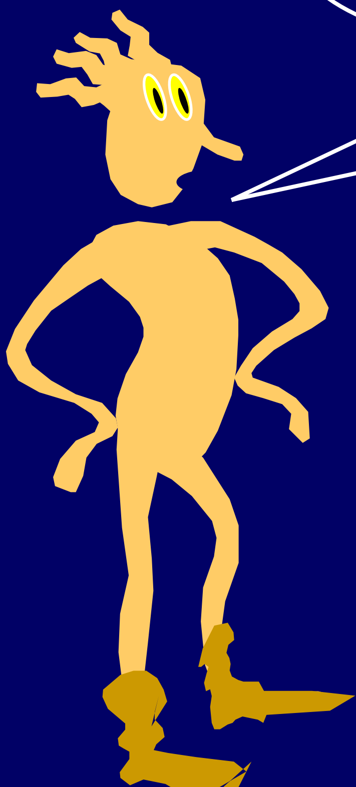
k in binary $\Rightarrow$ |k| bits

fat binary!

In fat-binary, $D(n) \le 2n$

Now $D(n) \le n + 2 (\lfloor \log n \rfloor + 1)$

Can you do better?



better-than-Fat-Binary-code(k)
    concatenates

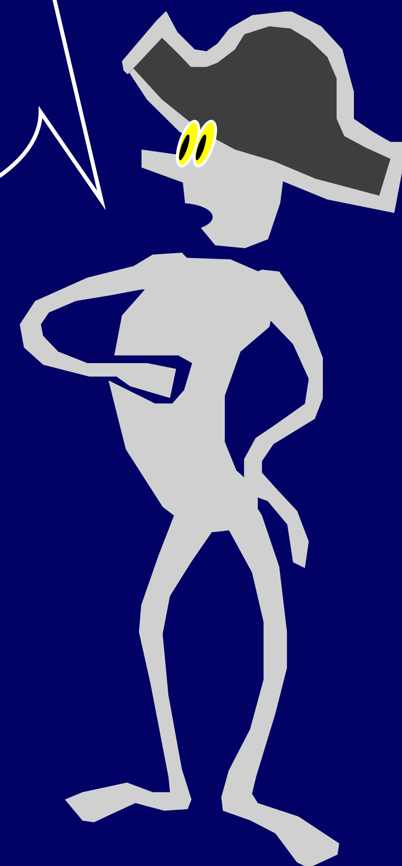
length of k in fat binary $\Rightarrow$ 2||k|| bits

k in binary $\Rightarrow$ |k| bits

Hey, wait!

In a better prefix-free code

RecursiveCode(k) concatenates
RecursiveCode(|k|) & k in binary

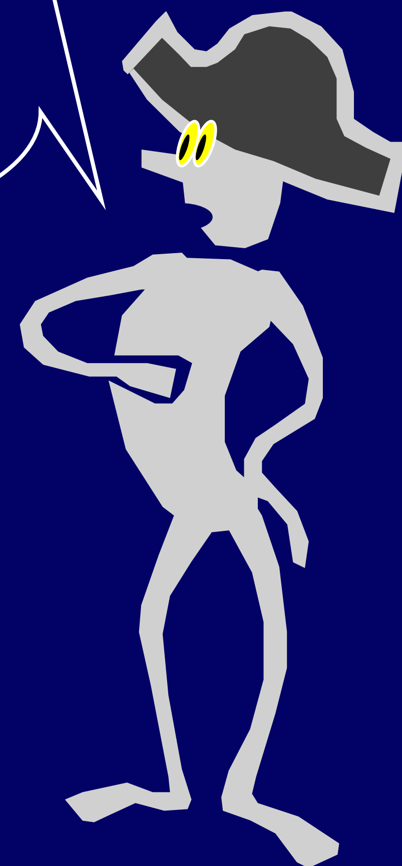


better-t-better-thanFB
~~better than Fat Binary~~ code

better-t-FB ||k|| + 2|||k|||
|k| in ~~fat binary~~ ⇒ ~~2||k|| bits~~
k in binary ⇒ |k| bits

Oh, I need to remember how many levels of recursion r(k)

In the final code
F(k) = F(r(k)) . RecursiveCode(k)

