Great Theoretical Ideas In Computer Science

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CS 15-251 Spring 2003

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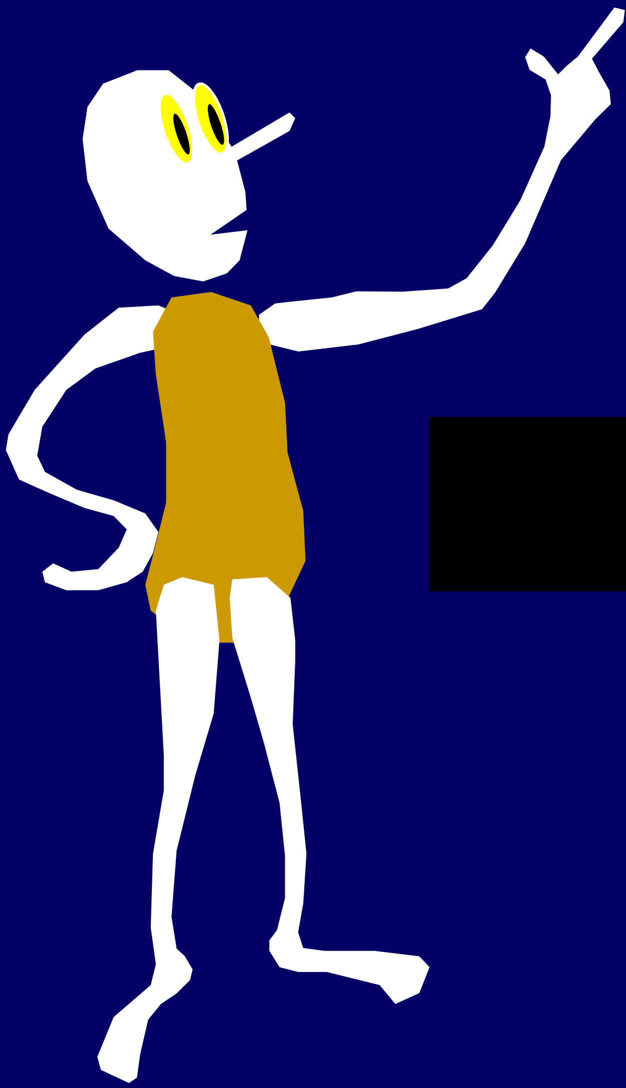
Counting III: Pascal's Triangle, Polynomials, and Vector Programs

$$1 + X^1 + X^2 + X^3 + \dots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$

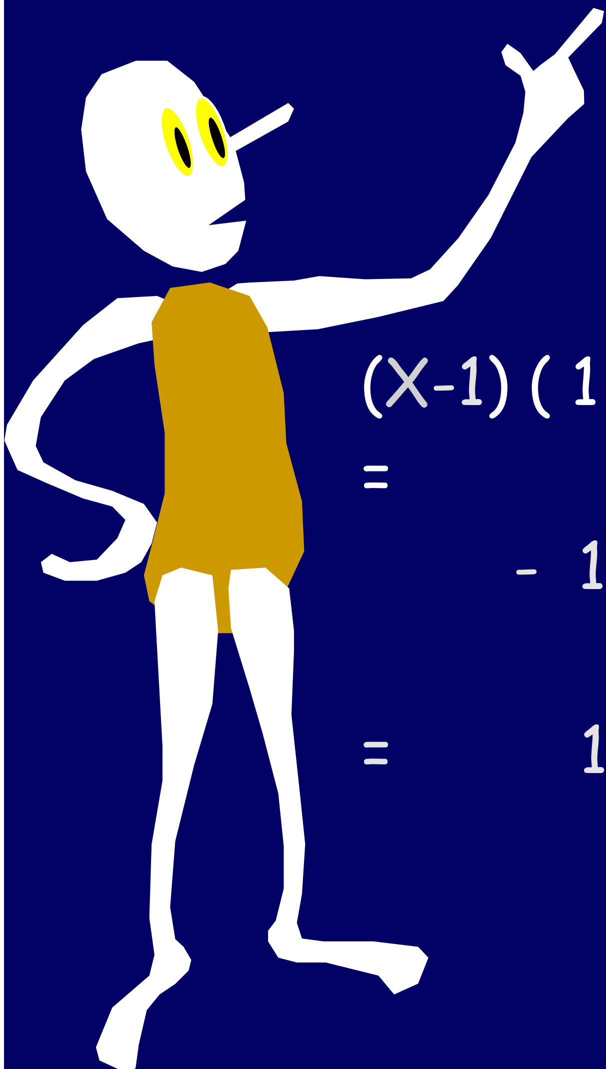


The Geometric Series

$$1 + X^1 + X^2 + X^3 + \dots + X^n + \dots\dots = \frac{1}{1 - X}$$

The Infinite Geometric Series

$$1 + X^1 + X^2 + X^3 + \dots + X^n + \dots\dots = \frac{1}{1 - X}$$



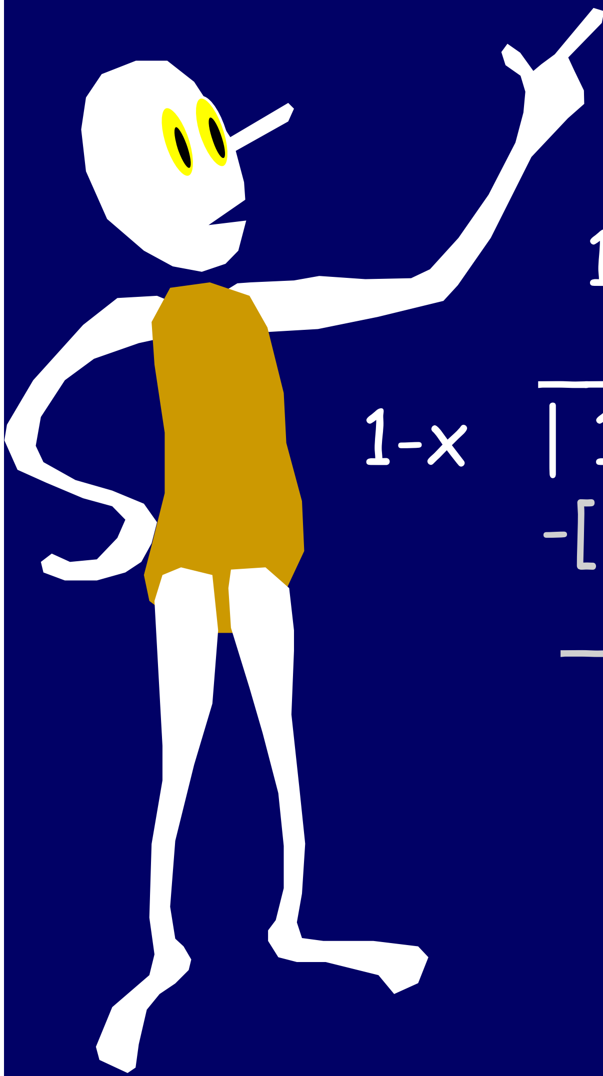
$$(X-1)\,(\,1 + X^1 + X^2 + X^3 + \dots + X^n + \dots\,)$$

$$= \quad X^1 + X^2 + X^3 + \dots \qquad + X^n + X^{n+1} + \dots\dots$$

$$-\;1 - X^1 - X^2 - X^3 - \dots \;- X^{n-1} - X^n - X^{n+1} - \dots$$

$$= \quad 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$

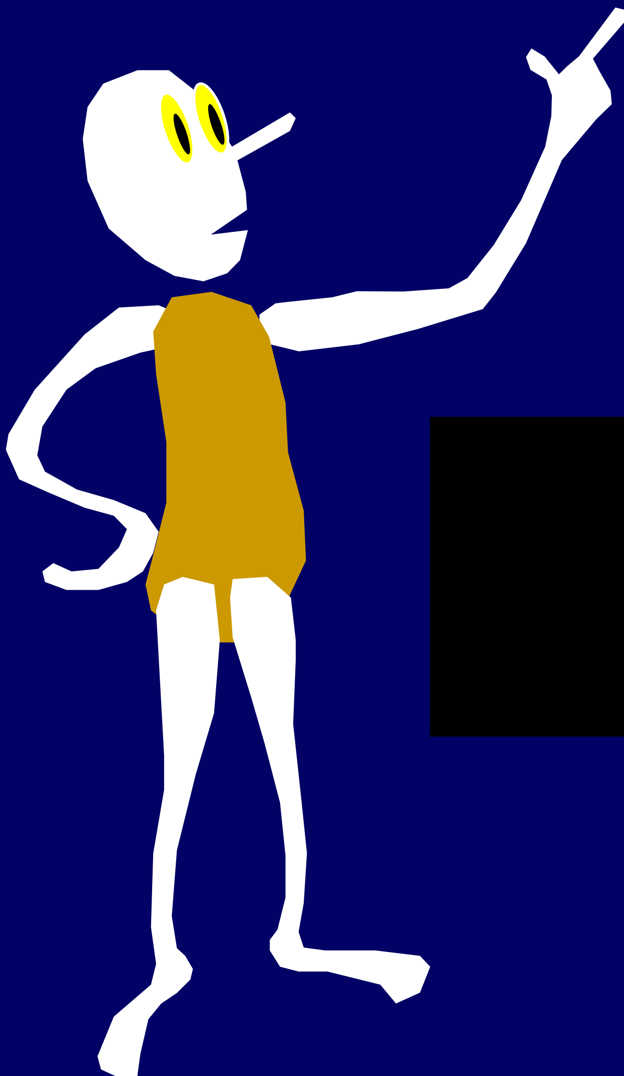


$$
\begin{array}{r|l}
 & 1 + x + \ldots\ldots \\
\hline
1-x & 1 \\
 & -[1 - x] \\
\cline{2-2}
 & \quad x \\
 & \quad -[x - x^2\ ] \\
\cline{2-2}
 & \qquad x^2 \\
 & \qquad -\ldots\ldots
\end{array}
$$

$$1 + aX^1 + a^2X^2 + a^3X^3 + \ldots + a^nX^n + \ldots\ldots = \frac{1}{1 - aX}$$

Geometric Series (Linear Form)

$$\left(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..\right)\left(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..\right) =$$

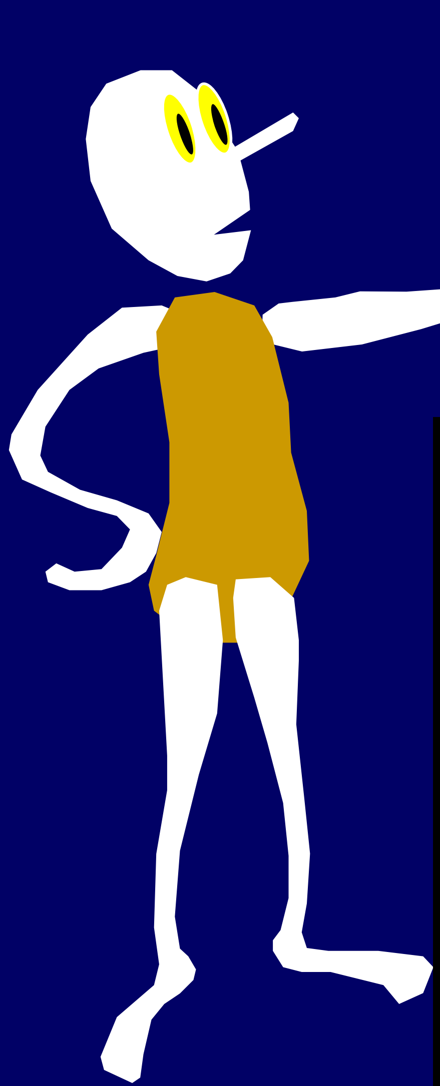


$$\frac{1}{(1 - aX)(1-bX)}$$

Geometric Series (Quadratic Form)

$$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\dots)(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\dots) =$$

$$1 + c_1X^1 + .. + c_k X^k + \dots$$

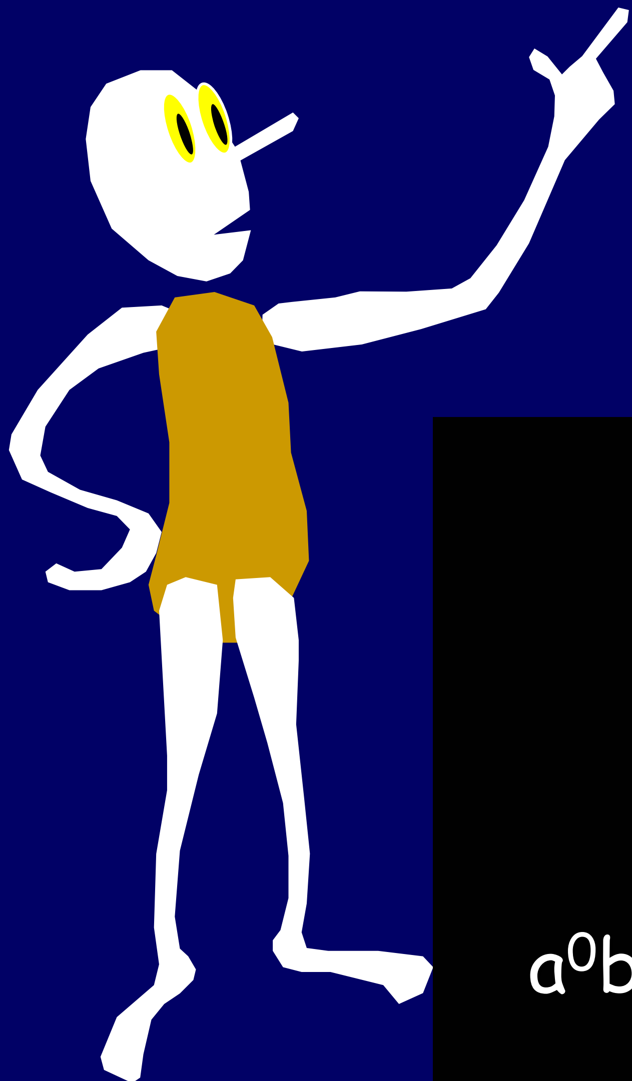


Suppose we multiply this out to get a single, infinite polynomial.

