Great Theoretical Ideas In Computer Science

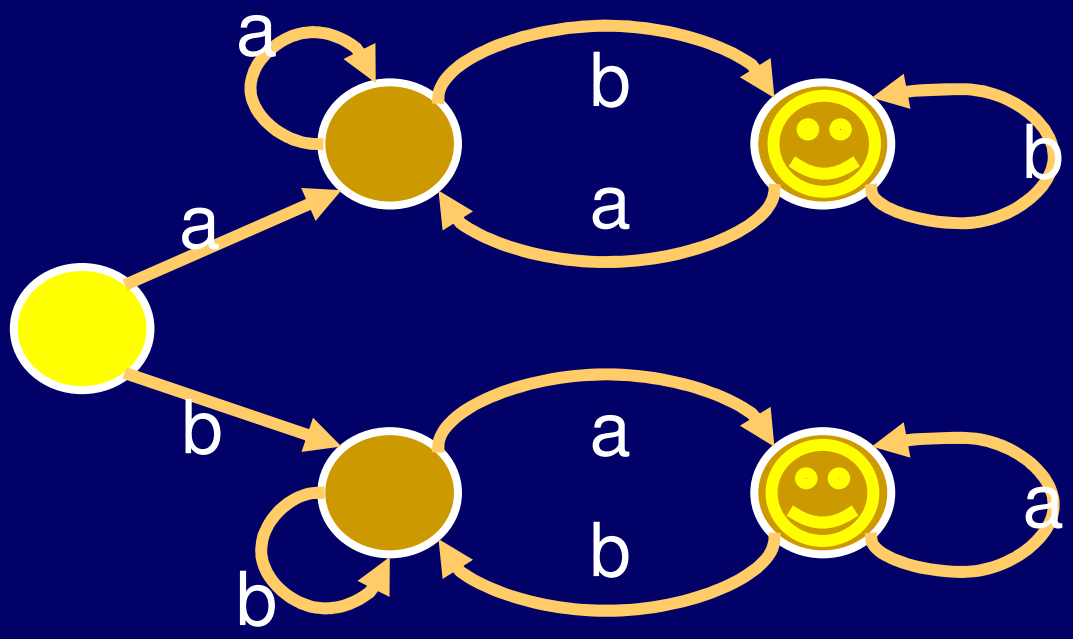
Steven Rudich

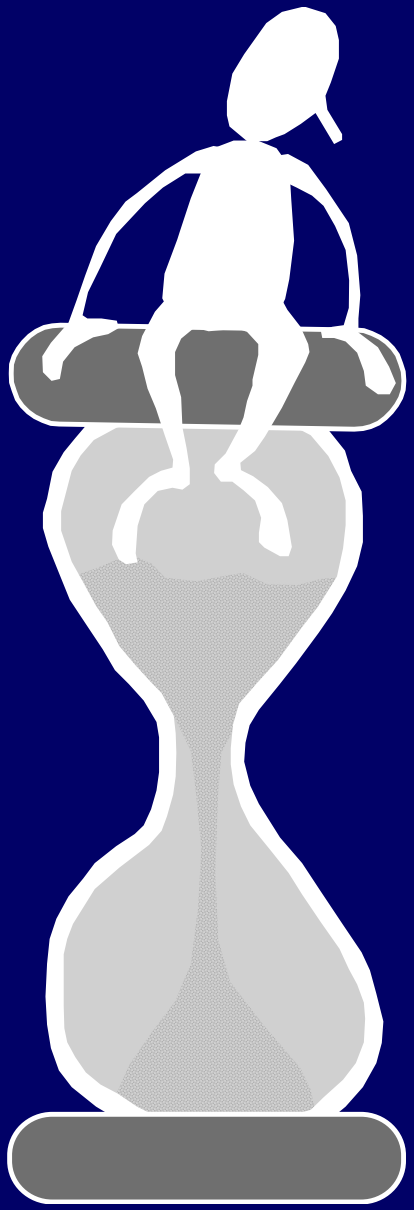
CS 15-251 Spring 2004

Lecture 12 Feb 19. 2004

Carnegie Mellon University

One Minute To Learn Programming: Finite Automata





Let me teach you a programming language so simple that you can learn it in less than a minute.

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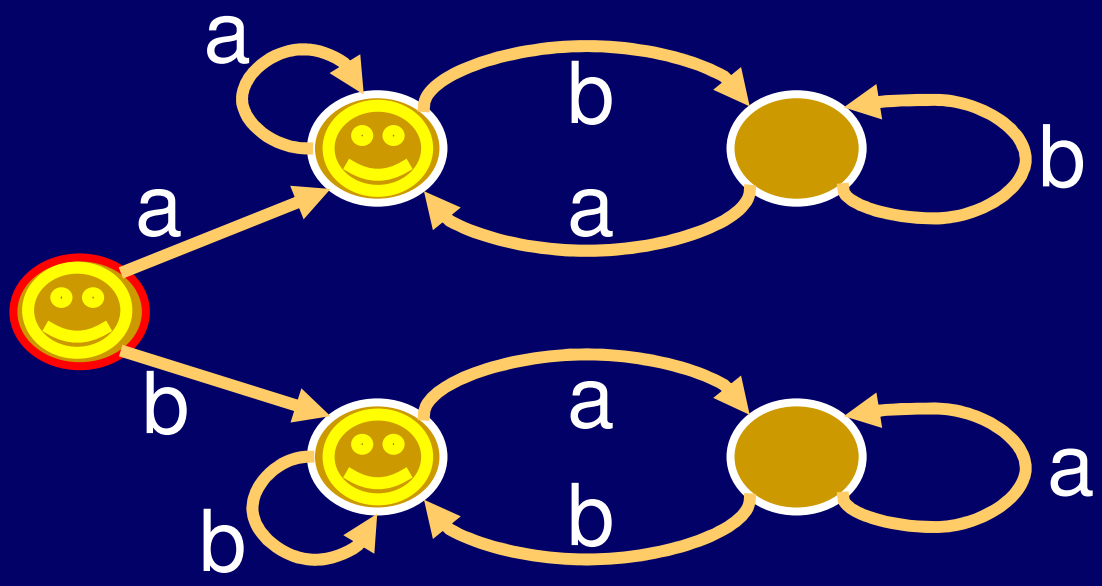
Meet "ABA" The Automaton!



Input String	Result
aba	Accept
aabb	Reject
aabba	Accept
ε	Accept

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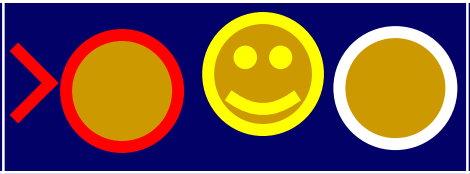
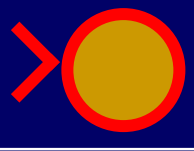
The Simplest Interesting Machine:

Finite State Machine
OR
Finite Automaton

Finite Automaton

Finite set of states		$Q = \{q_o, q_1, q_2, \ldots, q_k\}$
A start state		q_o
A set of accepting states		$F = \{q_{i_1}, q_{i_2}, \ldots, q_{i_r}\}$
A finite alphabet	$a \;\; b \;\; \#$ $x \;\; 1$	$\sum$
State transition instructions		$\partial : Q \times \Sigma \to Q$ $\partial(q_i, a) = q_j$

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How Machine M operates.

M "reads" one letter at a time from the input string (going from left to right)

M starts in state q_0.

If M is in state q_i reads the letter a then

If $\delta(q_i, a)$ is undefined then CRASH.

Otherwise M moves to state $\delta(q_i,a)$

Let $M=(Q,\Sigma,F,\delta)$ be a finite automaton.

M accepts the string *x* if when *M* reads *x* it ends in an accepting state.

M rejects the string *x* if when *M* reads *x* it ends in a non-accepting state.

M crashes on *x* if *M* crashes while reading *x*.

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The set (or language) accepted by M is:

$$L_M = \{x \in \Sigma^* \mid M \text{ accepts } x\}$$

$\Sigma^k \equiv$ All length k strings over the alphabet Σ

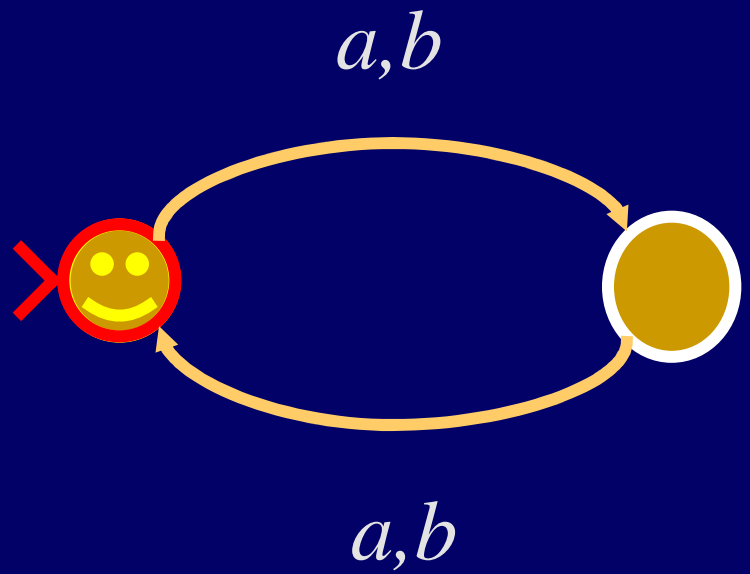
$$\Sigma^* \equiv \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \ldots$$

Notice that this is $\{\varepsilon\}$

What is the language accepted by this machine?

$L = \{a,b\}^*$ = all finite strings of a's and b's

What is the language accepted by this machine?



L = all even length strings of a's and b's

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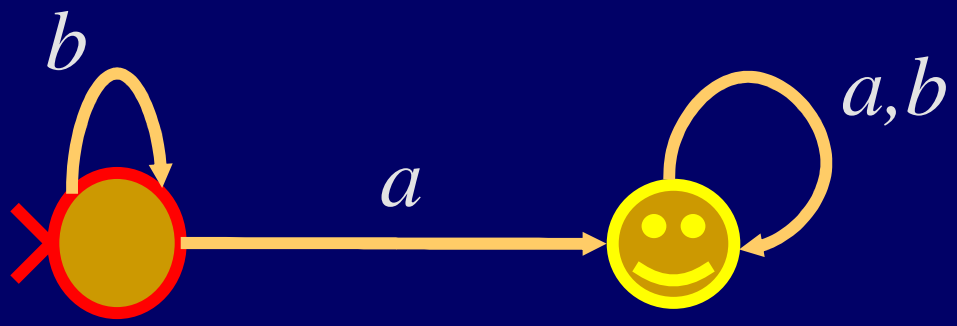
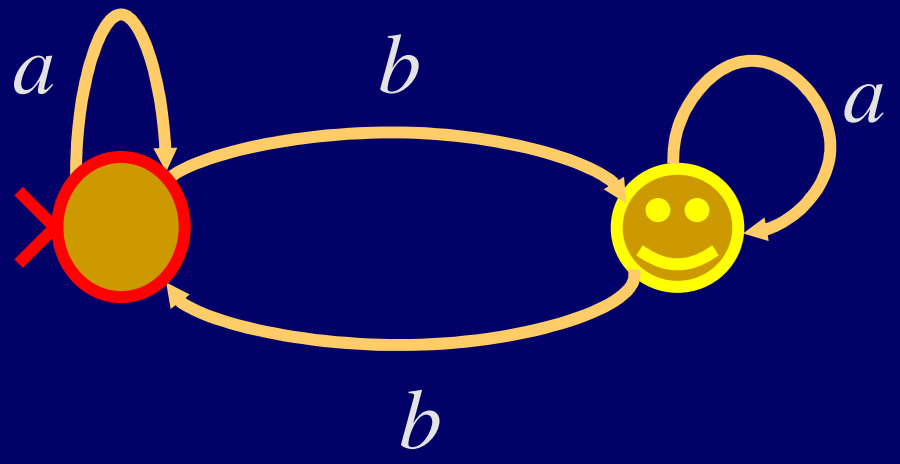
What machine accepts this language?