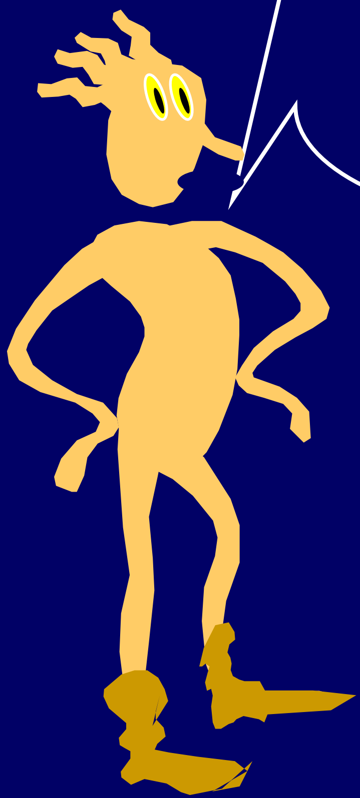


r(k) = log* k

Hence, length of F(k)
= |k| + ||k|| + |||k||| + … + 1
+ | log*k | + …

Good, Bonzo! I had thought you had fallen asleep.

Your code is sometimes called the Ladder code!!



Maybe I can do better…

Can I get a prefix code for k with length $\approx$ log k ?

No!

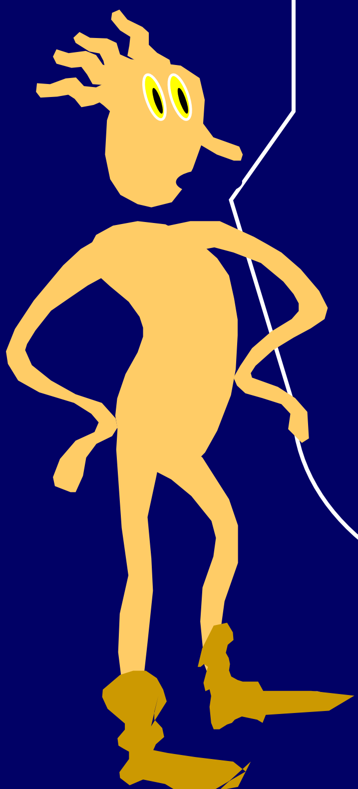
Let me tell you why
length ≈ log k
is not possible

Decision trees have a natural probabilistic interpretation.

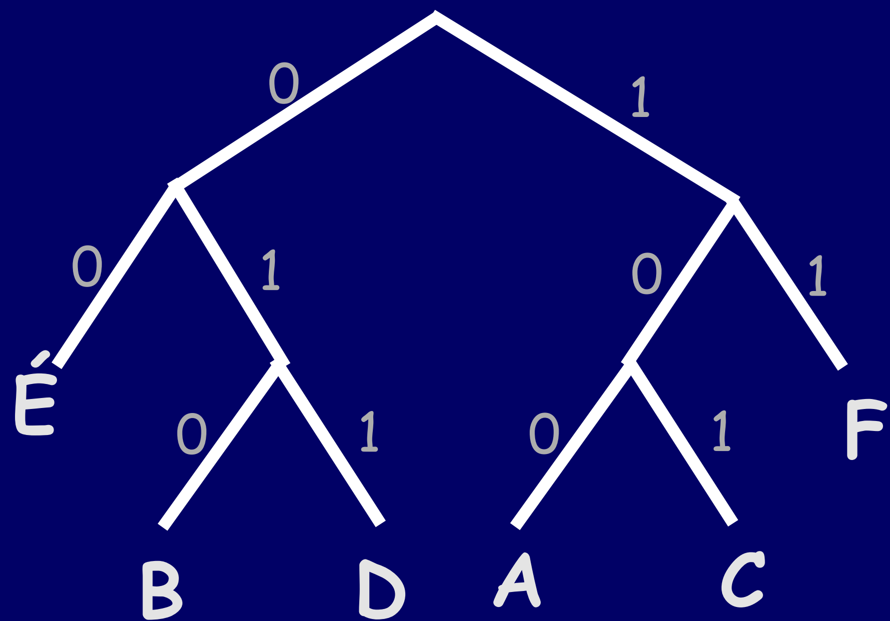
Let T be a decision tree for S.

Start at the root, flip a fair coin at each decision, and stop when you get to a leaf.