What is an expression for C_n?

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots\ldots)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots\ldots) =$$

$$1 + c_1X^1 + \ldots + c_k\, X^k + \ldots$$

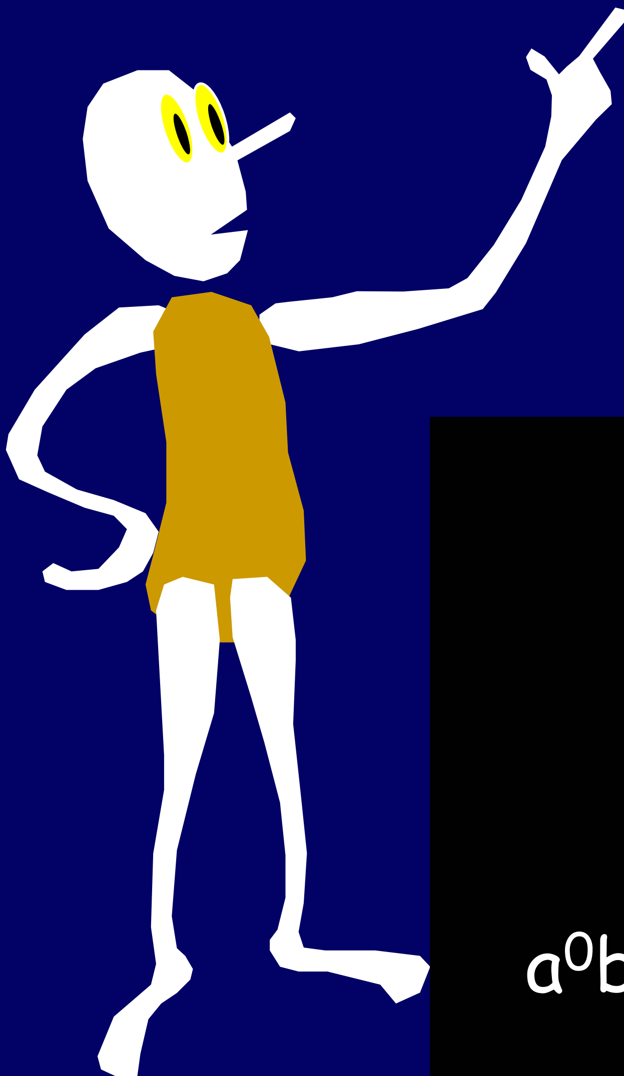


$$c_n =$$

$$a^0b^n + a^1b^{n-1} + \ldots\, a^ib^{n-i} \ldots + a^{n-1}b^1 + a^nb^0$$

$$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\dots)\,(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\dots) =$$

$$1 + c_1X^1 + .. + c_k\,X^k + \dots$$

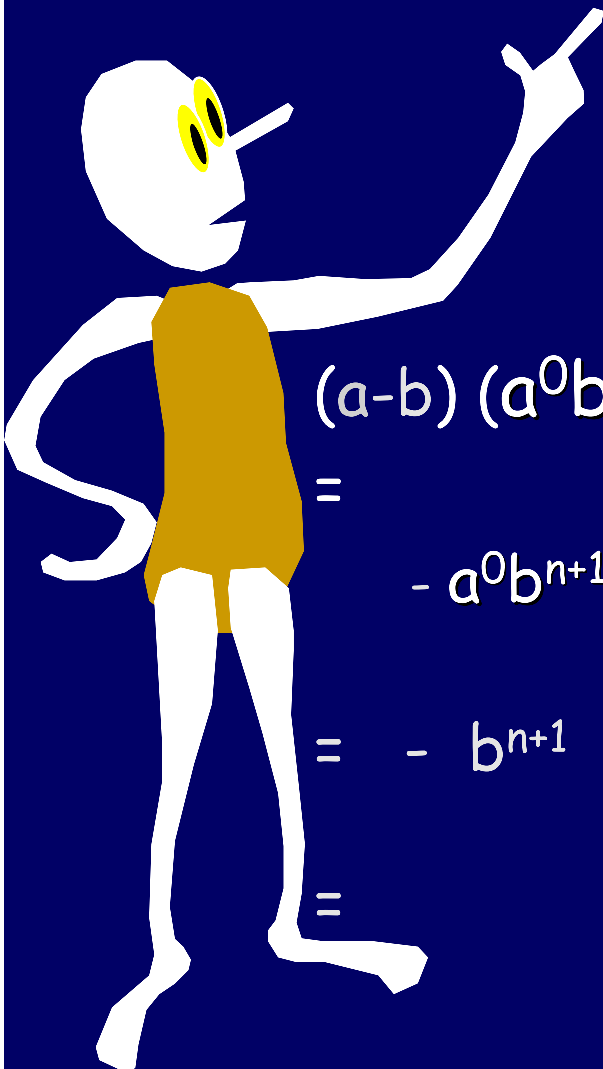


If $a = b$ then

$$c_n = (n+1)(a^n)$$

$$a^0b^n + a^1b^{n-1} + \dots\ a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0$$

$$a^0b^n + a^1b^{n-1} + \dots\ a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0 = \frac{a^{n+1} - b^{n+1}}{a - b}$$



$$(a-b)\ (a^0b^n + a^1b^{n-1} + \dots\ a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0)$$

$$= \quad a^1b^n + \dots\ a^{i+1}b^{n-i} \dots \quad + a^nb^1 + a^{n+1}b^0$$

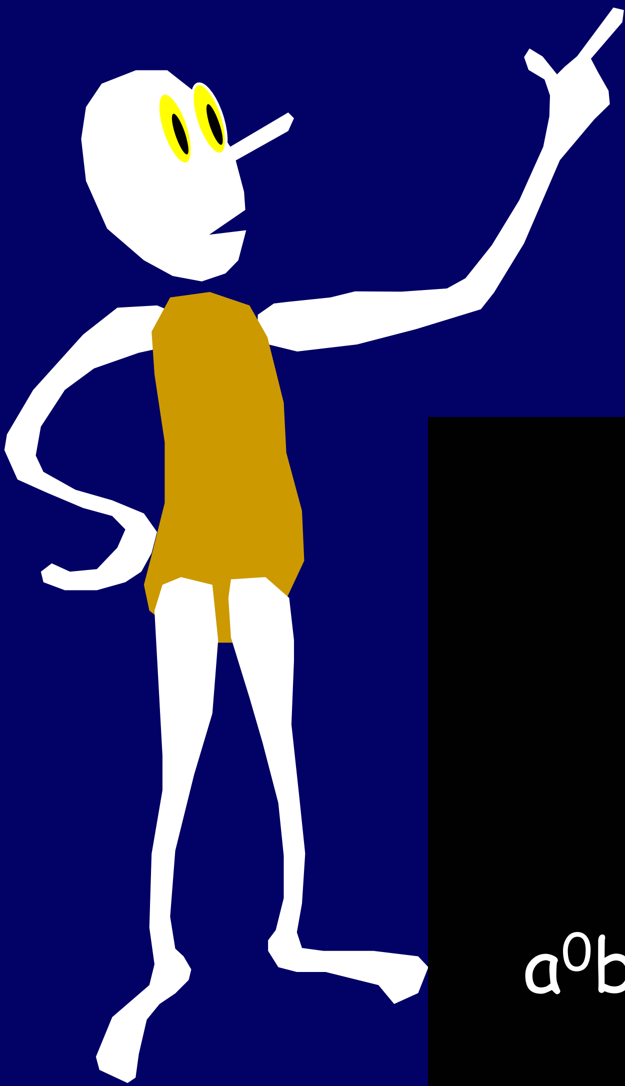
$$- a^0b^{n+1} \quad - a^1b^n \dots \quad a^{i+1}b^{n-i} \dots \quad - a^{n-1}b^2 - a^nb^1$$

$$= \quad - b^{n+1} \quad + \quad a^{n+1}$$

$$= \quad a^{n+1} - b^{n+1}$$

$$\left(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\right)\left(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\right) =$$

$$1 + c_1X^1 + .. + c_k X^k + \dots$$

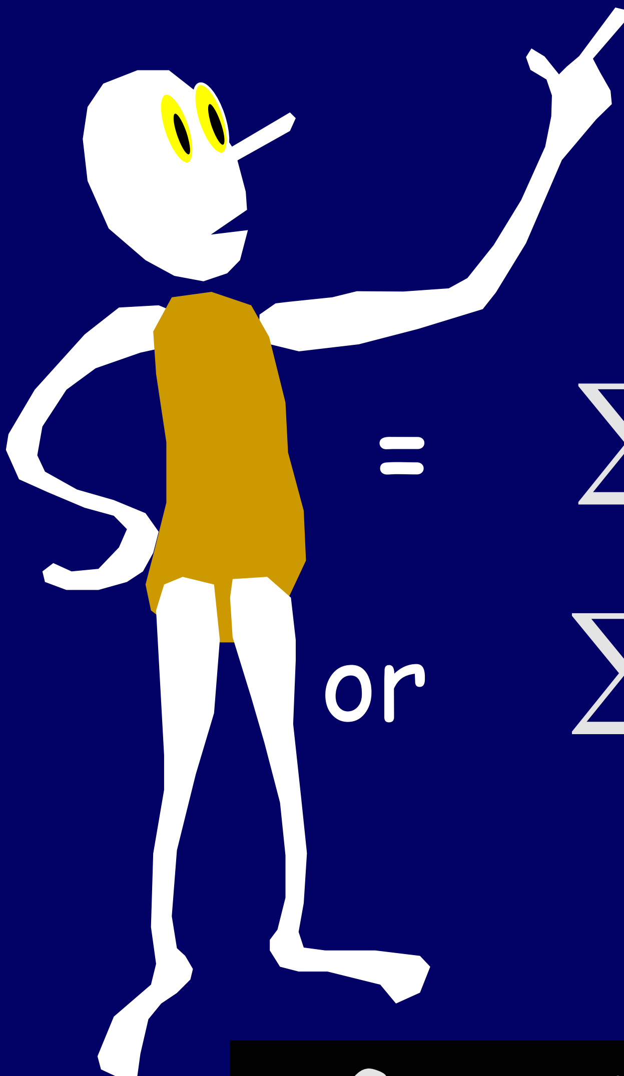


if $a \neq b$ then

$$c_n = \frac{a^{n+1} - b^{n+1}}{a - b}$$

$$a^0b^n + a^1b^{n-1} + \dots a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0$$