L = all strings in $\{a,b\}^*$ that contain at least one a

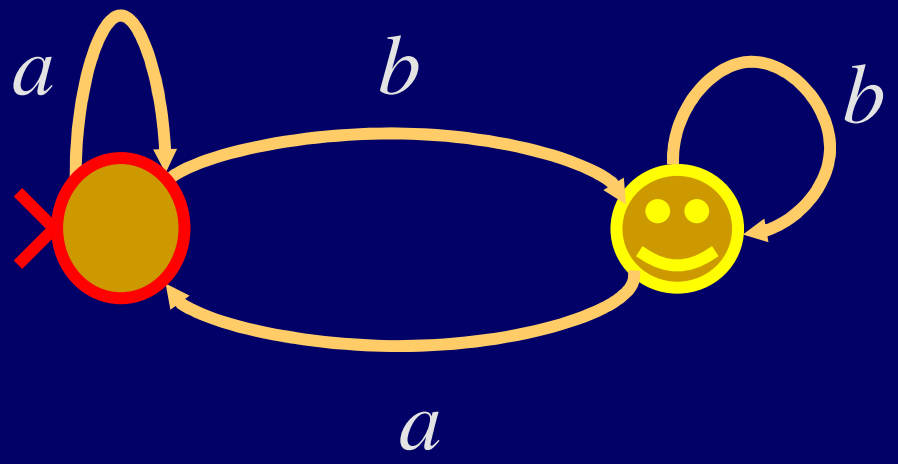


What machine accepts this language?

L = strings with an odd number of b's and any number of a's

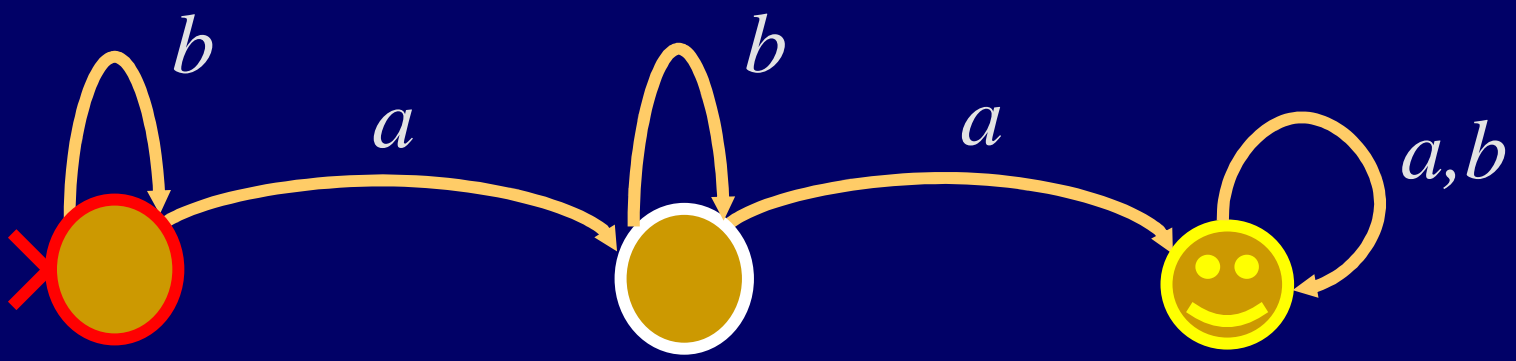


What is the language accepted by this machine?



L = any string ending with a b

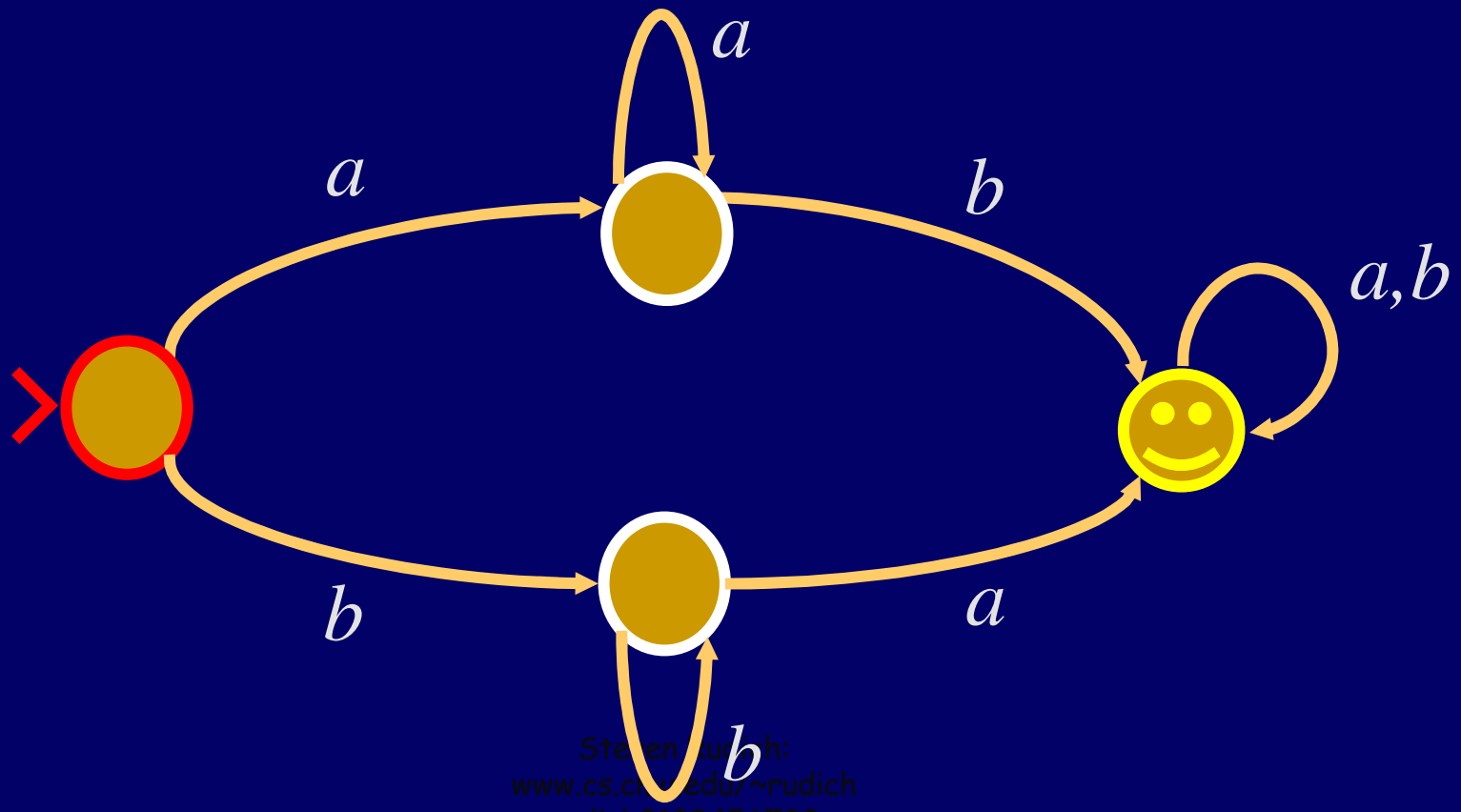
What is the language accepted by this machine?



L = any string with at least two a's

What machine accepts this language?

L = any string with an a and a b



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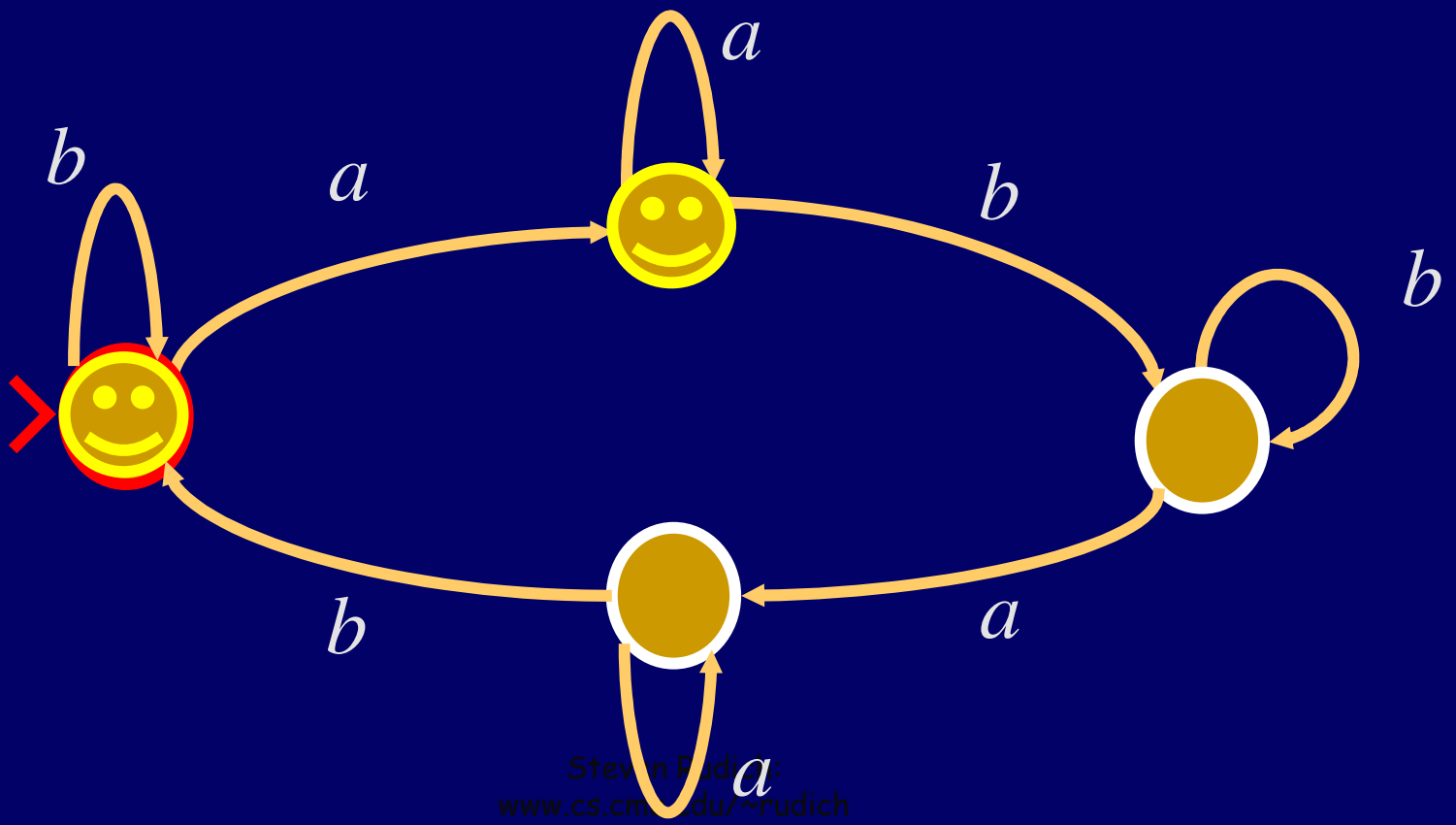
What machine accepts this language?