Each sequence w in S will be hit with probability $1/2^{|w|}$



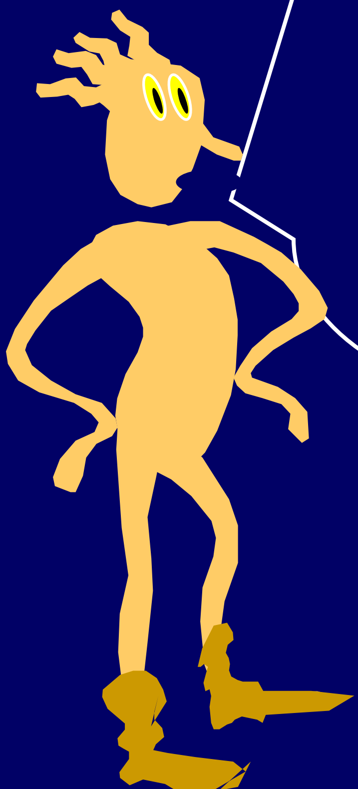
Random walk down the tree



Each sequence w in S will be hit with probability $1/2^{|w|}$

Hence, Pr(F) = ¼, Pr(A) = 1/8, Pr(C) = 1/8, …

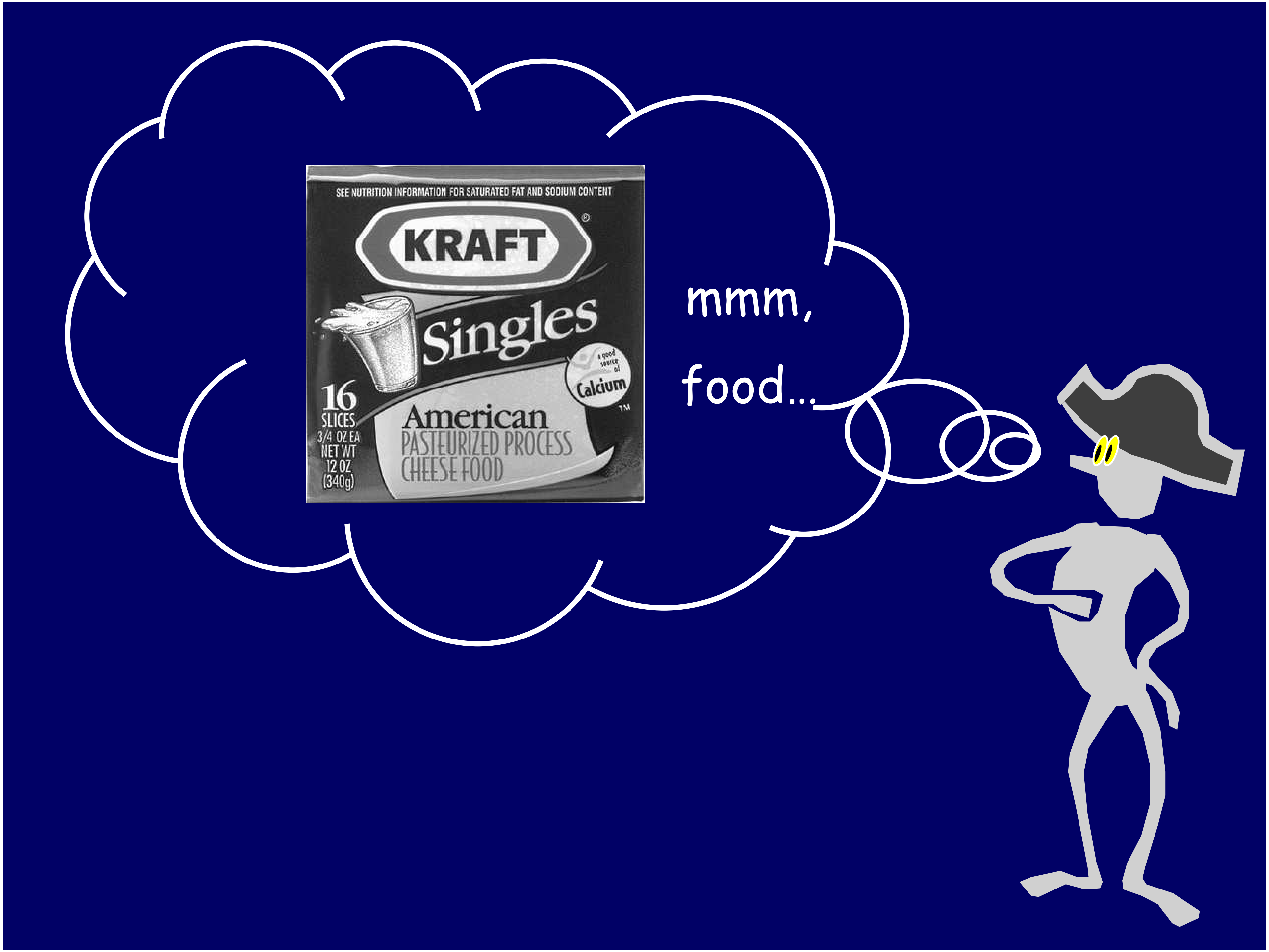
Let T be a decision tree for S (possibly countably infinite set)