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots\ldots)\,(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots\ldots) =$$



$$= \frac{1}{(1 - aX)(1-bX)}$$

$$= \sum_{n=0..\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

or

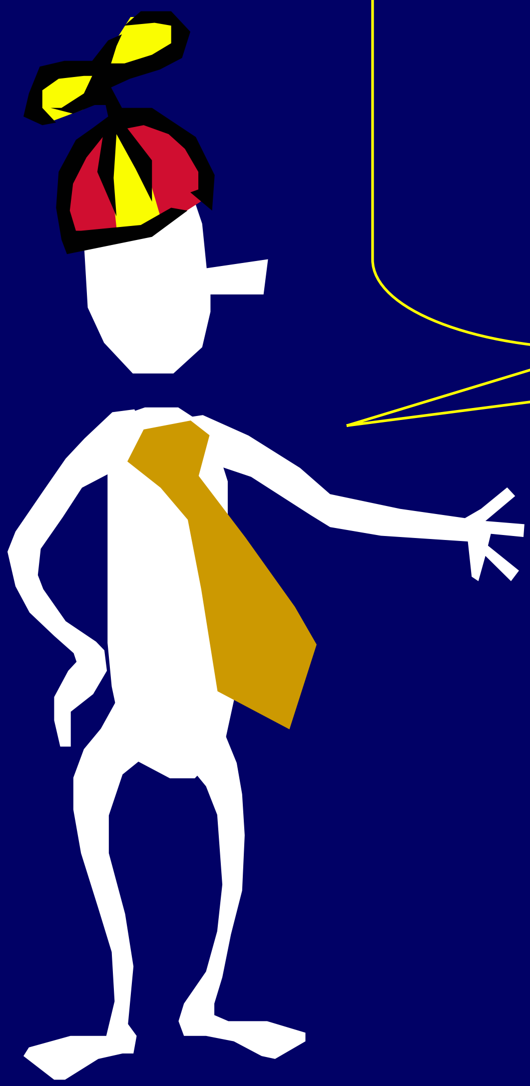
$$\sum_{n=0..\infty} (n+1)a^n \; X^n$$

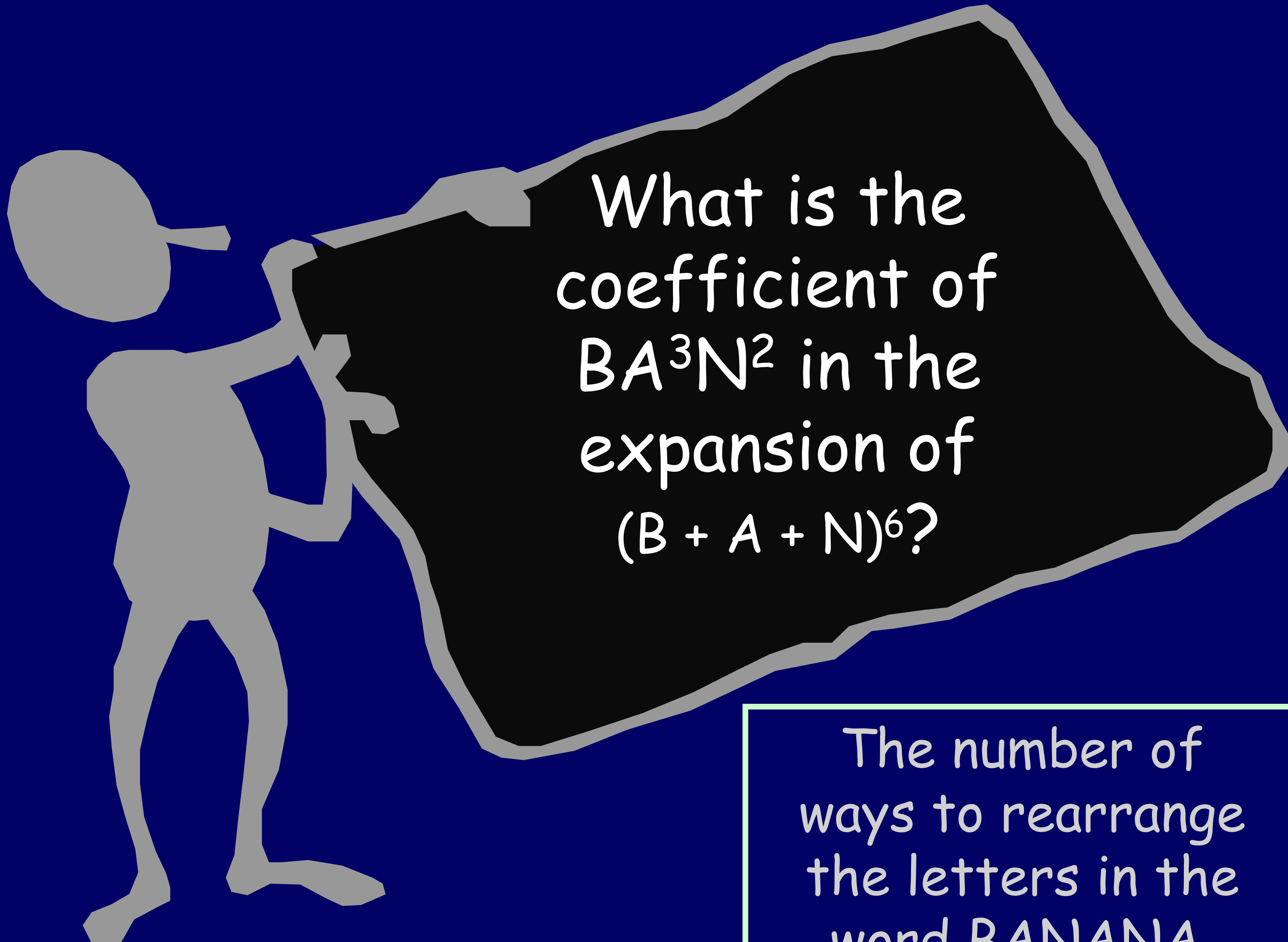
when $a=b$

Geometric Series (Quadratic Form)

Previously, we saw that

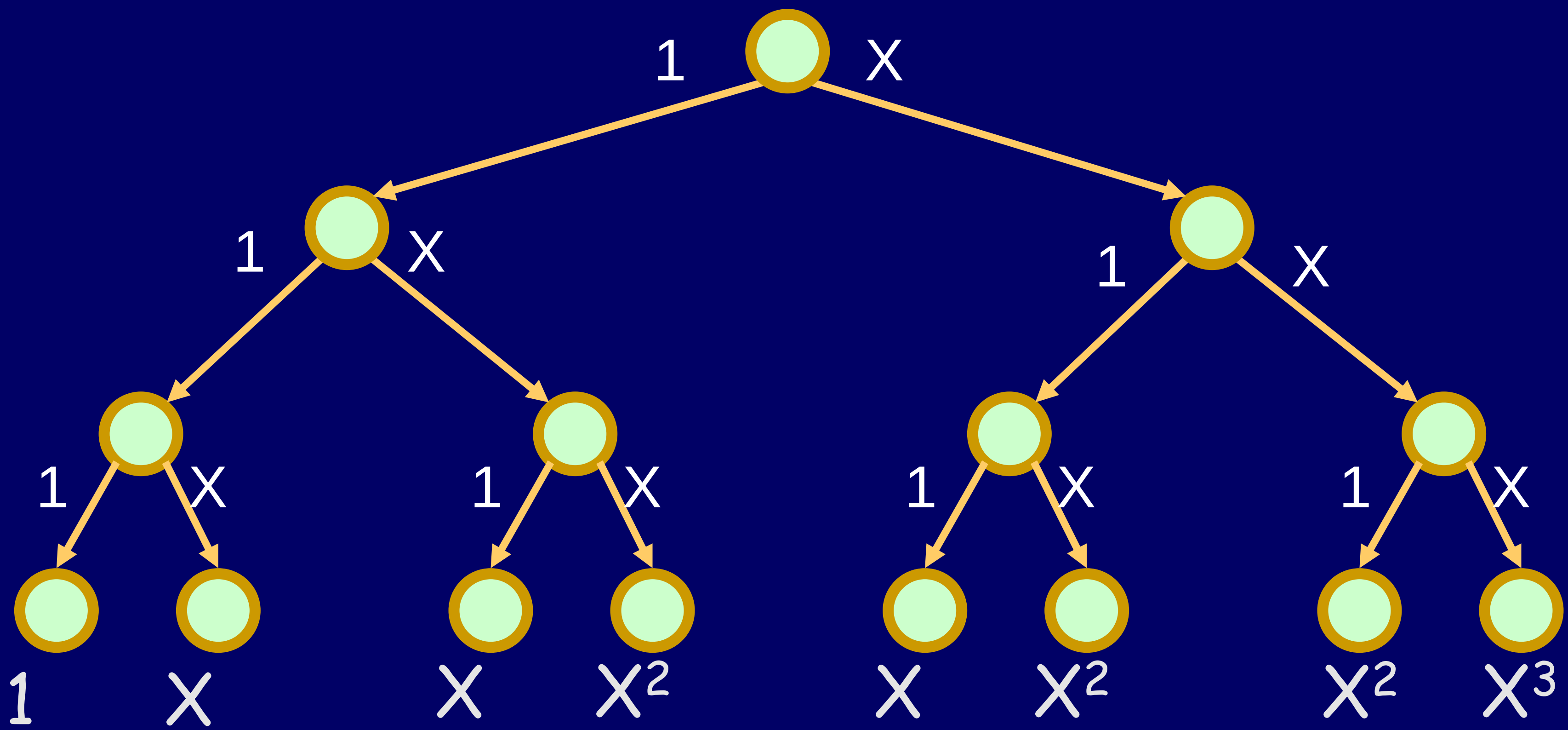
Polynomials Count!





The number of ways to rearrange the letters in the word BANANA.

Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

The Binomial Formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

One polynomial, two representations

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

"Product form" or
"Generating form"

"Additive form" or
"Expanded form"

Power Series Representation

$$\boxed{(1+x)^n} = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$= \boxed{\sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k}$$

Since $\binom{n}{k} = 0$ if $k > n$

"Closed form" or "Generating form"

"Power series" ("Taylor series") expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x=1$.

$$2^n = \underbrace{\sum_{k=0}^{n} \binom{n}{k}}_{\text{The number of subsets of an } n\text{-element set}}$$

By varying x, we can discover new identities

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let *x= -1*.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

The number of even-sized subsets of an *n* element set is the same as the number of odd-sized subsets.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

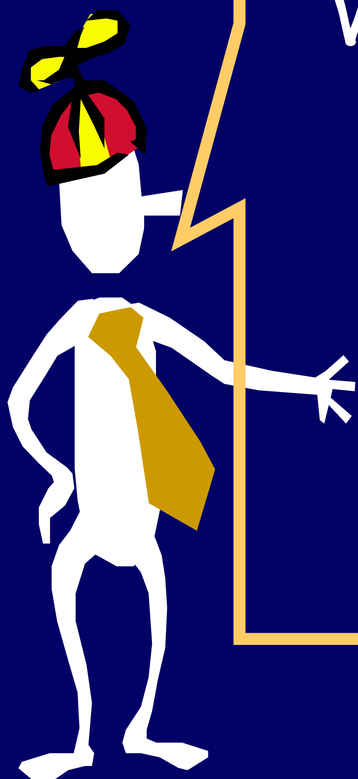
Let *x= -1*.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

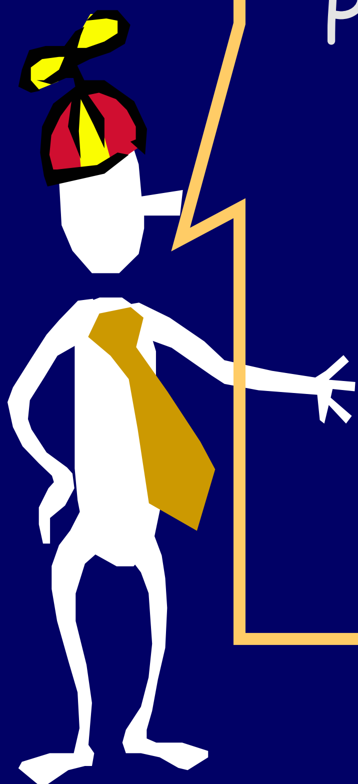
$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



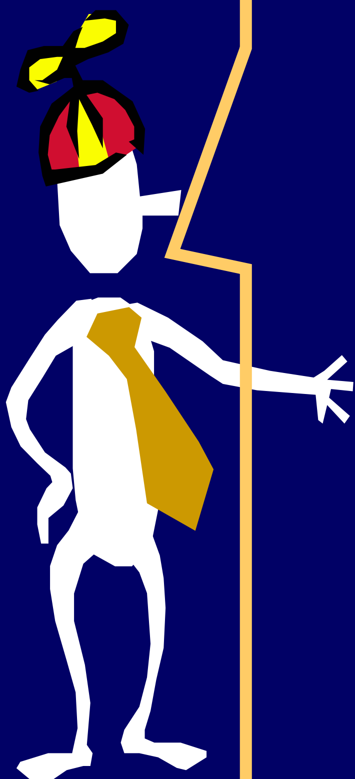
We could discover new identities by substituting in different numbers for X. One cool idea is to try complex roots of unity, however, the lecture is going in another direction.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



Proofs that work by manipulating algebraic forms are called "algebraic" arguments. Proofs that build a 1-1 onto correspondence are called "combinatorial" arguments.

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$



Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We gave an algebraic proof that

$$|O_n| = |E_n|$$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

A <u>combinatorial</u> proof must construct a one-to-one correspondence between O_n and E_n

An attempt at a correspondence

Let f_n be the function that takes an n-bit string and flips all its bits.

f_n is clearly a one-to-one and onto function

for odd *n*. *E.g.* in f_7 we have

0010011 → 1101100

1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100

101010 → 010101

Uh oh. Complementing maps evens to evens!