L = strings with an even number of ab pairs



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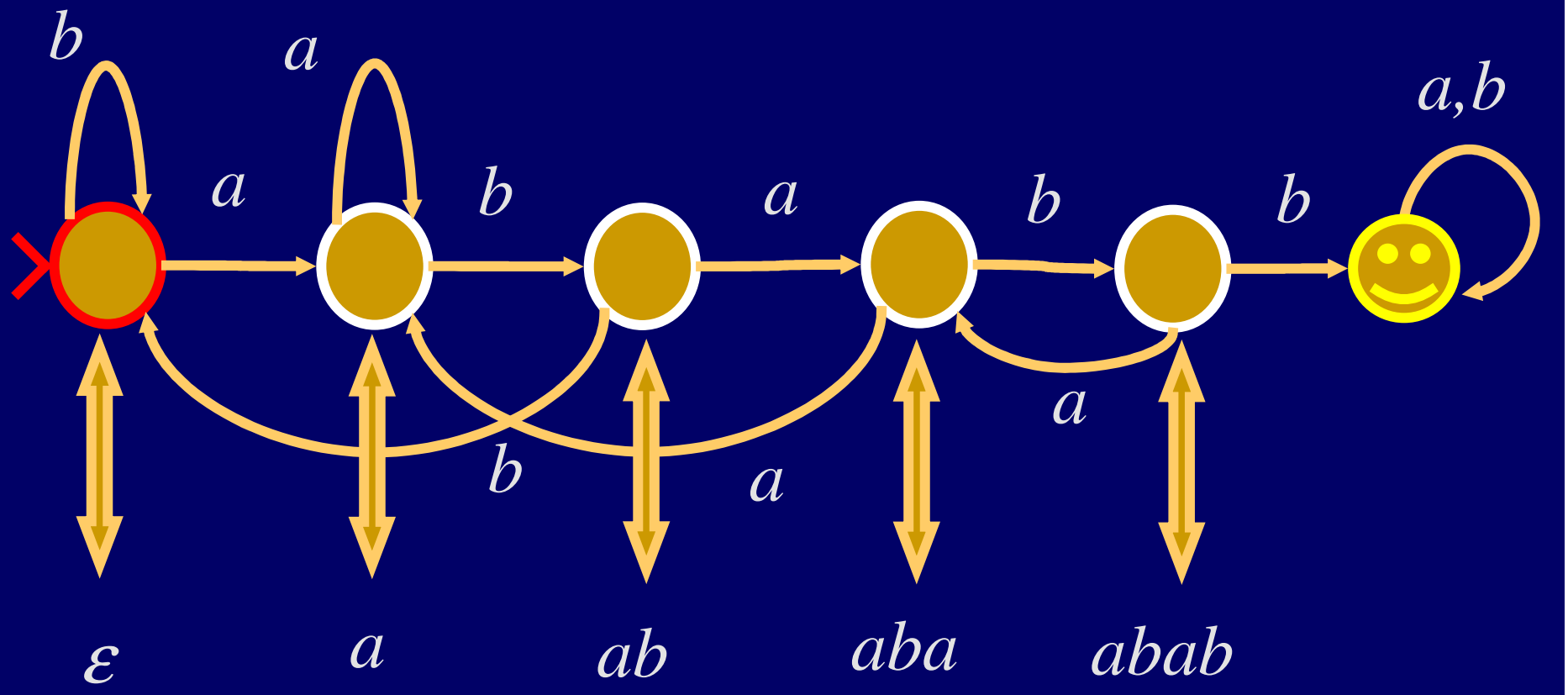
L = all strings containing $ababb$ as a consecutive substring



Invariant: I am state s exactly when s is the longest suffix of the input (so far) that forms a prefix of $ababb$.

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The "grep" Problem

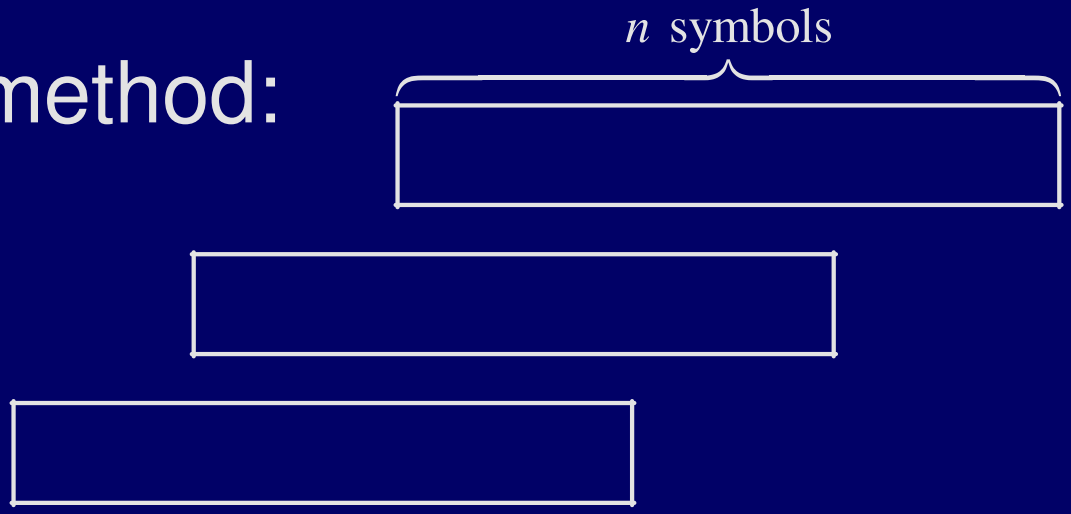
Input:

- text T of length t
- string S of length n

Problem:

Does the string S appear inside the text T ?

Naïve method:



$$a_1, a_2, a_3, \ldots\ldots\ldots\ldots\ldots, a_t$$

Cost: $O(nt)$ comparisons

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Automata Solution

- Build a machine M that accepts any string with S as a consecutive substring.
- Feed the text to M.
- Cost: t comparisons + time to build M.
- As luck would have it, the Knuth, *Morris*, Pratt algorithm builds M quickly.
- By the way, it can be done with fewer than t comparisons in the worst case!

Real-life uses of finite state machines

- grep
- coke machines
- thermostats (fridge)
- elevators
- train track switches
- lexical analyzers for parsers

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Any $L \subseteq \Sigma^*$ is defined to be a language.

L is just a set of strings. It is called a language for historical reasons.

Let $L \subseteq \Sigma^*$ be a language.

L is called a regular language if there is some finite automaton that accepts L.

In this lecture we have seen many regular languages.

- Σ^*
- even length strings
- strings containing *ababb*

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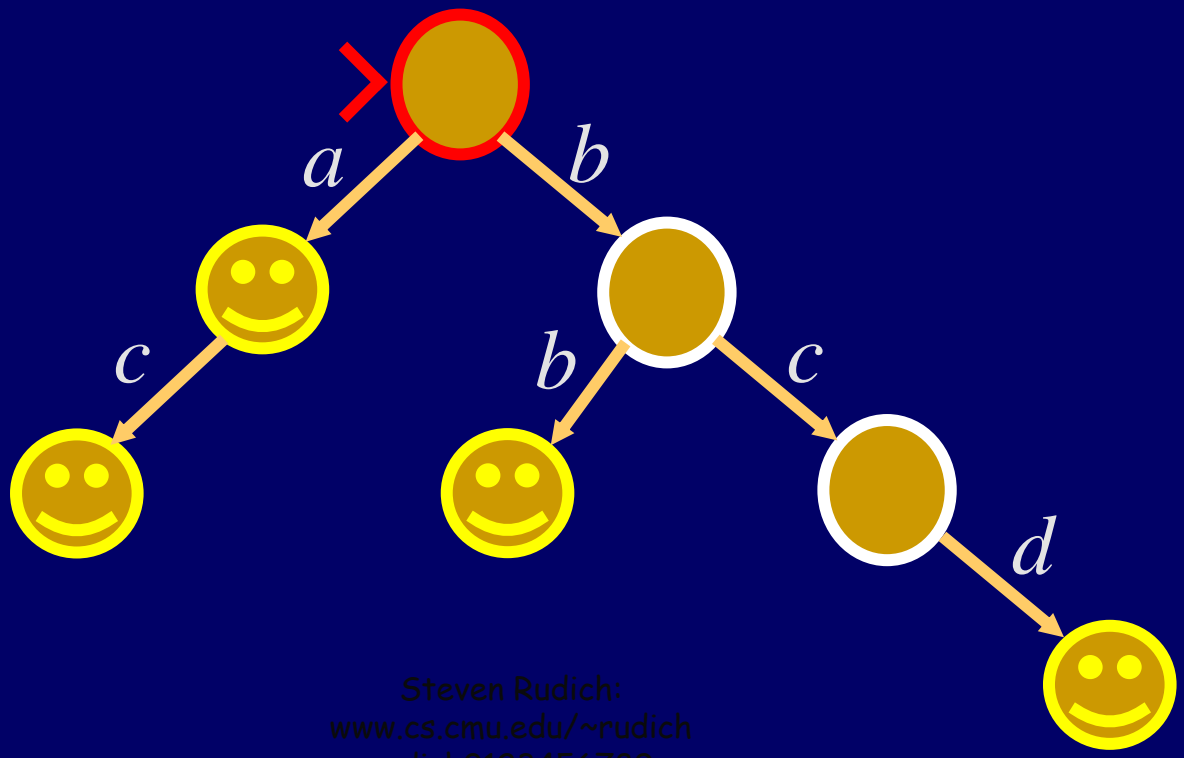
Theorem: Any finite langage is regular.

Proof: Make a machine with a “path” for each string in the language.

Example: $L = \{a, bcd, ac, bb\}$



Are all languages regular?

Consider the language

$$a^n b^n = \{\varepsilon, ab, aabb, aaabbb, \ldots\}$$

i.e., a bunch of a's followed
by an equal number of b's

No finite automaton accepts this language.

Can you prove this?

$a^n b^n$ is not regular. No machine has enough states to keep track of the number of a's it might encounter.