The probability that some element in S is hit by a random walk down from the root is

$$\sum_{w\in S} 1/2^{|w|} \leq 1$$

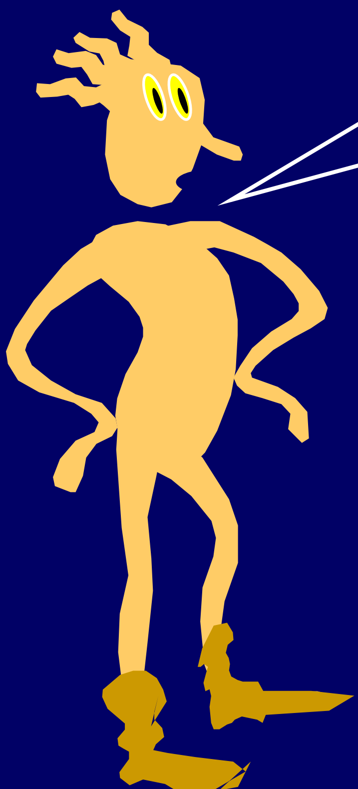
Kraft Inequality


SEE NUTRITION INFORMATION FOR SATURATED FAT AND SODIUM CONTENT
KRAFT
Singles
Calcium
16 SLICES 3/4 OZ EA NET WT 12 OZ (340g)
American
PASTEURIZED PROCESS CHEESE FOOD
mmm,
food...

Let S be any prefix-free code.

Kraft Inequality:

$$\sum_{w\in S} 1/2^{|w|} \le 1$$



Fat Binary:

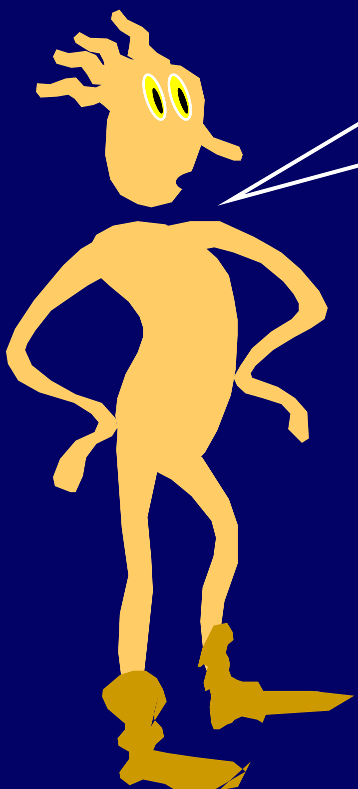
f(k) has $2|k| \approx 2 \log k$ bits

$$\sum_{k\in\mathbb{N}} 1/k^2 \le \sum_{k\in\mathbb{N}} \tfrac{1}{2}^{|f(k)|} \le 1$$

Let S be any prefix-free code.

Kraft Inequality:

$$\Sigma_{w \in S} 1/2^{|w|} \leq 1$$



Better Code:

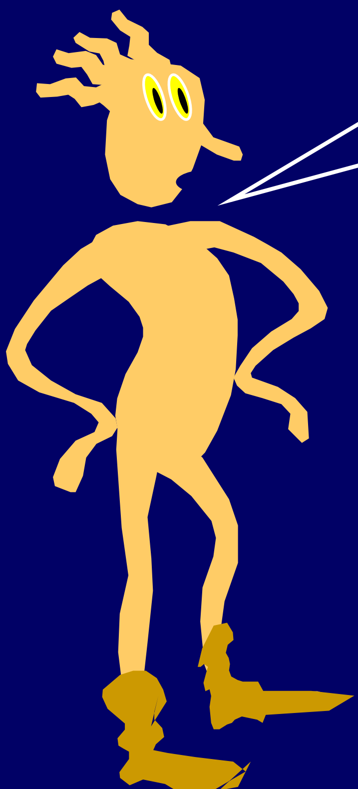
f(k) has $|k| + 2||k||$ bits

$$\Sigma_{k \in \mathbb{N}} 1/(k (\log k)^2) \leq 1$$

Let S be any prefix-free code.

Kraft Inequality:

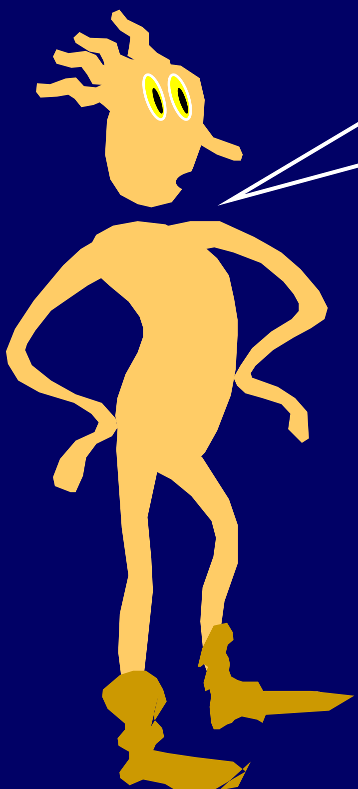
$$\sum_{w \in S} 1/2^{|w|} \leq 1$$



Ladder Code: k is represented by

$|k| + ||k|| + |||k||| + \ldots$ bits

$$\sum_{k \in \mathbb{N}} 1/(k \log k \log\log k \ldots) \leq 1$$



Let S be any prefix-free code.