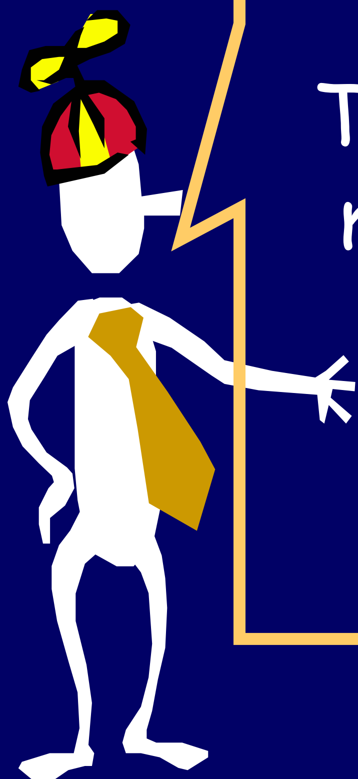
A correspondence that works for all *n*

Let f_n be the function that takes an n-bit string and flips only *the first bit*. For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle:
k^{th} row are the coefficients of $(1+X)^k$

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

k^{th} Row Of Pascal's Triangle:

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{k}, \ldots \binom{n}{n}$$

$$(1+X)^0 = 1$$

$$(1+X)^1 = 1 + 1X$$

$$(1+X)^2 = 1 + 2X + 1X^2$$

$$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$$

$$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$$

Inductive definition of kth entry of nth row:
Pascal(n,0) = Pacal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n,k)

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

"Pascal's Triangle"

$$\binom{0}{0}=1$$

$$\binom{1}{0}=1 \qquad \binom{1}{1}=1$$

$$\binom{2}{0}=1 \qquad \binom{2}{1}=2 \qquad \binom{2}{2}=1$$

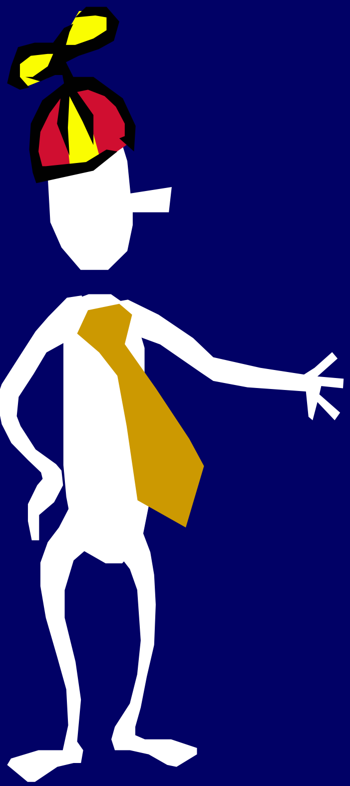
$$\binom{3}{0}=1 \qquad \binom{3}{1}=3 \qquad \binom{3}{2}=3 \qquad \binom{3}{3}=1$$

Al-Karaji, Baghdad 953-1029

Chu Shin-Chieh 1303
The Precious Mirror of the Four Elements
. . . Known in Europe by 1529

Blaise Pascal 1654

Pascal's Triangle



1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

Summing The Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1	=1
1 + 1	=2
1 + 2 + 1	=4
1 + 3 + 3 + 1	=8
1 + 4 + 6 + 4 + 1	=16
1 + 5 + 10 + 10 + 5 + 1	=32
1 + 6 + 15 + 20 + 15 + 6 + 1	=64

$\vdots$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

6+20+6 = 1+15+15+1

Summing on 1st Avenue

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$$\sum_{i=k}^{n} i = \sum_{i=k}^{n} \binom{i}{1} = \binom{n+1}{2} = \frac{n \cdot (n+1)}{2}$$

Summing on kth Avenue

1

1 1

1 2 1

1 3 3 1

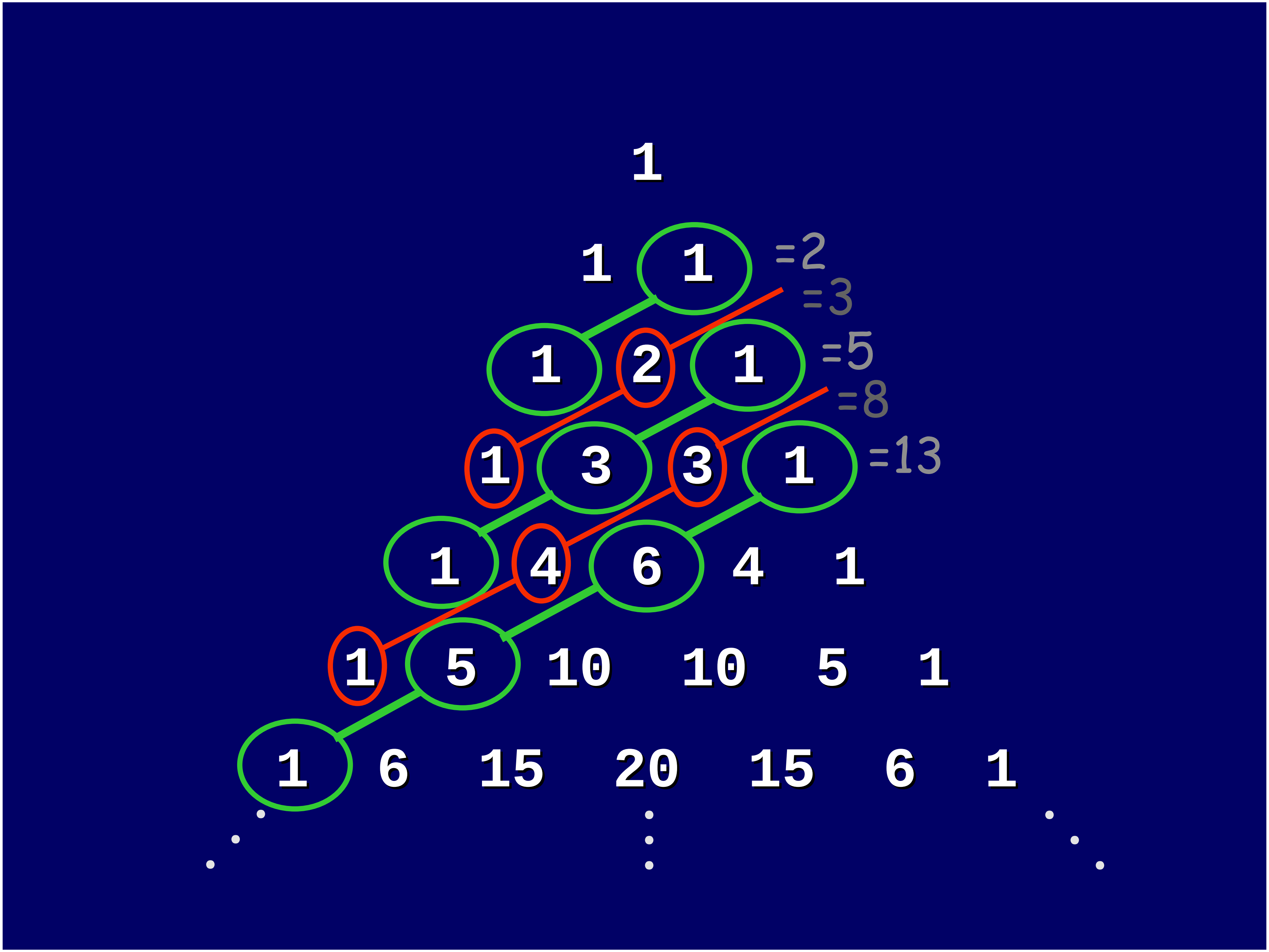
1 4 6 4 1

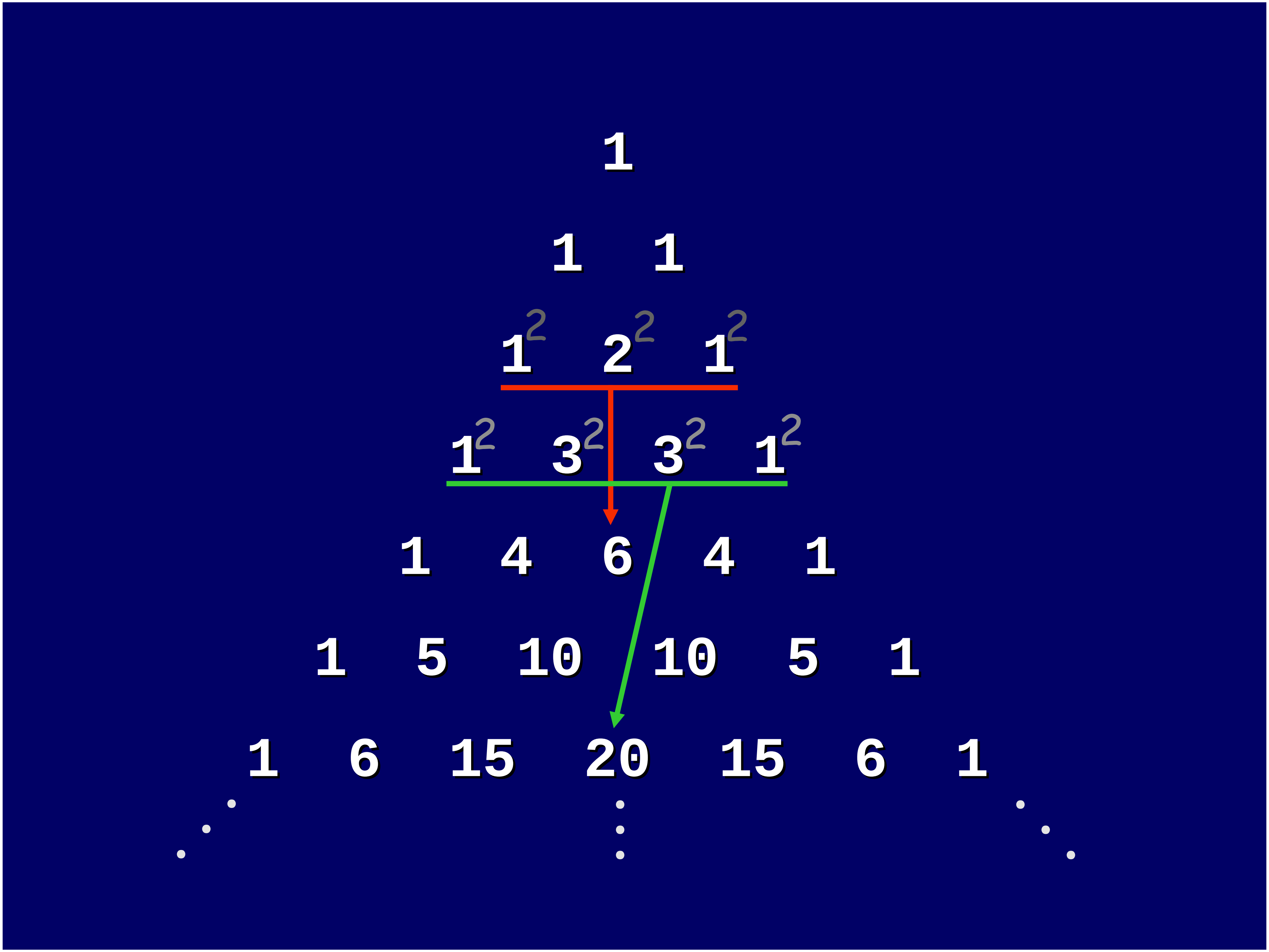
1 5 10 10 5 1

1 6 15 20 15 6 1

$\vdots \qquad \ddots$

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

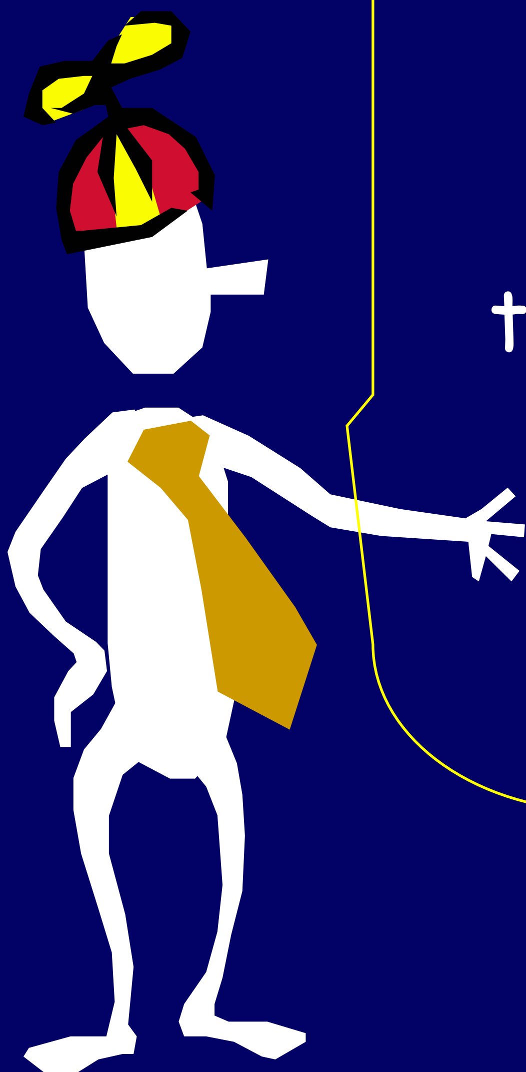

1
1 1 =2
=3
1 2 1 =5
=8
1 3 3 1 =13
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1