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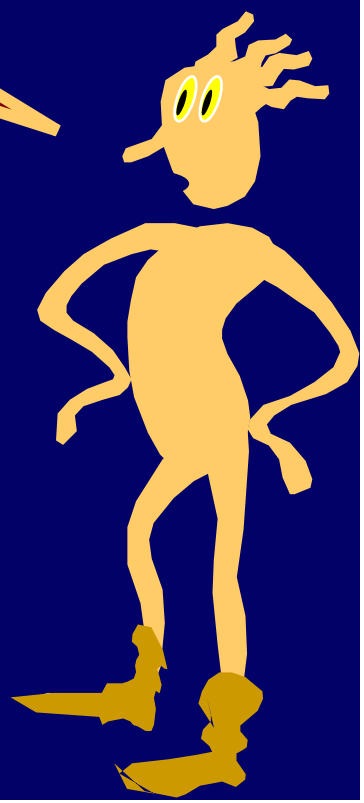
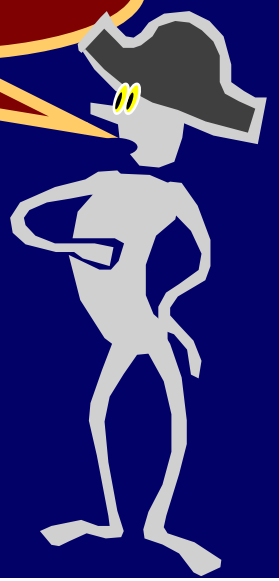
That is a fairly weak argument. Consider the following example…

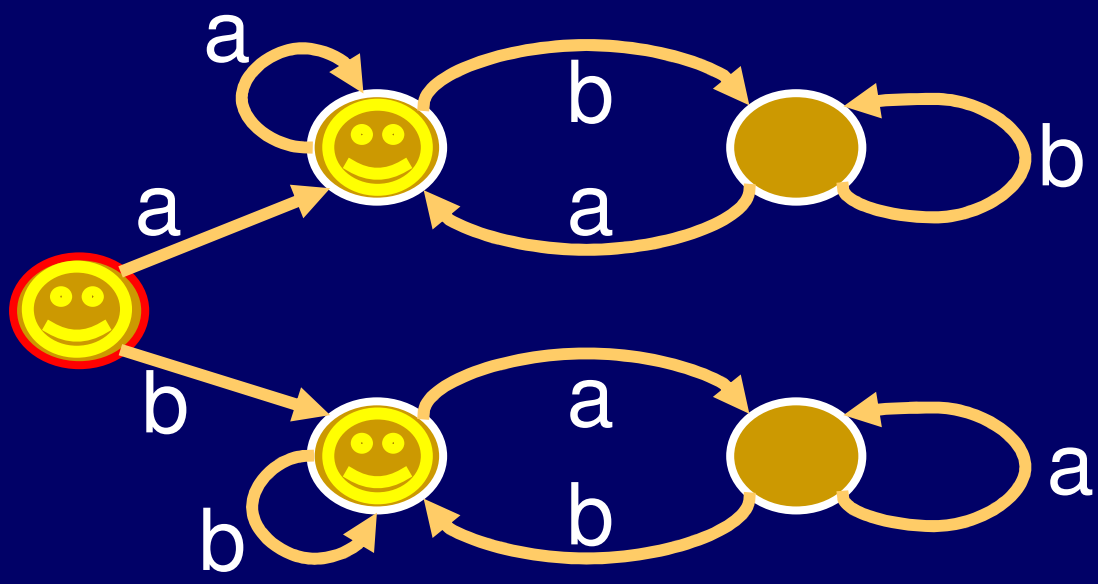


L = strings where the # of occurrences of the pattern ab is equal to the number of occurrences of the pattern ba

Can't be regular. No machine has enough states to keep track of the number of occurrences of ab.

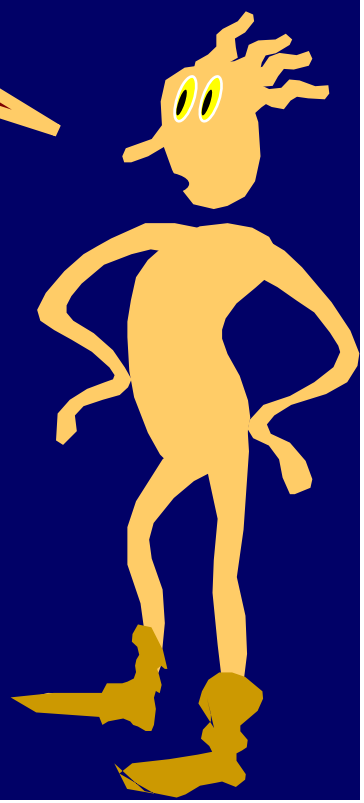


Remember "ABA"?



ABA accepts only the strings with an equal number of ab's and ba's!

Let me show you a professional strength proof that $a^n b^n$ is not regular….



Professional Strength Proof

Theorem: $a^n b^n$ is not regular.

Proof: Assume that it is. Then $\exists$ M with k states that accepts it.

For each $0 \leq i \leq k$, let S_i be the state M is in after reading a^i.

$\exists i,j \leq k$ s.t. $S_i = S_j$, but $i \neq j$

M will do the same thing on $a^i b^i$ and $a^j b^i$.

But a valid M must reject $a^j b^i$ and accept $a^i b^i$.

$\Rightarrow\Leftarrow$

MORAL:

Finite automata can't count.

Advertisement

You can learn much more about these creatures in the FLAC course.

Formal Languages, Automata, and Computation

- **There is a unique smallest automaton for any regular language**
- **It can be found by a fast algorithm.**
- **……**

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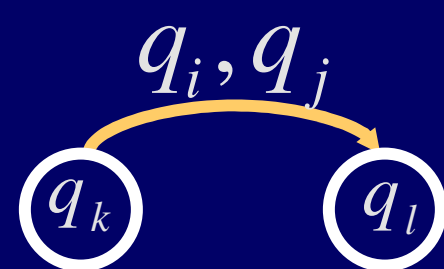
Cellular Automata

•Line up a bunch of ***identical*** finite automata in a straight line.

•Transitions are based on the states of the machine's two neighbors or an indicator that a neighbor is missing. (There is no other input.)

$$Q \times Q \times Q \rightarrow Q$$

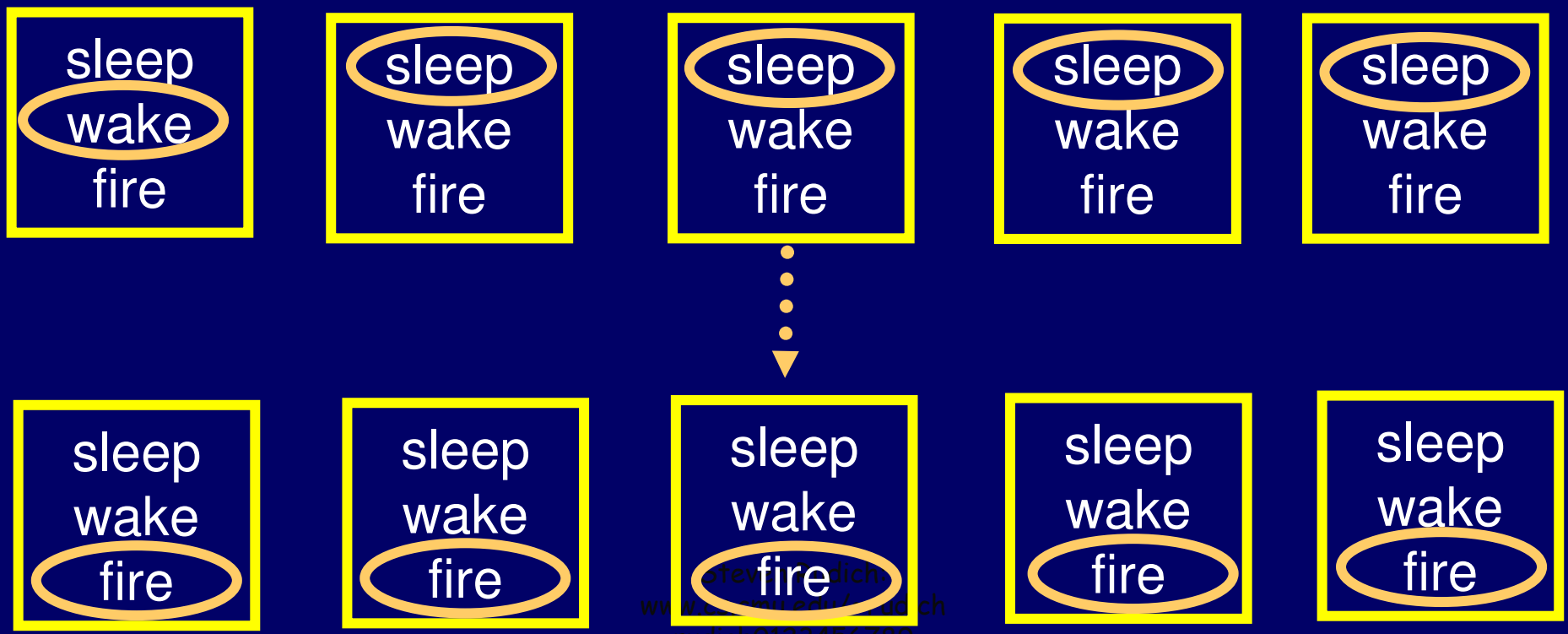
$$(q_k, q_i, q_j) = q_l$$



•All cells move to their next states at the same time: synchronous transition

The Firing Squad Problem

- Five "soldiers" all start in the sleep state. You change the one on the left to the wake state.
- All five must get to the fire state at the same time (for the first time).



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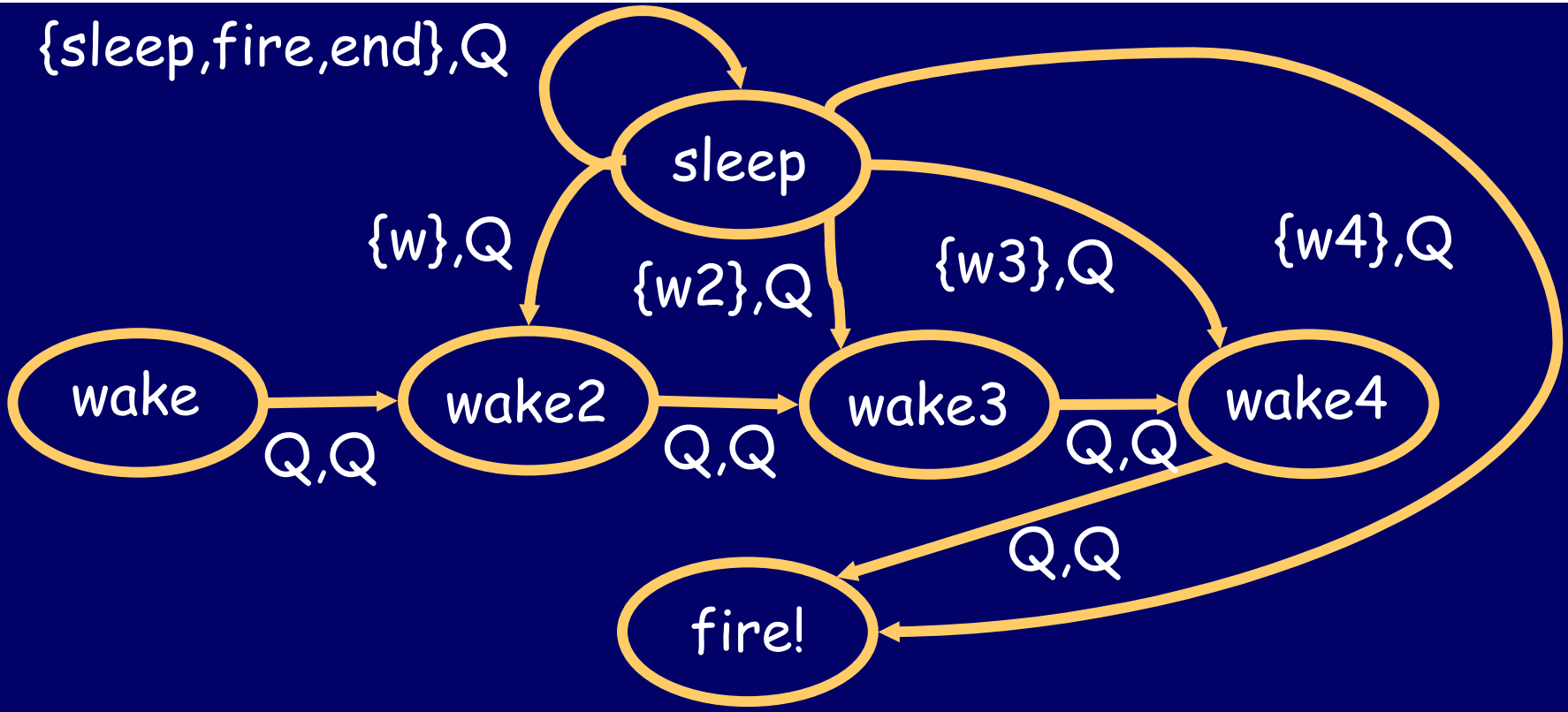
Shorthand

Means use this transition when your left neighbor is in any state at all and your right neighbor is in state a,b, or d.