Kraft Inequality:

$$\sum_{w\in S} 1/2^{|w|} \leq 1$$

Can a code that represents k by $|k| = \log k$ bits exist?

No, since $\sum_{k\in\mathbb{N}} 1/k$ diverges !!

So you can't get log n, Bonzo...


Mmm...

Back to compressing words

The optimal-depth decision tree
for any set S with (k+1) elements has depth
$\lfloor \log k \rfloor + 1$

$\Downarrow$

The optimal prefix-free code
for A-Z + "space" has length
$\lfloor \log 26 \rfloor + 1 = 5$

English Letter Frequencies

But in English, different letters occur with different *frequencies*.

A 8.1%	F 2.3%	K .79%	P 1.6%	U 2.8%	Z .04%
B 1.4%	G 2.1%	L 3.7%	Q .11%	V .86%	
C 2.3%	H 6.6%	M 2.6%	R 6.2%	W 2.4%	
D 4.7%	I 6.8%	N 7.1%	S 6.3%	X .11%	
E 12%	J .11%	O 7.7%	T 9.0%	Y 2.0%	

ETAONIHSRDLUMWCFGYPBVKQXJZ

short encodings!

Why should we try to minimize
the maximum length of a codeword?

If encoding A-Z, we will be happy if
the "<u>average codeword</u>" is short.

Morse Code

A .-	F ..-.	K -.-	P .--.	U ..-	Z --..
B -...	G --.	L .-..	Q --.-	V ...-	
C -.-.	H	M --	R .-.	W .--	
D -..	I ..	N -.	S ...	X -..-	
E .	J .---	O ---	T -	Y -.--	

ETAONIHSRDLUMWCFGYPBVKQXJZ

Given frequencies for A-Z, what is the optimal prefix-free encoding of the alphabet?

I.e., one that minimizes the <u>average</u> code length

Huffman Codes: Optimal Prefix-free Codes Relative to a Given Distribution

Here is a Huffman code based on the English letter frequencies given earlier:

A 1011	F 101001	K 10101000	P 111000	U 00100
B 111001	G 101000	L 11101	Q 1010100100	V 1010101
C 01010	H 1100	M 00101	R 0011	W 01011
D 0100	I 1111	N 1000	S 1101	X 1010100101
E 000	J 1010100110	O 1001	T 011	Y 101011
				Z 1010100111

But we are talking English!

Huffman coding uses only letter frequencies.

But we can use correlations!

E.g., Q is almost always followed by U…

Can we use this?

Yes…

Let us see how these correlations help

Random words

Randomly generated letters from A-Z, space not using the frequencies at all:

XFOML RXKHRJFFJUJ ALPWXFWJXYJ FFJEYVJCQSGHYD QPAAMKBZAACIBZLKJQD

Random words

Using only single character frequencies:

OCRO HLO RGWR NMIELWIS EU LL NBNESEBYA TH EEI ALHENHTTPA OOBTTVA NAH BRL

Random words

Each letter depends on the previous letter:

ON IE ANTSOUTINYS ARE T INCTORE ST BE S DEAMY ACHIN D ILONASIVE TUCOOWE AT TEASONARE FUSO TIZIN ANDY TOBE SEACE CTISBE

Random words

Each letter depends on 2 previous letters:

IN NO IST LAT WHEY CRATICT FROURE BIRS GROCID PONDENOME OF DEMONSTURES OF THE REPTAGIN IS REGOACTIONA OF CRE

Random words

Each letter depends on 3 previous letters:

THE GENERATED JOB PROVIDUAL BETTER TRAND THE DISPLAYED CODE, ABOVERY UPONDULTS WELL THE CODERST IN THESTICAL IT DO HOCK BOTHEMERG.

(INSTATES CONS ERATION. NEVER ANY OF PUBLE AND TO THEORY. EVENTIAL CALLEGAND TO ELAST BENERATED IN WITH PIES AS IS WITH THE)

References

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Elements of Information Theory, by T. Cover and J. Thomas