1
1 1
1² 2² 1²
1² 3² 3² 1²
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Al-Karaji Squares

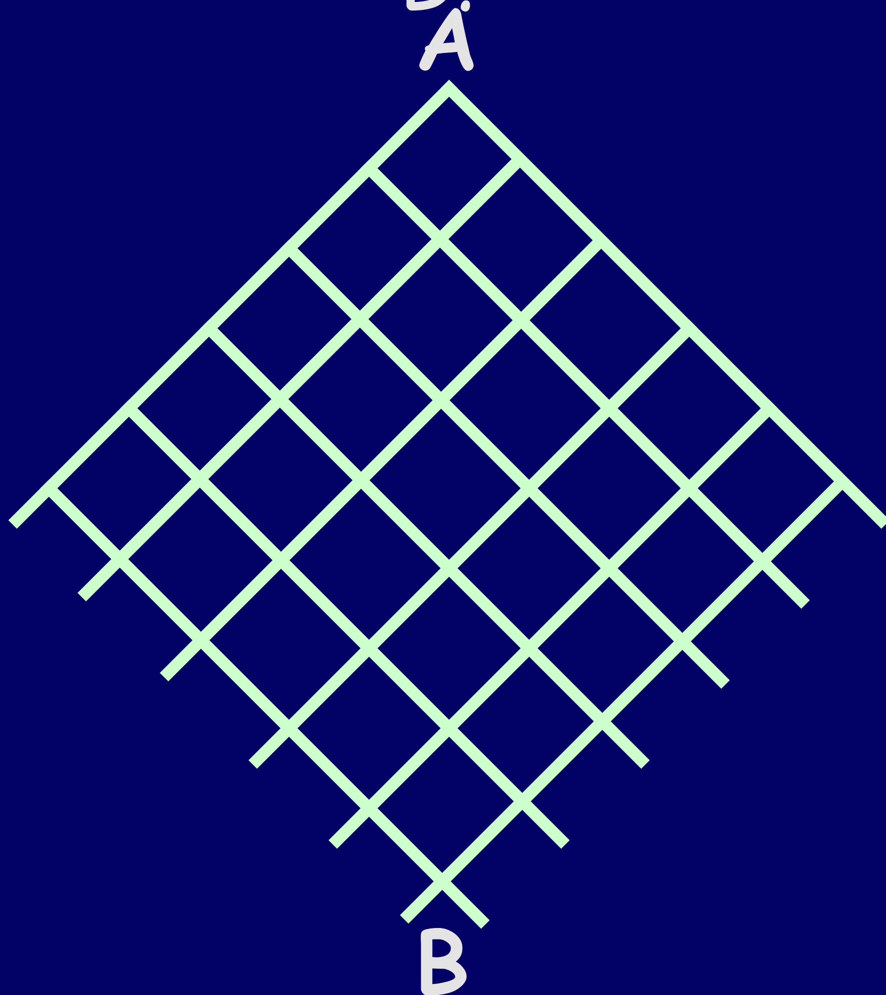
```
                    1                              =0
                 1     1                           =1
              1     2 + 2*1                        =4
           1     3 + 2*3     1                     =9
        1     4 + 2*6   4     1                    =16
     1     5 + 2*10    10     5     1              =25
  1     6 + 2*15    20    15     6     1           =36
```

.
.
.



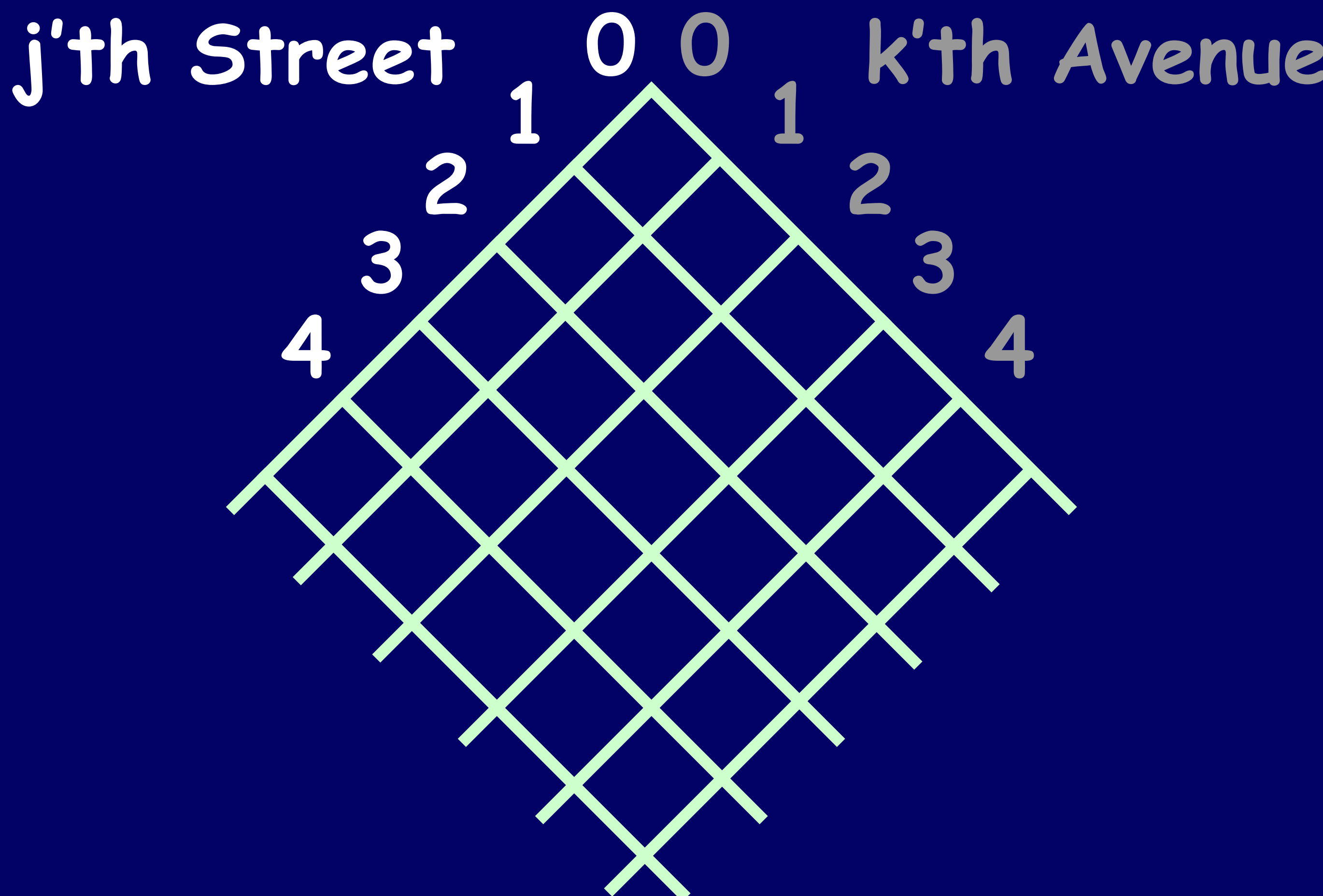
All these properties can be proved inductively and algebraically. We will give combinatorial proofs using the Manhattan block walking representation of binomial coefficients.

How many shortest routes from A to B?