	1	2	3	4	5
	sleep	sleep	sleep	sleep	sleep
	sleep	sleep	sleep	sleep	sleep
	sleep	sleep	sleep	sleep	sleep
	sleep	sleep	sleep	sleep	sleep
	sleep	sleep	sleep	sleep	sleep

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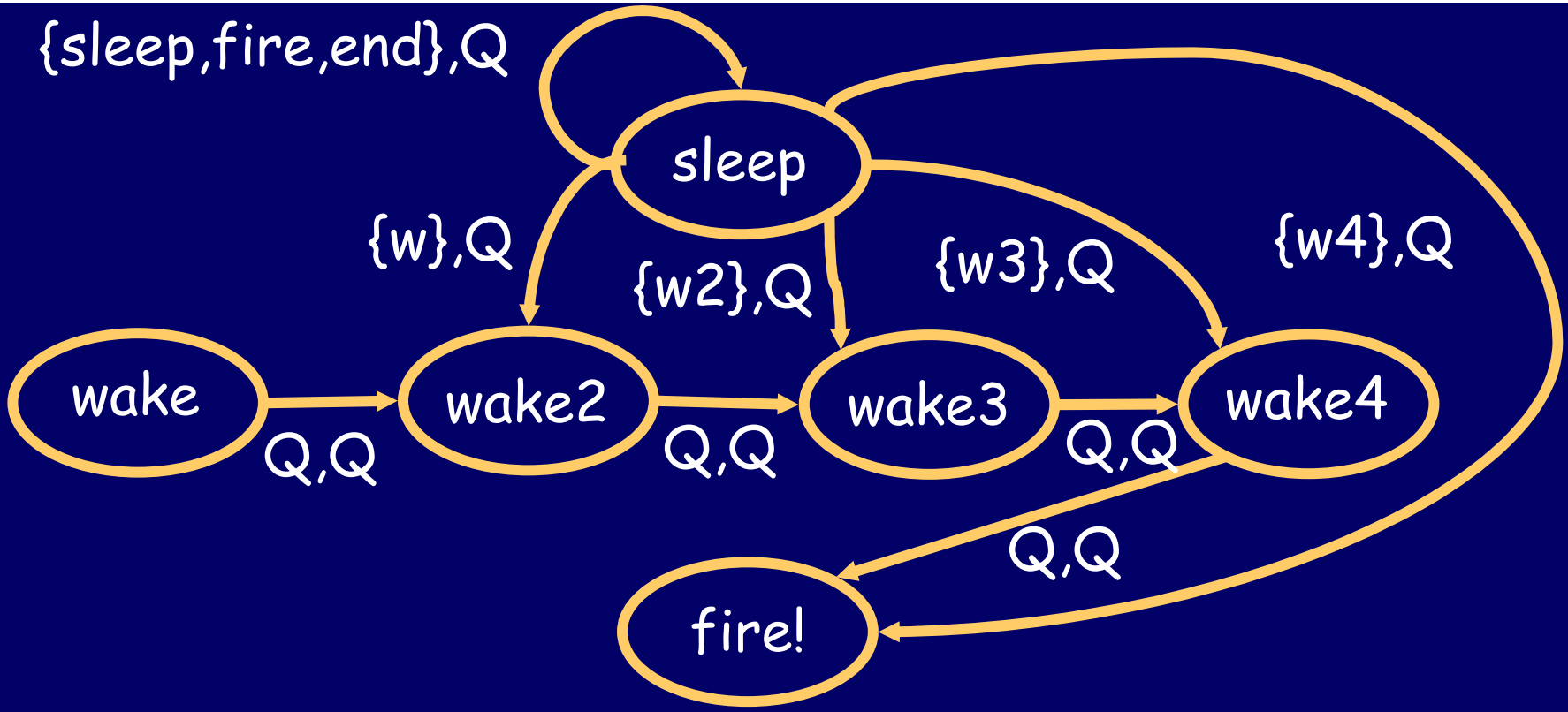

	1	2	3	4	5
	wake	sleep	sleep	sleep	sleep

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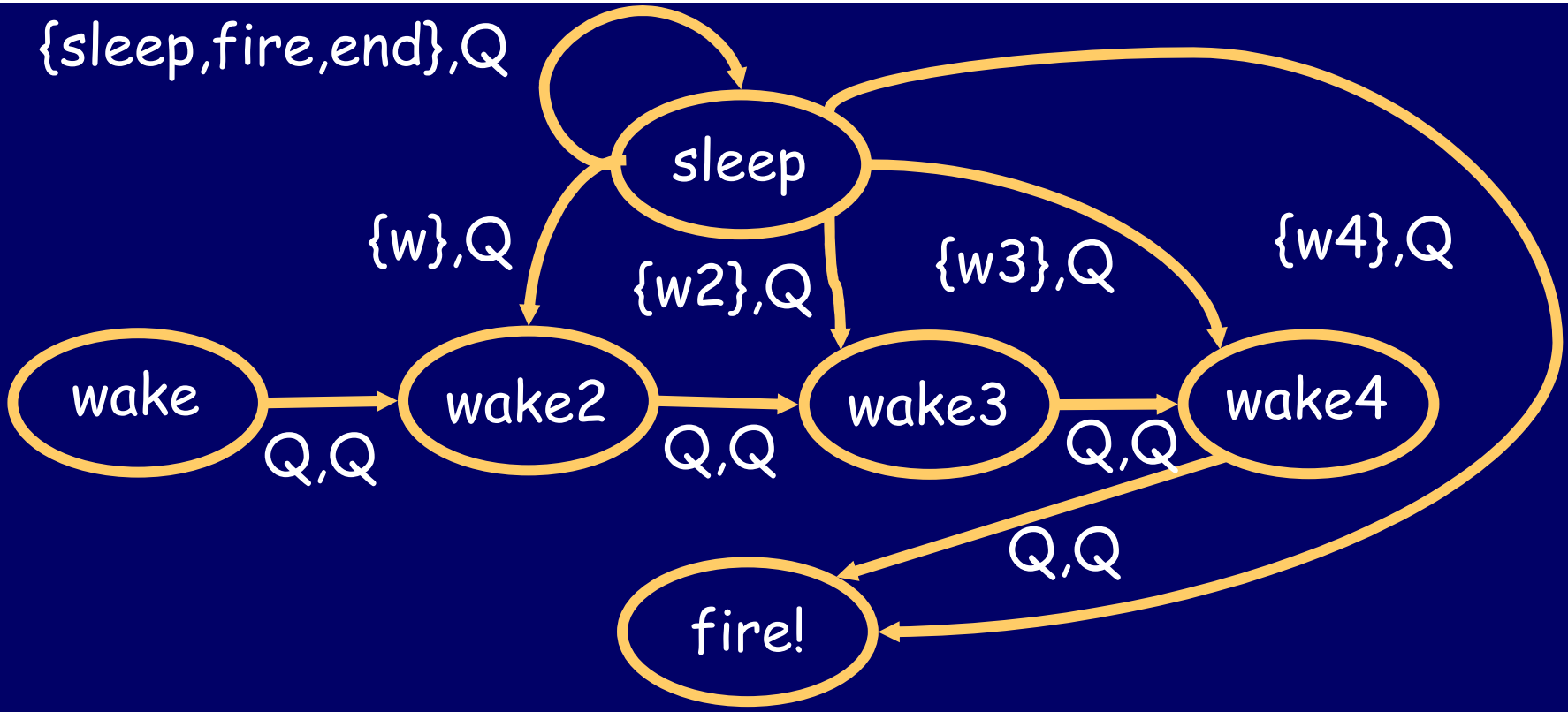

	1	2	3	4	5
	wake	sleep	sleep	sleep	sleep
	wake2	wake2	sleep	sleep	sleep

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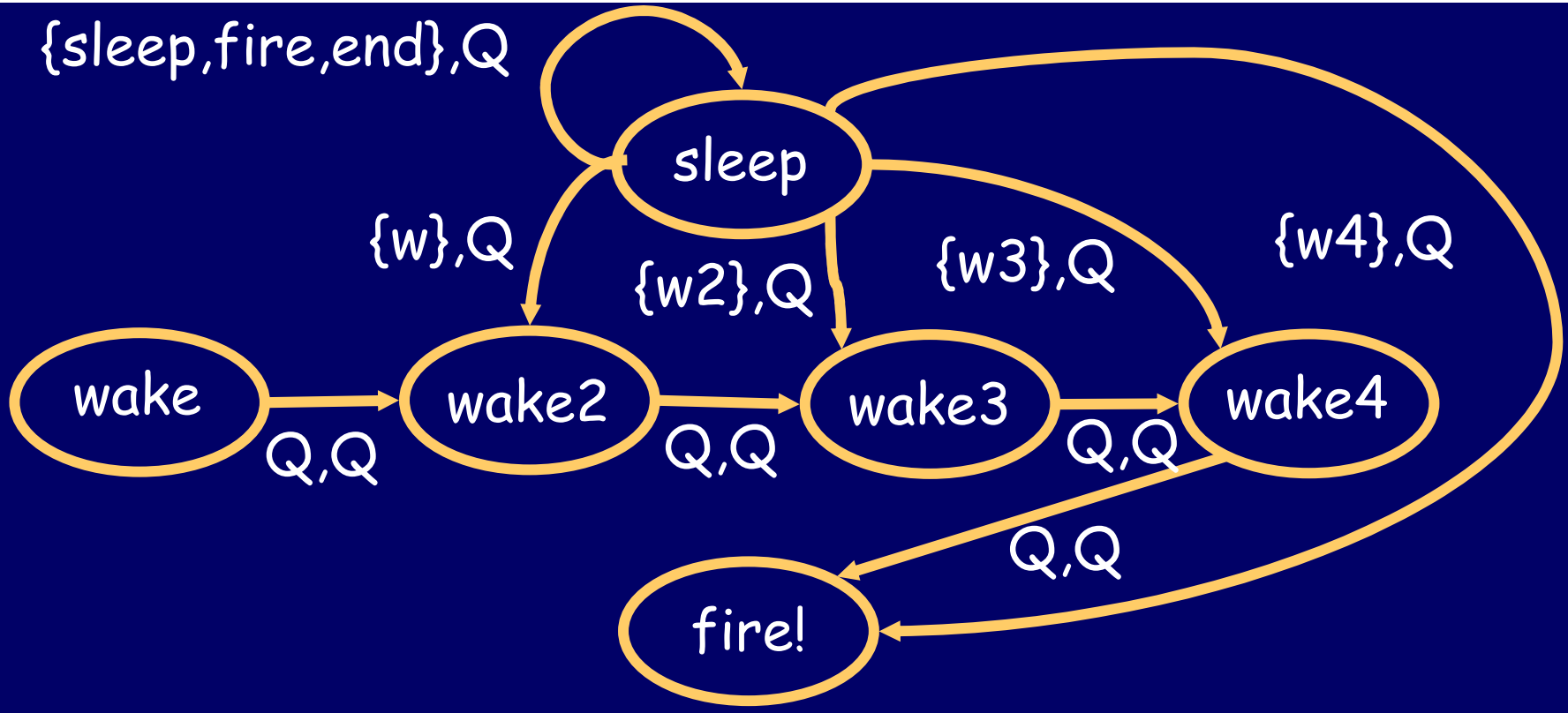