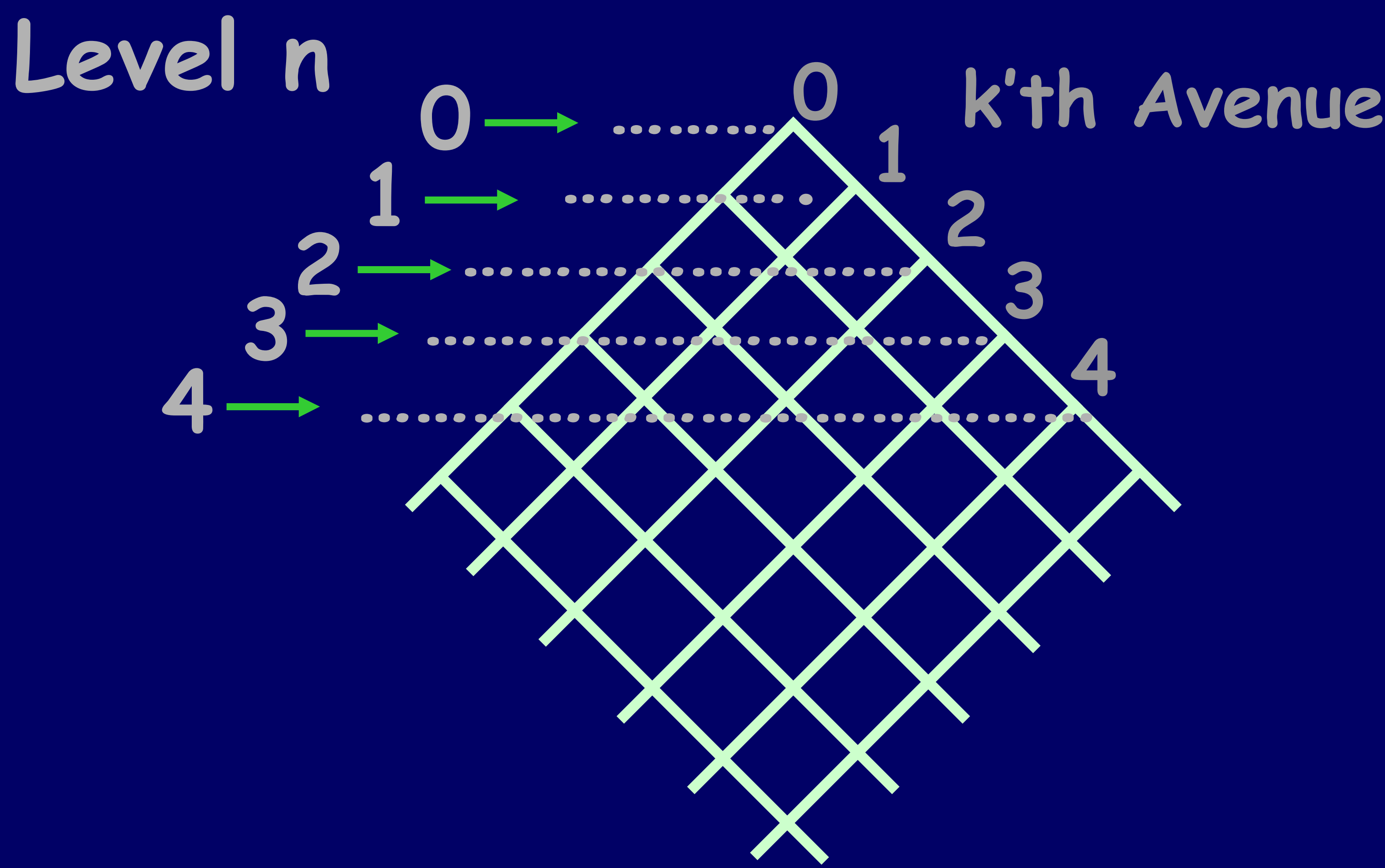
$$\binom{10}{5}$$

Manhattan



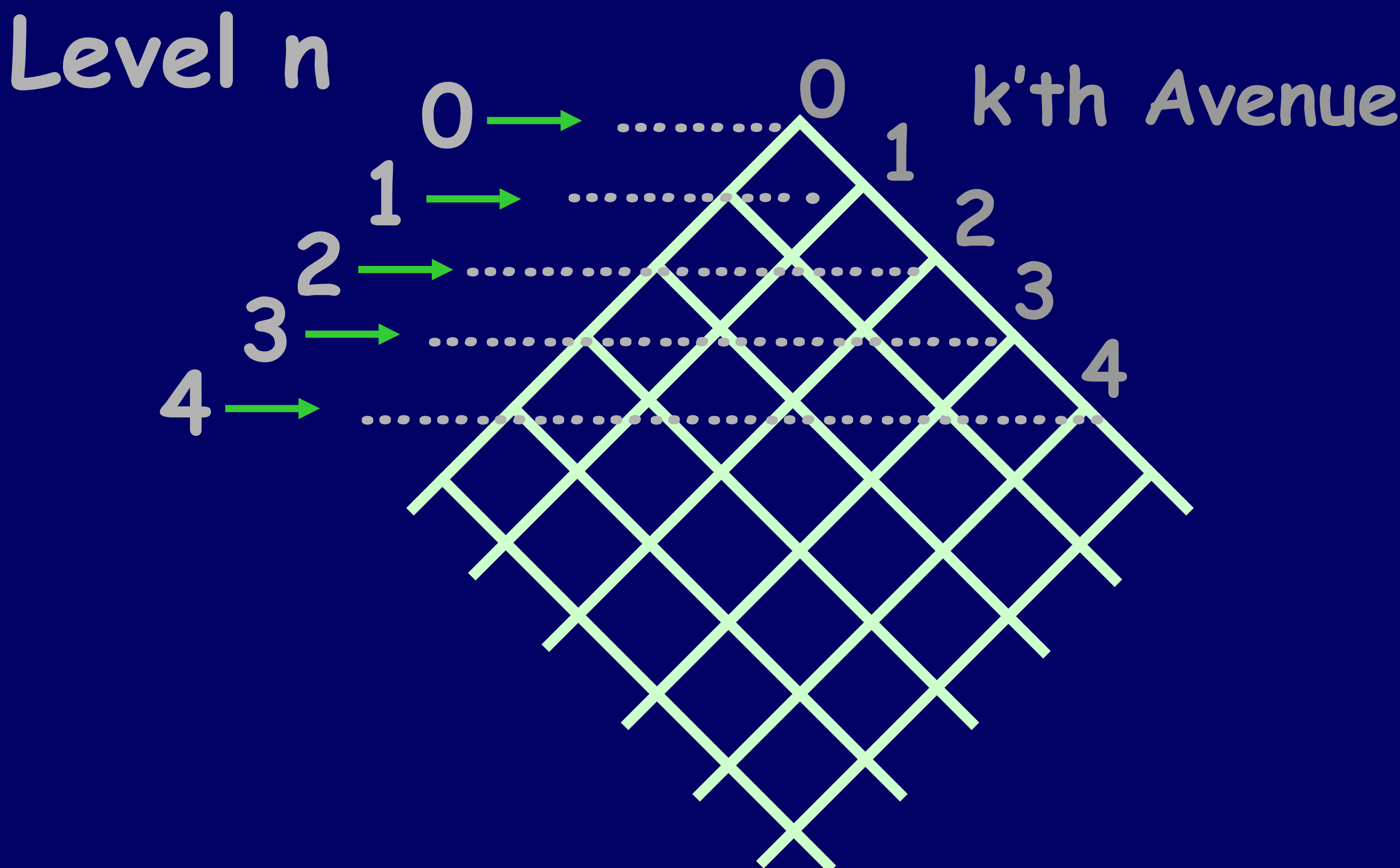
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k).

Manhattan

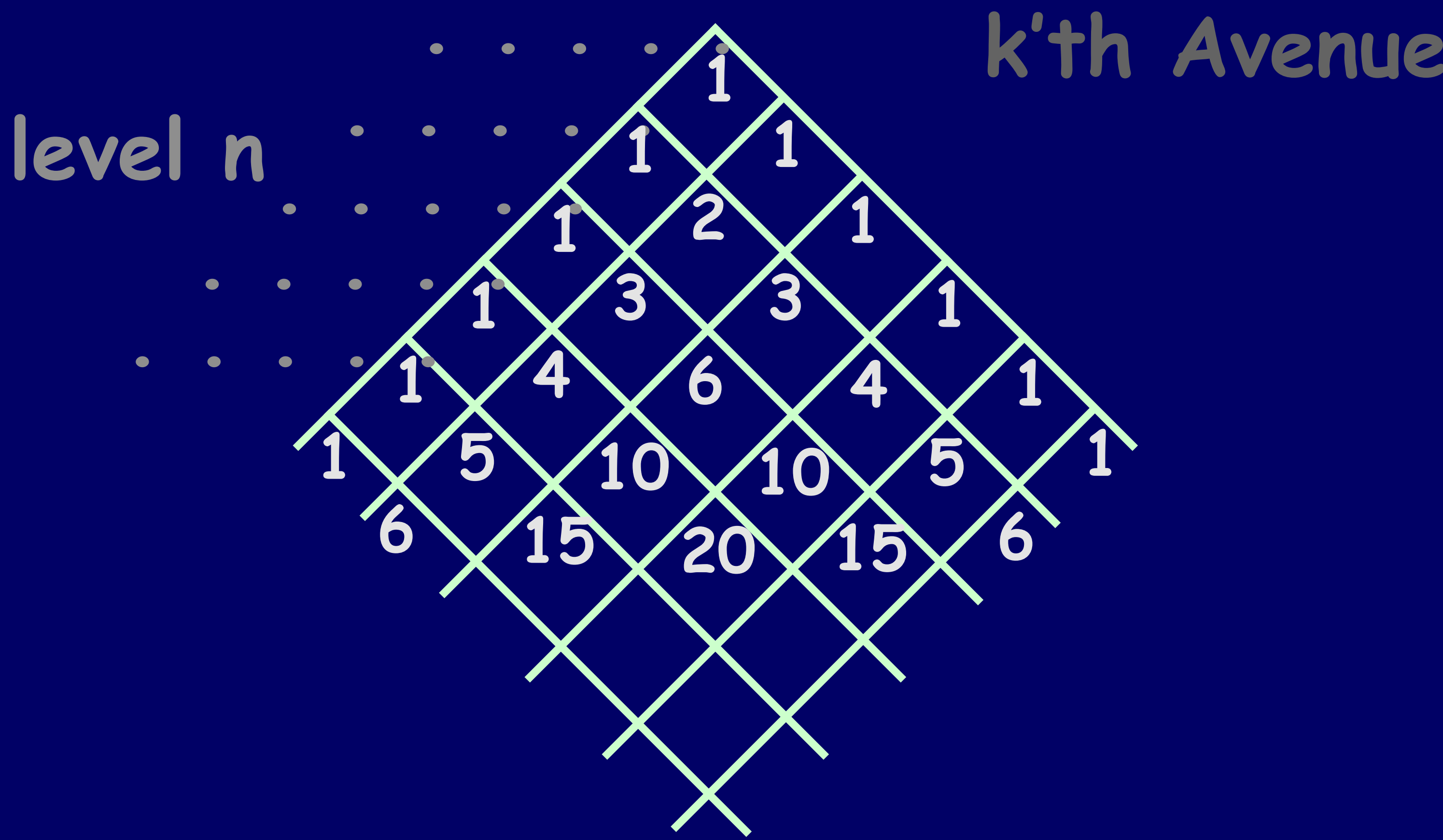


There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k).

Manhattan



There are $\binom{n}{k}$ shortest routes from (0,0) to Level n and k^{th} Avenue.


k'th Avenue
level n
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6

k'th Avenue

level n

1

1 1

1 2 1

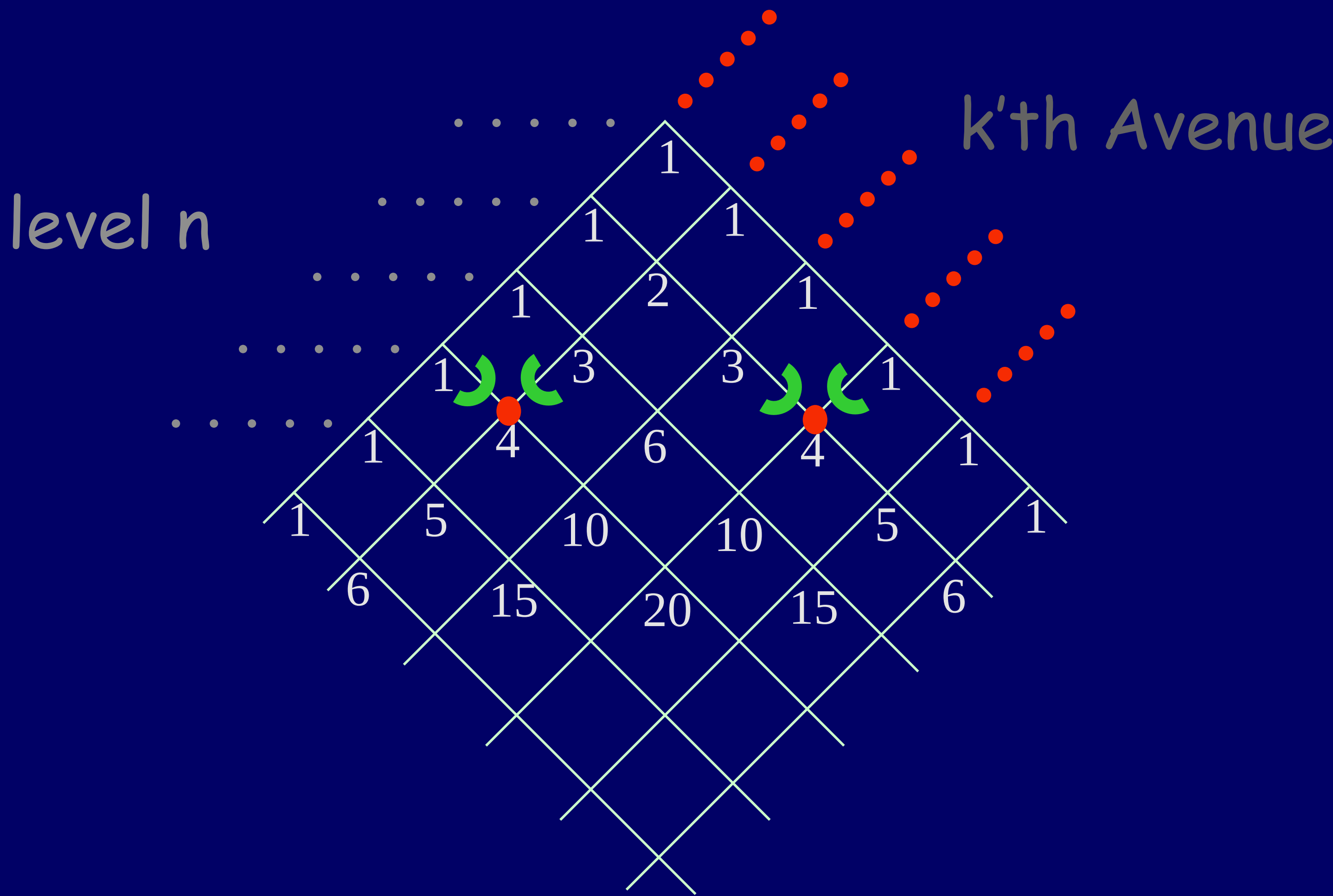
1 3 3 1

1 4 6 4 1

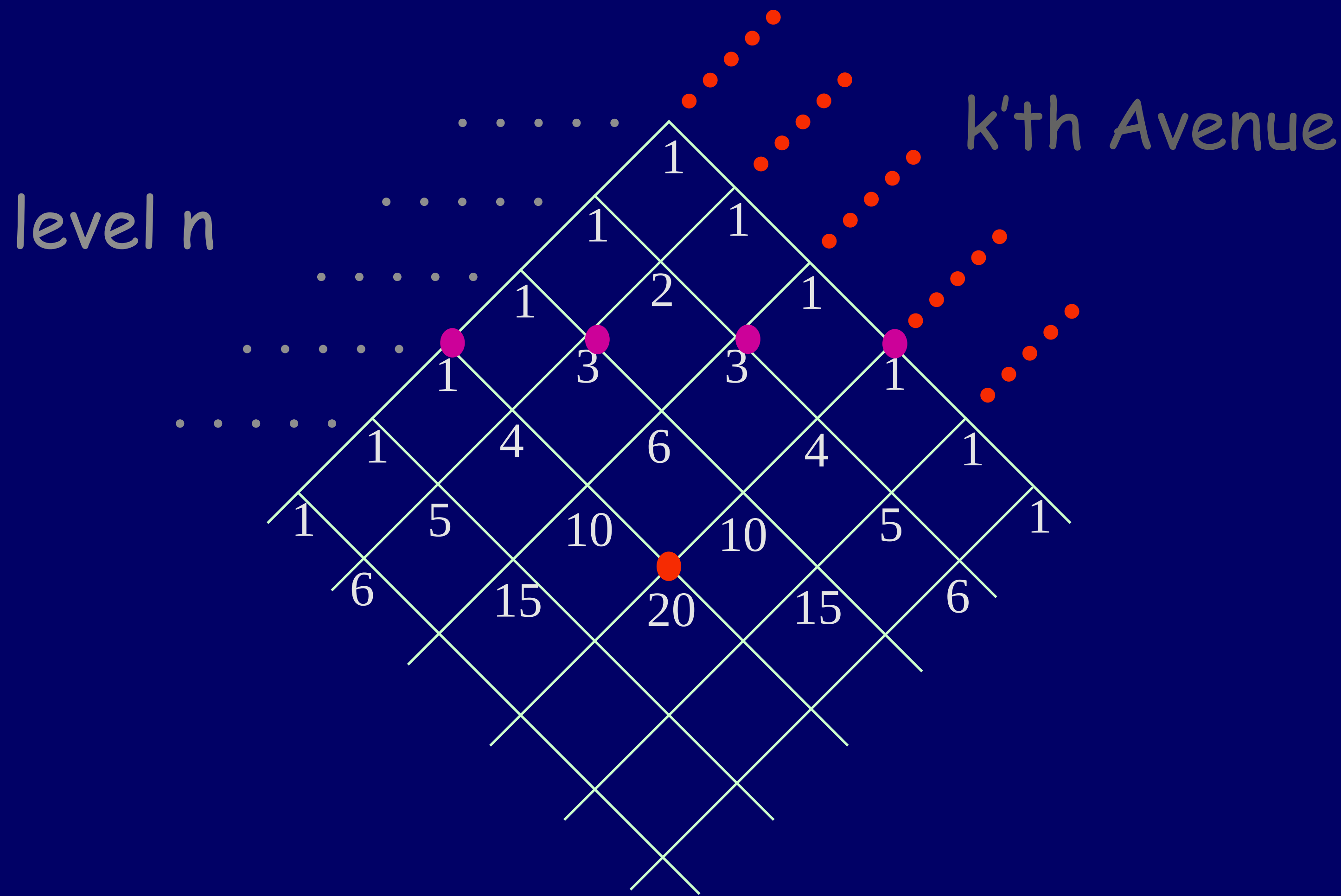
1 5 10 10 5 1

6 15 20 15 6

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



$$\sum_{k\ \text{even}} \binom{n}{k} = 2^{n-1}$$



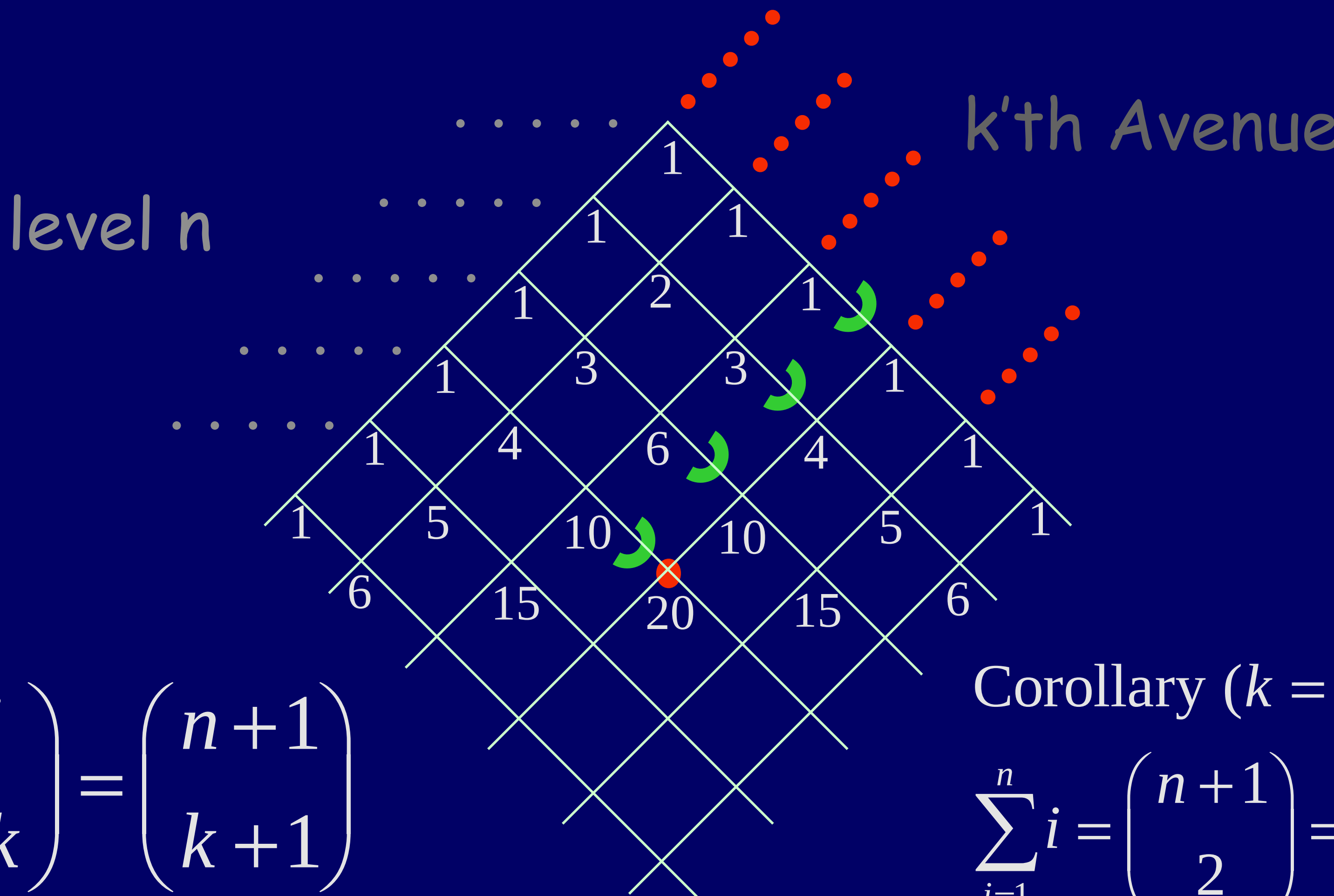
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

By convention:

$$0! = 1$$ (empty product = 1)

$$\binom{n}{k} = 1 \quad \text{if } k = 0$$

$$\binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n$$



$$\sum_{i=1}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Corollary ($k = 1$)

$$\sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

Application (Al-Karaji):

$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$= (1\cdot 0 + 1) + (2\cdot 1 + 2) + (3\cdot 2 + 3) + \cdots + (n(n-1) + n)$$

$$= 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + n(n-1) + \sum_{i=1}^{n} i$$

$$= 2\left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n}{2}\right] + \binom{n+1}{2}$$

$$= 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{(2n+1)(n+1)n}{6}$$

Vector Programs

Let's define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable $V^{\rightarrow}$ can be thought of as:

< *, *, *, *, *, *, ... >

0 1 2 3 4 5 . . . .

Vector Programs

Let k stand for a scalar constant

$\langle k \rangle$ will stand for the vector $\langle k,0,0,0,\ldots \rangle$

$\langle 0 \rangle = \langle 0,0,0,0,\ldots. \rangle$

$\langle 1 \rangle = \langle 1,0,0,0,\ldots \rangle$

$\vec{V} + \vec{T}$ means to add the vectors position-wise.

$\langle 4,2,3,\ldots \rangle + \langle 5,1,1,\ldots. \rangle = \langle 9,3,4,\ldots \rangle$

Vector Programs

RIGHT($V^{\rightarrow}$) means to shift every number in $V^{\rightarrow}$ one position to the right and to place a 0 in position 0.

RIGHT(<1,2,3, …>) = <0,1,2,3,. …>

Vector Programs

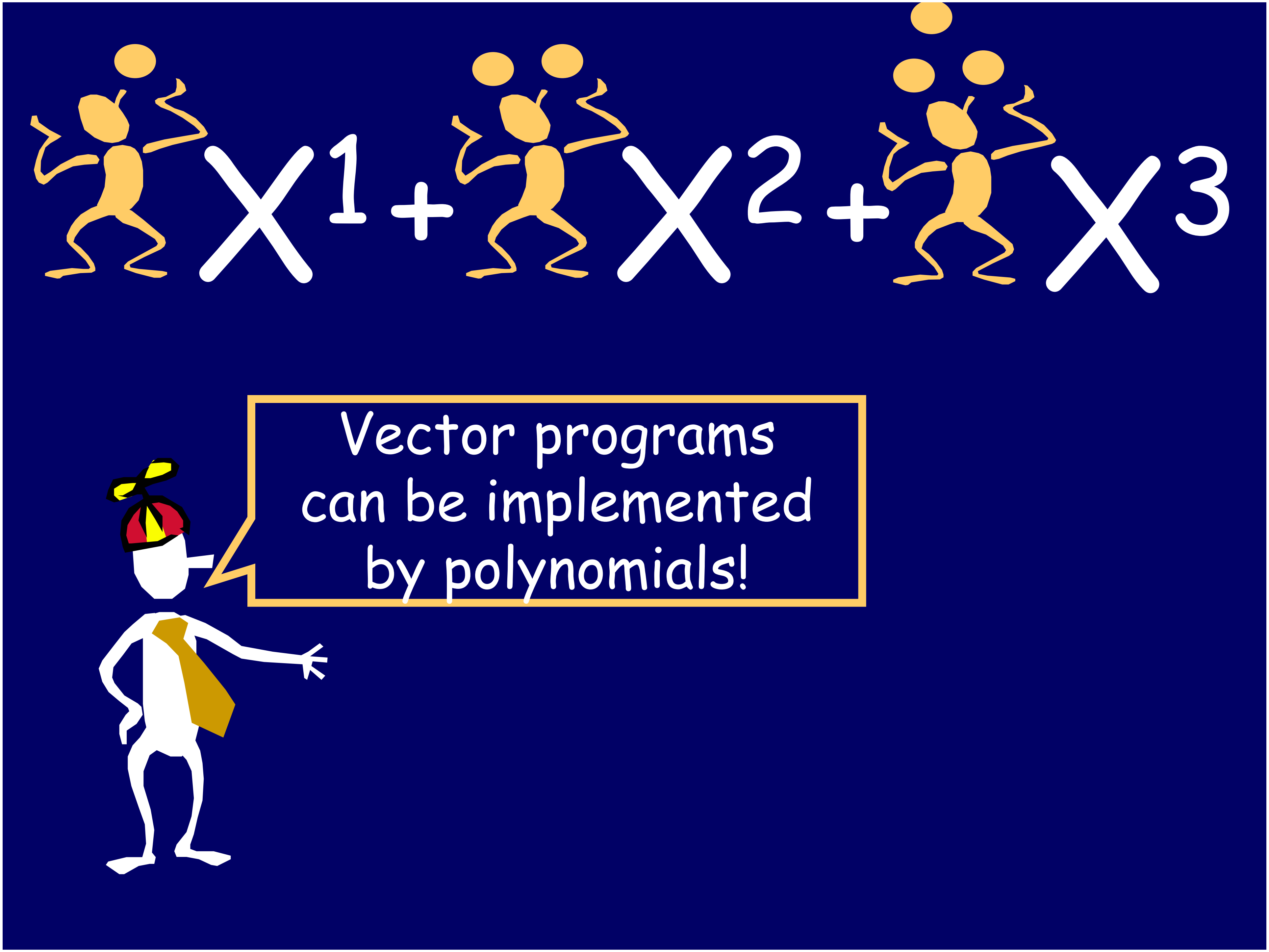
Example:	Stare
$V^{\rightarrow} := \langle 6 \rangle;$	$V^{\rightarrow} = \langle 6,0,0,0,.. \rangle$
$V^{\rightarrow} := \mathrm{RIGHT}(V^{\rightarrow}) + \langle 42 \rangle;$	$V^{\rightarrow} = \langle 42,6,0,0,.. \rangle$
$V^{\rightarrow} := \mathrm{RIGHT}(V^{\rightarrow}) + \langle 2 \rangle;$	$V^{\rightarrow} = \langle 2,42,6,0,.. \rangle$
$V^{\rightarrow} := \mathrm{RIGHT}(V^{\rightarrow}) + \langle 13 \rangle;$	$V^{\rightarrow} = \langle 13,2,42,6,. \rangle$

$$V^{\rightarrow} = \langle\, 13,\ 2,\ 42,\ 6,\ 0,\ 0,\ 0,\ .\ .\ .\ \rangle$$

Vector Programs

Example:	Stare
$V^{\to} := \langle 1 \rangle;$	$V^{\to} = \langle 1,0,0,0,.. \rangle$
Loop n times:	$V^{\to} = \langle 1,1,0,0,.. \rangle$
$V^{\to} := V^{\to} + \mathrm{RIGHT}(V^{\to});$	$V^{\to} = \langle 1,2,1,0,.. \rangle$
	$V^{\to} = \langle 1,3,3,1,. \rangle$

$V^{\to}$ = n^{th} row of Pascal's triangle.


$X^1 + X^2 + X^3$
Vector programs can be implemented by polynomials!