	1	2	3	4	5
	wake	sleep	sleep	sleep	sleep
	wake2	wake2	sleep	sleep	sleep
	wake3	wake3	wake3	sleep	sleep

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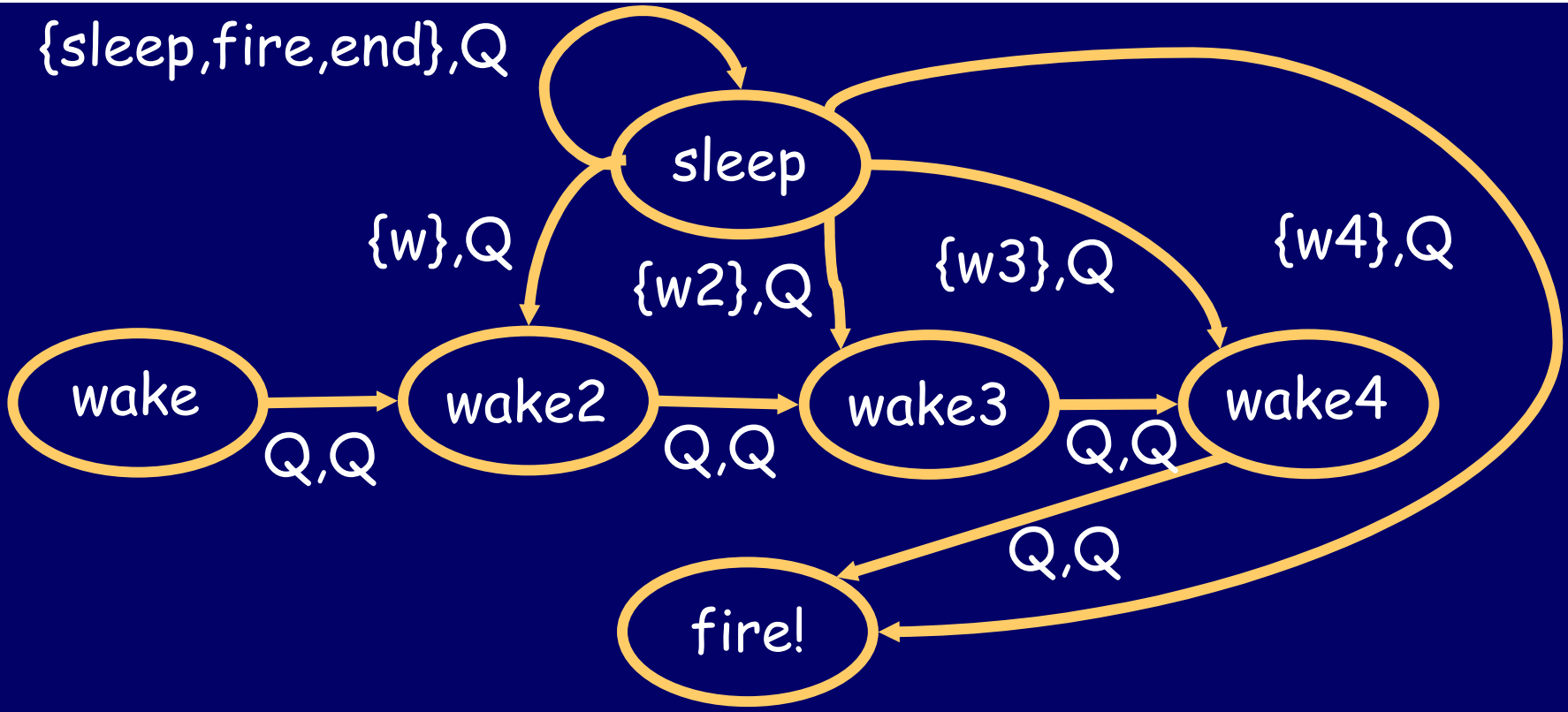

	1	2	3	4	5
	wake	sleep	sleep	sleep	sleep
	wake2	wake2	sleep	sleep	sleep
	wake3	wake3	wake3	sleep	sleep
	wake4	wake4	wake4	wake4	sleep

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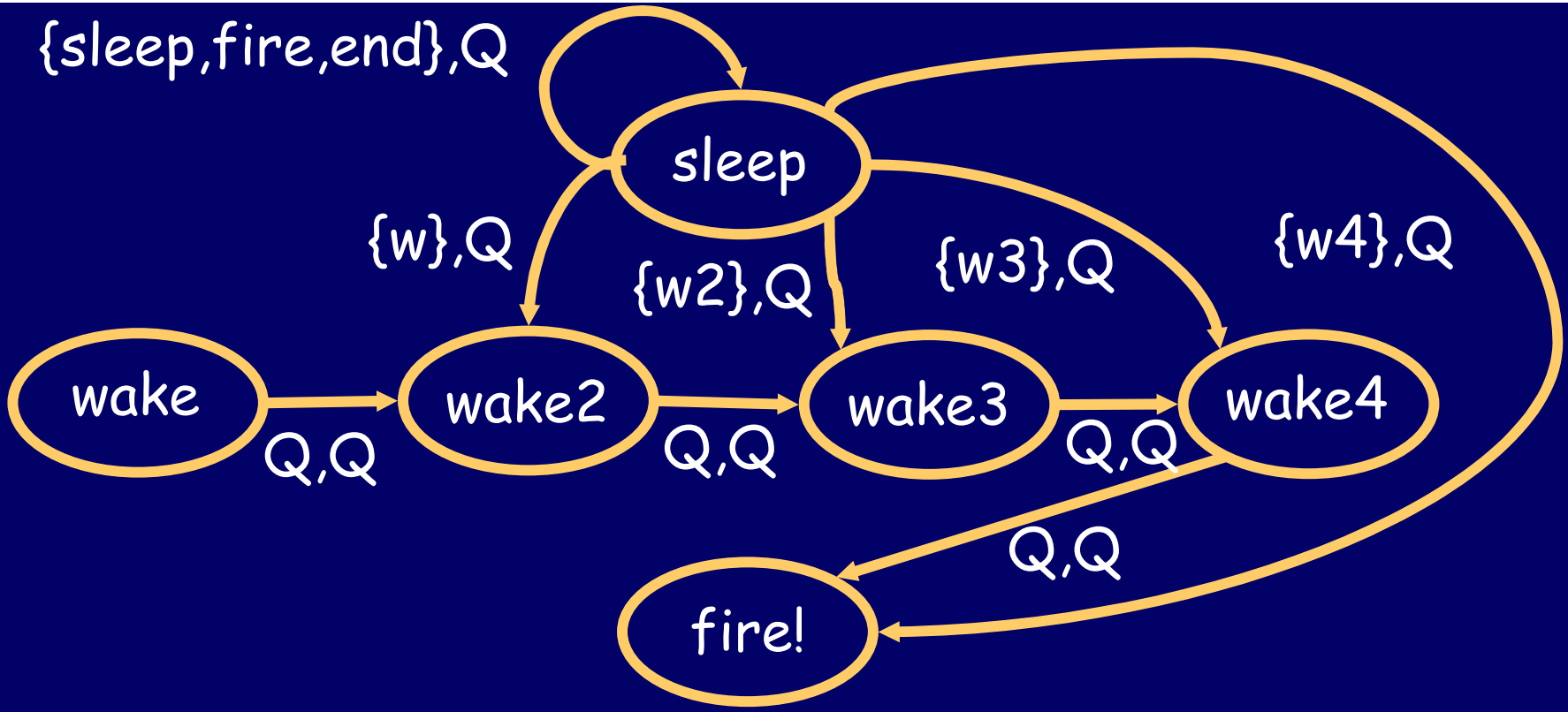

	1	2	3	4	5
	wake	sleep	sleep	sleep	sleep
	wake2	wake2	sleep	sleep	sleep
	wake3	wake3	wake3	sleep	sleep
	wake4	wake4	wake4	wake4	sleep
	fire	fire	fire	fire	fire

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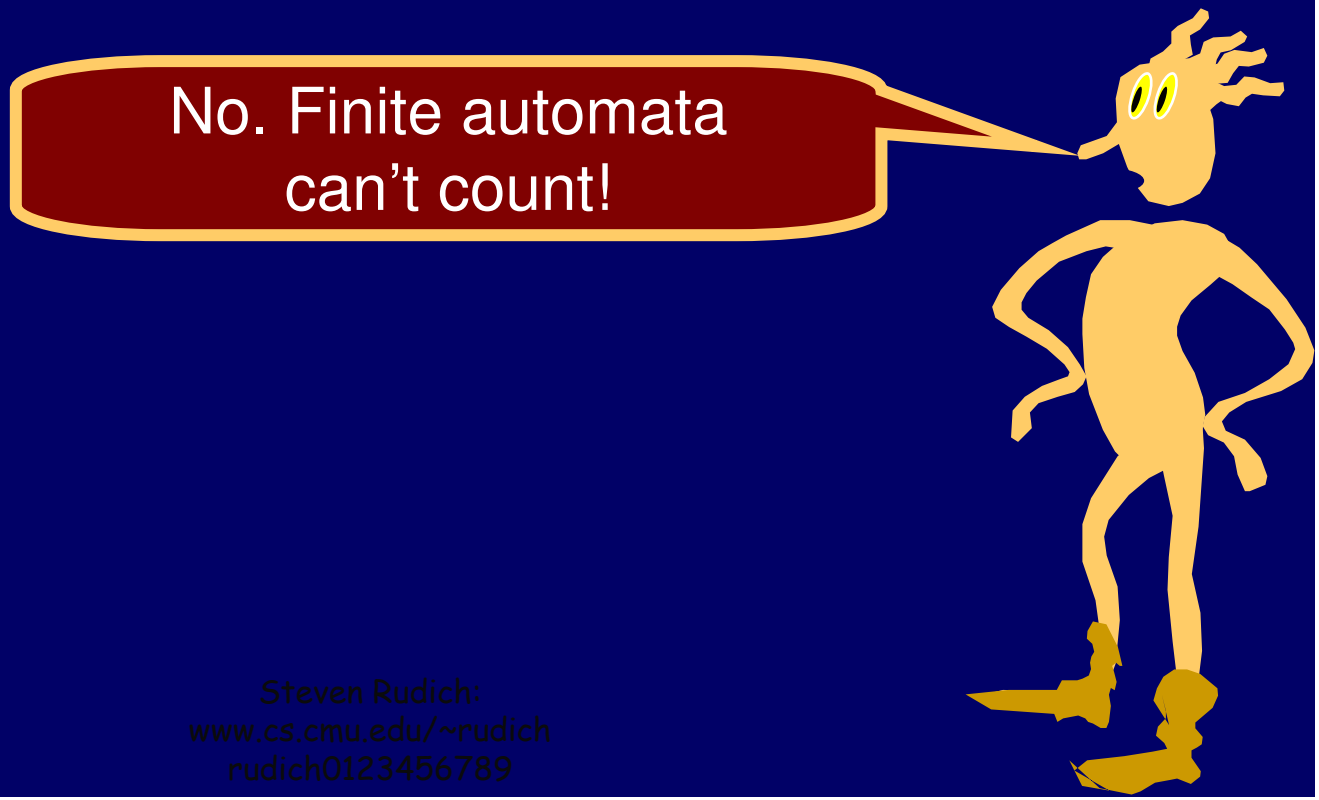
Question

Can you build the soldier's finite automaton brain before you know how many soldiers will be in the line?



Don’t jump to conclusions! It *is* possible to design a single cellular automaton that works for any number of soldiers!

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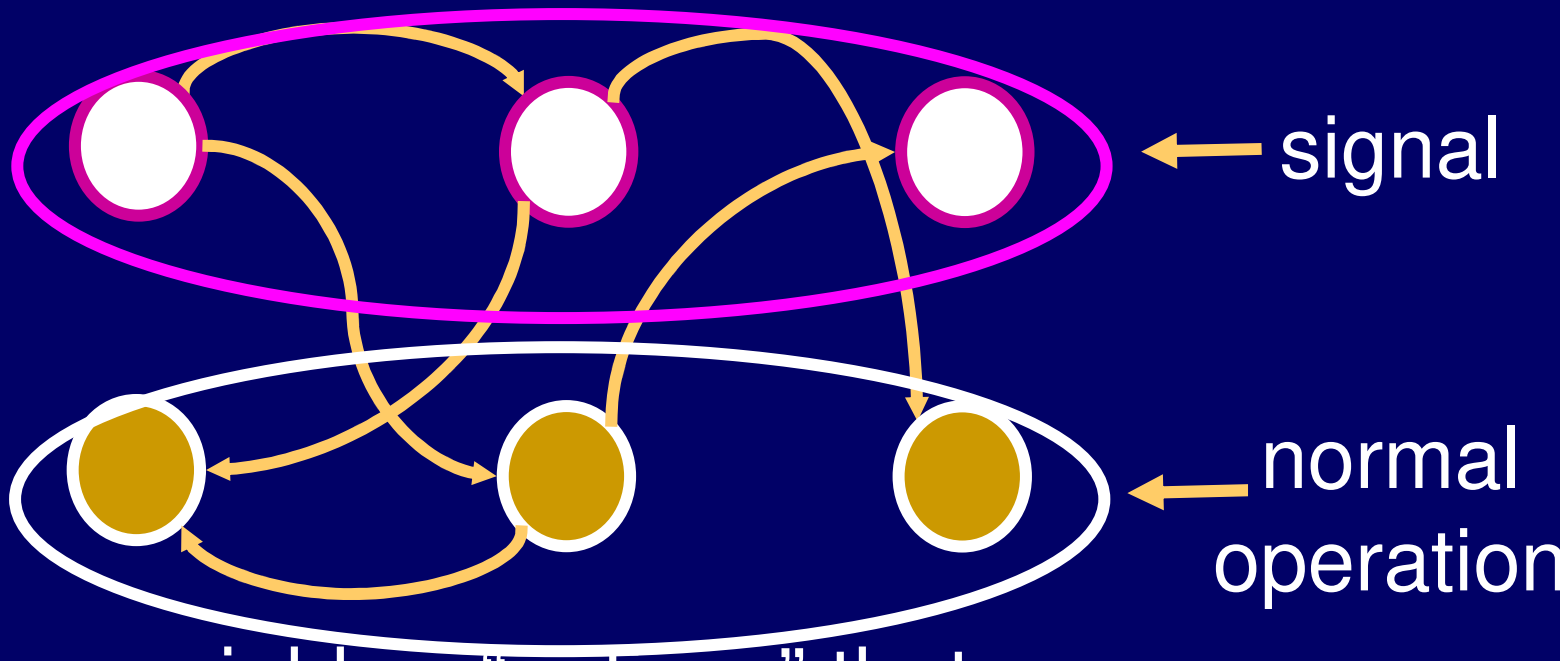
An automaton can “signal” to its neighbors.

Certain subsets of its states will correspond to signaling.



Your neighbor “notices” that you are in a signaling state.

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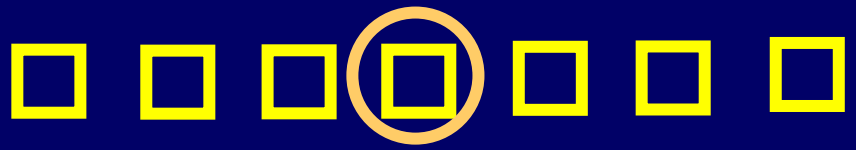
Define the *center* of the line to be:

- The guy in the middle if the # of soldiers is odd.



- The two guys in the middle if the the # of soldiers is even.

Useful subroutine:

method to find the center after leftmost soldier is awakened.

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A Method to Find the Center

- Send a signal back and forth from “end” to “end”.

- Each time it gets to an “end”, send it back, but tell the neighbor of the “end” that it is the “new end”.

- The center(s) will get a “new end” signal from both sides.

- # of steps: $O\left(n^2\right)$

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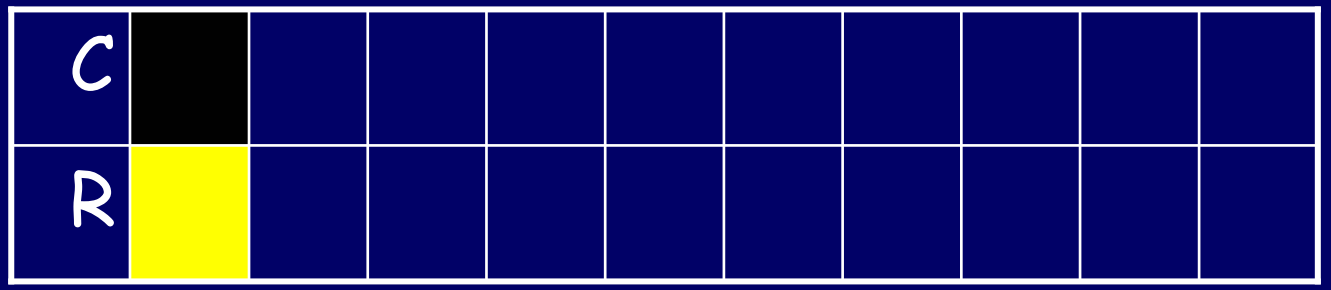
Coyote and Roadrunner.

Roadrunner is exactly
three times faster
than coyote.



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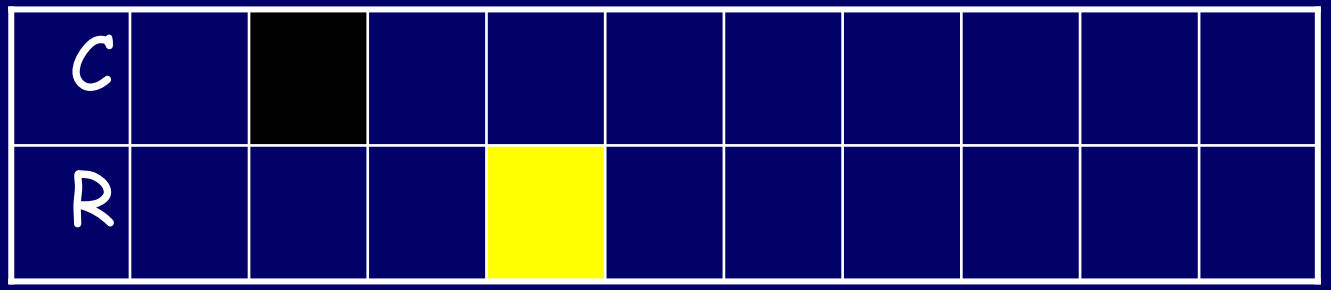

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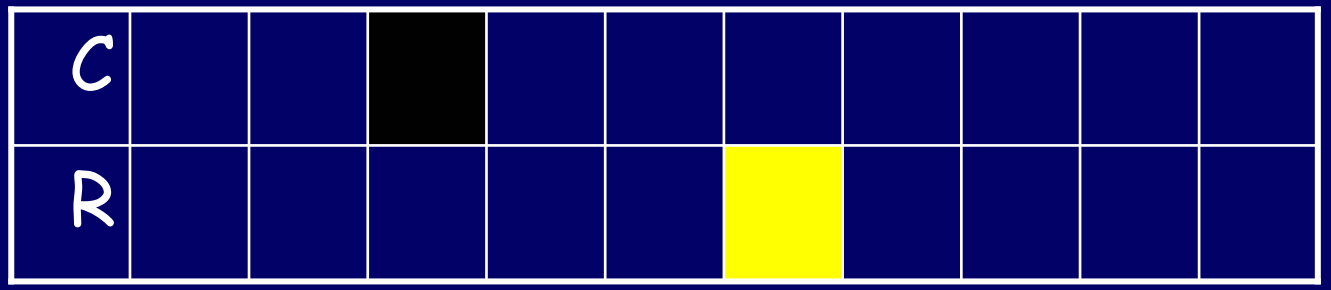

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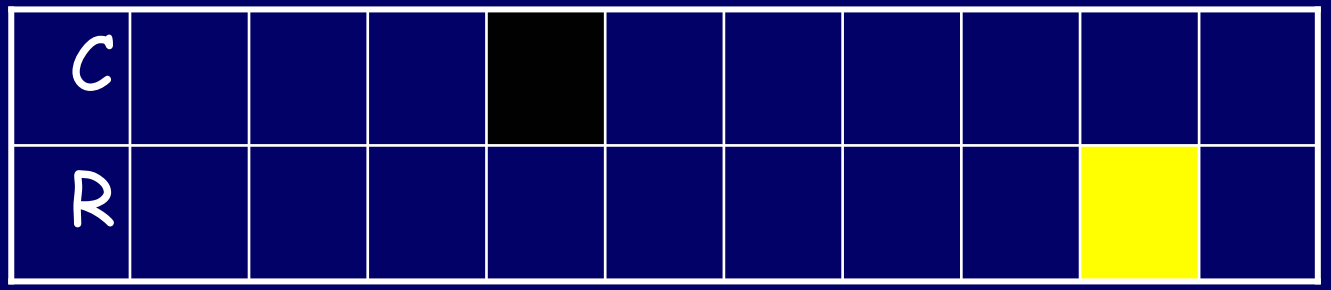

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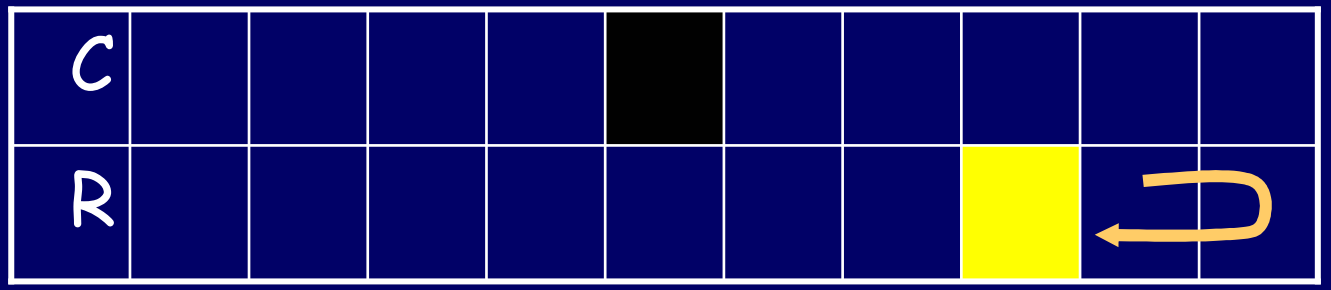

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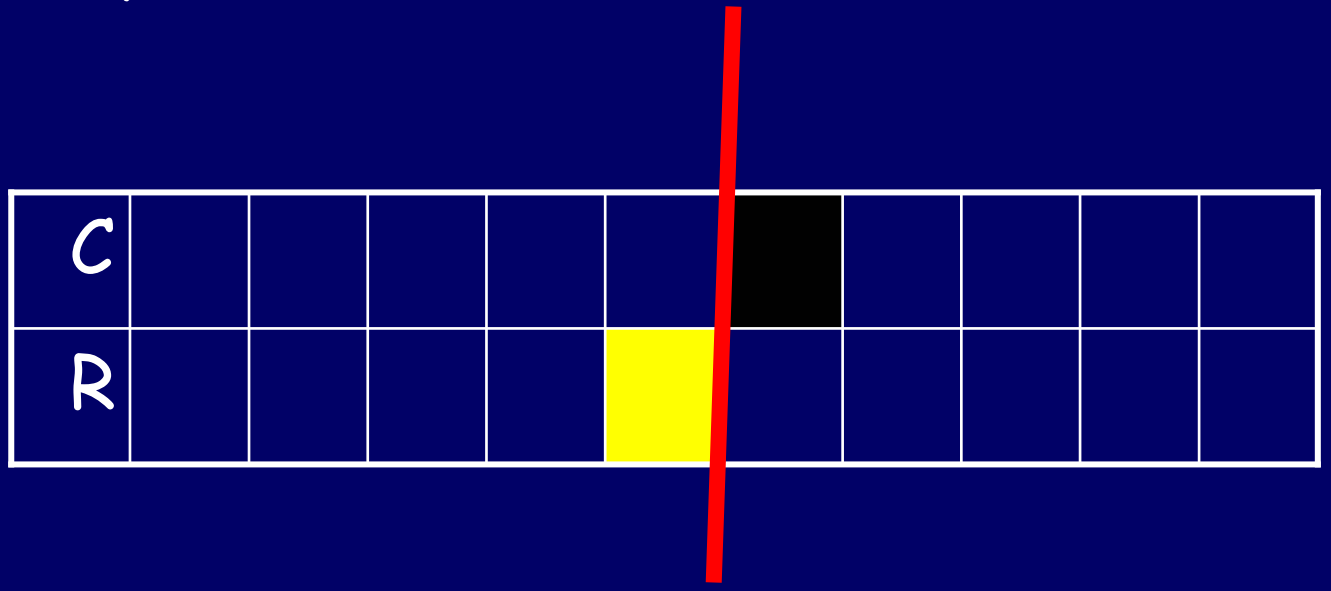
The move where coyote sees roadrunner along side or just behind him is the move across the center.

Roadrunner is exactly three times faster than coyote.



Roadrunner Coyote Center Algorithm

- Use two signals: one fast and one slow

- When you get the fast signal, pass it on (or bounce it back) on the next time step.

- When you get the slow signal, hold it for two steps, then pass it on the next time step.

- When you wake and you are leftmost, consider yourself to hold both the slow and fast signal.

- If you get one signal from each side, shout, “I am the center!”

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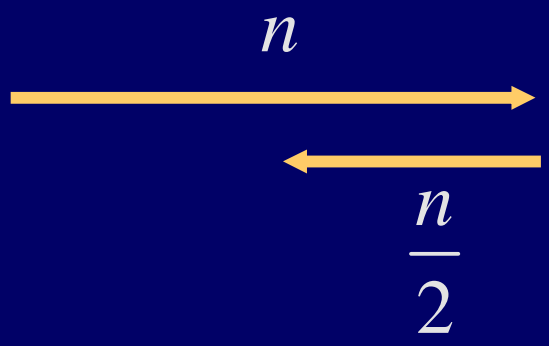
Firing Squad Problem

- Keep finding the middle recursively: If you're the center, mark yourself as an "end" and find the centers on each side of you.

- If you are an "end" and your neighbor is starting both signals, send back a "fire next step".

Firing Squad Analysis

How many steps to find the center?



$$\boxed{\frac{3}{2}n \text{ steps}}$$

How many steps to fire?

$$\frac{3}{2}n + \frac{3}{2}\frac{n}{2} + \frac{3}{2}\frac{n}{4} + \frac{3}{2}\frac{n}{8} + \cdots$$

$$= \frac{3}{2}n\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)$$

$$= \frac{3}{2}n \cdot 2 = \boxed{3n}$$

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The fact that firing squad can be solved in 3n steps is impressive!

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Can you show a matching lower bound?

Lower Bound

Can you see why any solution must use at least $2n - 2$ steps?

MATHCING UPPER BOUND!

It takes at least 2n-2 for a signal to reach one end and come back.

There is a way to solve firing squad in 2n-2 steps!

Wow! Firing squad can be solved in exactly 2n-2 time!

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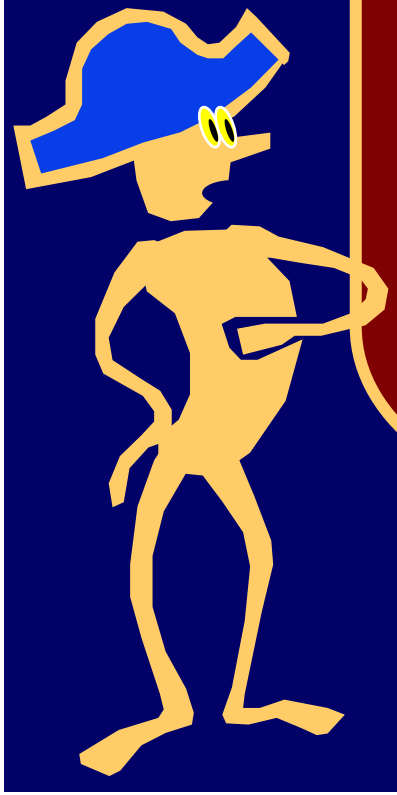
Yes, that was shown by
Abraham Waksman
Information and Control 9
(1966), Pages 66-78

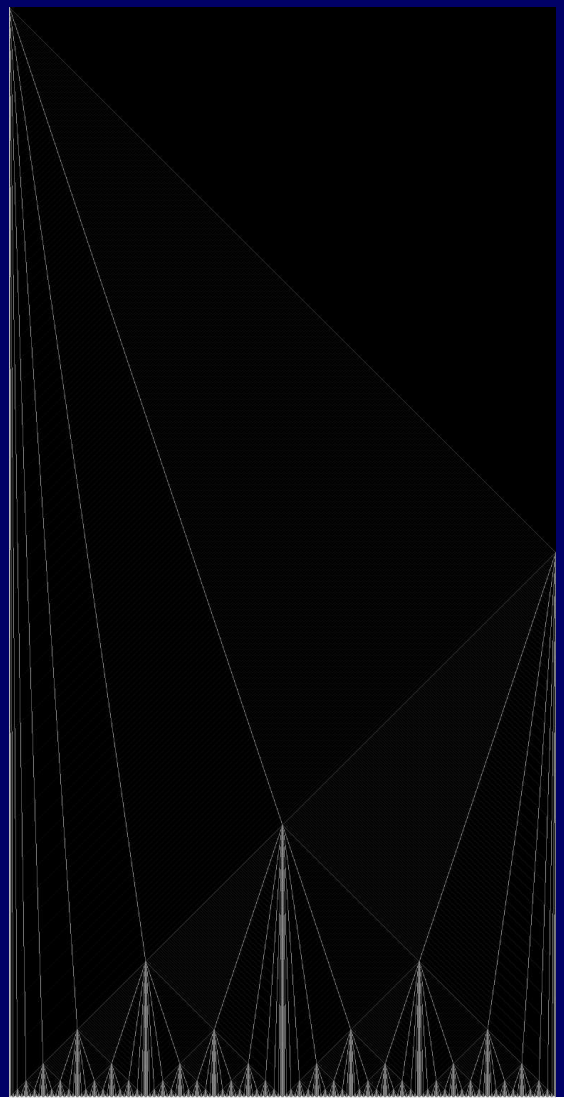
The solution used 16 states.



I have been reading:

http://xraysgi.ims.uconn.edu/fsquad/firing-solution.html

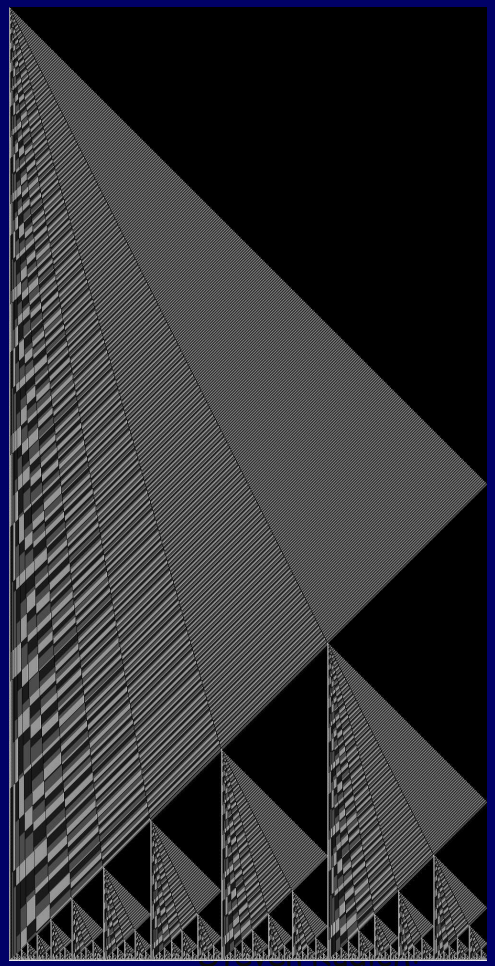
Waserman's Solution running on 768 soldiers.



Jacques Mazoyer (Theoretical Computer Science 50 (1987), pages 183-238)

Gives a 6 state 2n-2 time solution!

Mozoyer's Solution running on 1000 soldiers.



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It can be shown that there is no 4 state solution.

OPEN PROBLEM:

Is there a five state minimal time solution?

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Other Research

- Different topologies
- Self-stabilizing clocks
  - all cells fire simultaneously at regular intervals
  - even if you mess with the states of all the cells they get back on schedule (eventually)

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