Programs -----> Polynomials

The vector $\vec{V} = \langle a_0, a_1, a_2, \ldots \rangle$ will be represented by the polynomial:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

Formal Power Series

The vector $V^{\rightarrow} = \langle a_0, a_1, a_2, \ldots \rangle$ will be represented by the formal power series:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

$V^{\rightarrow} = \langle a_0, a_1, a_2, \ldots \rangle$

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

$\langle 0 \rangle$ is represented by 0

$\langle k \rangle$ is represented by k

$V^{\rightarrow} + T^{\rightarrow}$ is represented by $(P_V + P_T)$

RIGHT($V^{\rightarrow}$) is represented by $(P_V X)$

Vector Programs

Example:

$\vec{V} := \langle 1 \rangle;$ $\quad$ $P_V := 1;$

Loop n times:

$\vec{V} := \vec{V} + \mathrm{RIGHT}(\vec{V});$ $\quad$ $P_V := P_V + P_V X;$

$\vec{V}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

$V^{\rightarrow} := \langle 1 \rangle;$ $\qquad\qquad$ $P_V := 1;$

Loop n times:

$V^{\rightarrow} := V^{\rightarrow} + \mathrm{RIGHT}(V^{\rightarrow});$ $\qquad\qquad$ $P_V := P_V\,(1+X);$

$V^{\rightarrow}$ = n^{th} row of Pascal's triangle.

Vector Programs

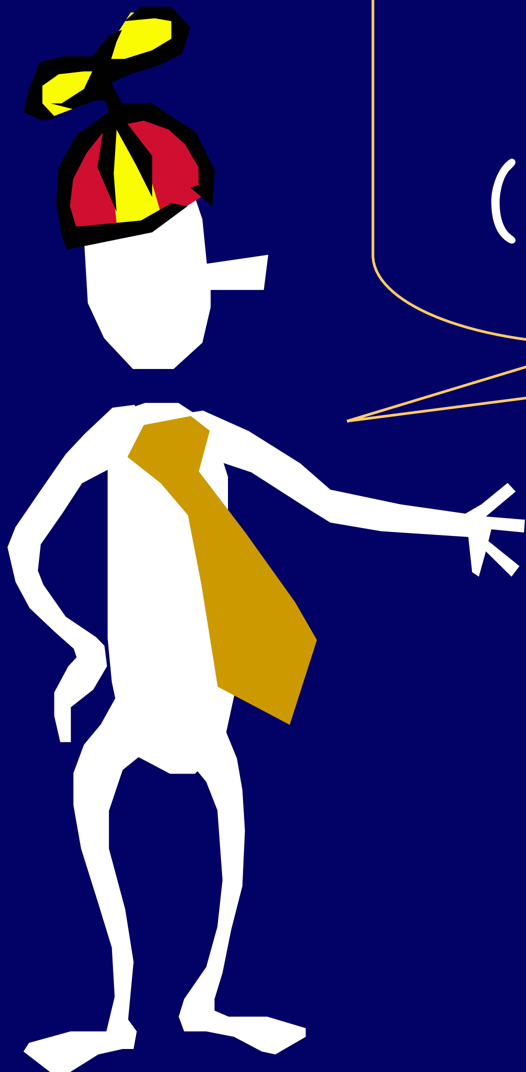
Example:

$$\left.\begin{array}{l} \vec{V} := \langle 1 \rangle; \\ \\ \text{Loop } n \text{ times:} \\ \quad \vec{V} := \vec{V} + \mathrm{RIGHT}(\vec{V}); \end{array}\right\} \quad P_V = (1+X)^n$$

$\vec{V}$ = n^{th} row of Pascal's triangle.

What is the coefficient of X^k in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + \ldots .)^n ?$$



Each path in the choice tree for the cross terms has n choices of exponent $e_1, e_2, \ldots, e_n \geq 0$. Each exponent can be any natural number.

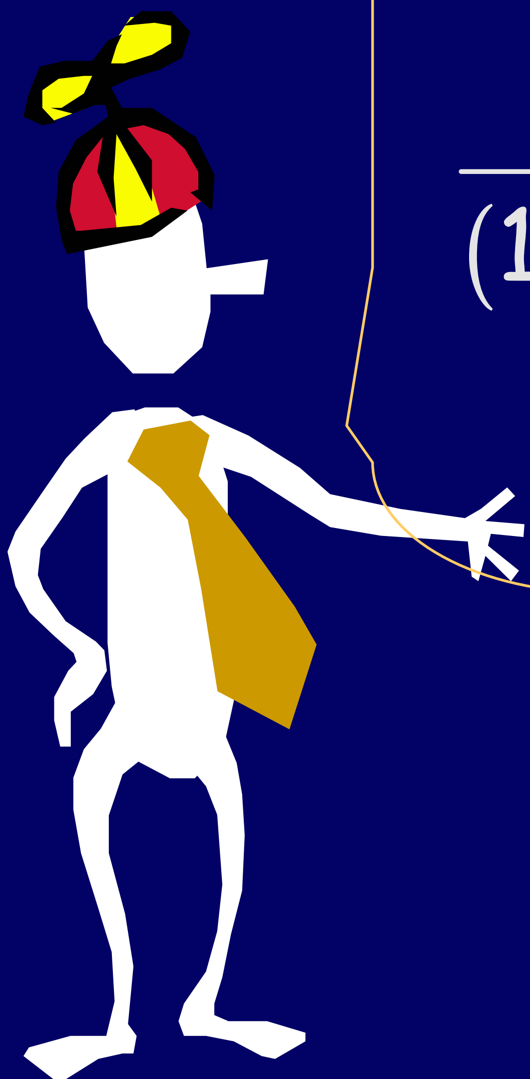
Coefficient of X^k is the number of non-negative solutions to:

$$e_1 + e_2 + \ldots + e_n = k$$

What is the coefficient of X^k in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + \ldots)^n\ ?$$

$$\binom{n+k-1}{n-1}$$



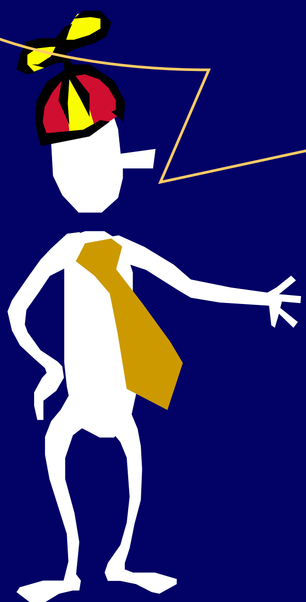
$$(1 + X + X^2 + X^3 + X^4 + \ldots\ldots)^n =$$

$$\frac{1}{(1-X)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} X^k$$

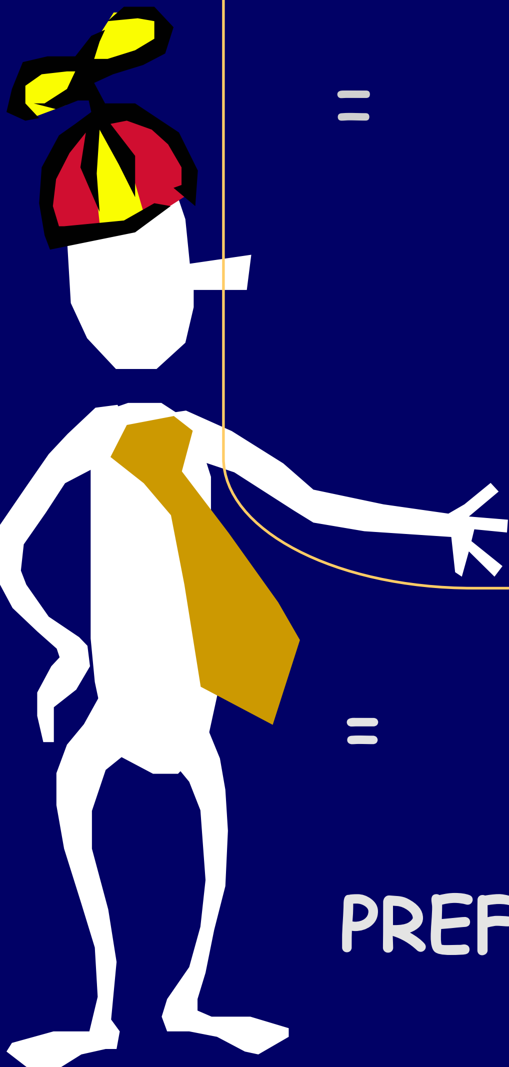
What is the coefficient of X^k in the expansion of:

$$(a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots)\ (1 + X + X^2 + X^3 + \ldots)$$

$$= (a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots)\ /\ (1 - X) \qquad ?$$



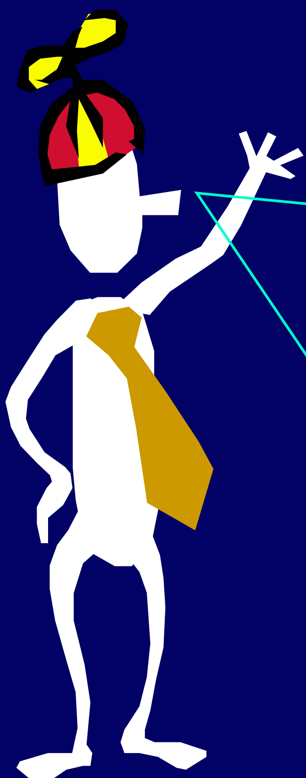
$$a_0 + a_1 + a_2 + \ldots + a_k$$



$$(a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots) / (1 - X)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{i=k} a_i\right)X^k$$

$$= \mathrm{PREFIXSUM}(a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots)$$



Let’s add an instruction called PREFIXSUM to our VECTOR language.

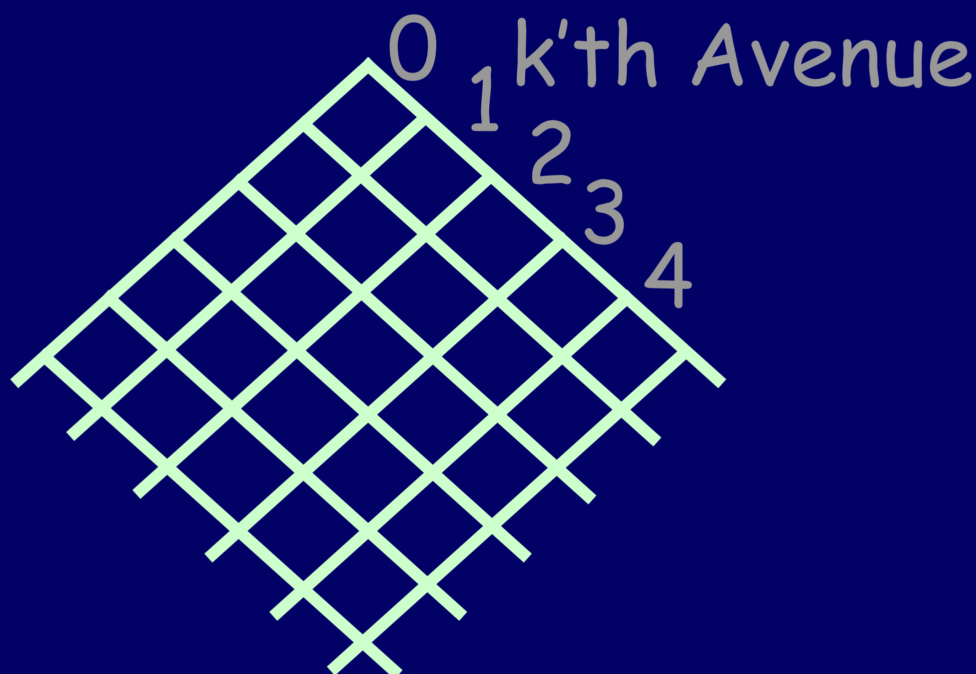
$$W^{\rightarrow} := \text{PREFIXSUM}(V^{\rightarrow})$$

means that the i^{th} position of W contains the sum of all the numbers in V from positions 0 to i.

What does this program output?

$\vec{V} := \vec{1}$;

Loop k times: $\vec{V}$:= PREFIXSUM($\vec{V}$) ;



Can you see how PREFIXSUM can be represented by a familiar polynomial expression?

$$\vec{W} := \mathrm{PREFIXSUM}(\vec{V})$$

is represented by

$$P_W = P_V \,/\, (1-X)$$
$$= P_V\,(1+X+X^2+X^3+ \ldots\ldots)$$

Al-Karaji Program

```
Zero_Ave    := PREFIXSUM(<1>);
First_Ave   := PREFIXSUM(Zero_Ave);
Second_Ave :=PREFIXSUM(First_Ave);

Output:=
     First_Ave + 2*RIGHT(Second_Ave)
```

OUTPUT$^{\rightarrow}$ = <1, 4, 9, 25, 36, 49, …. >

Al-Karaji Program

$\text{Zero\_Ave} = 1/(1-X);$

$\text{First\_Ave} = 1/(1-X)^2;$

$\text{Second\_Ave} = 1/(1-X)^3;$

Output =

$$1/(1-X)^2 + 2X/(1-X)^3$$

$$(1-X)/(1-X)^3 + 2X/(1-X)^3$$

$$= (1+X)/(1-X)^3$$

$(1+X)/(1-X)^3$

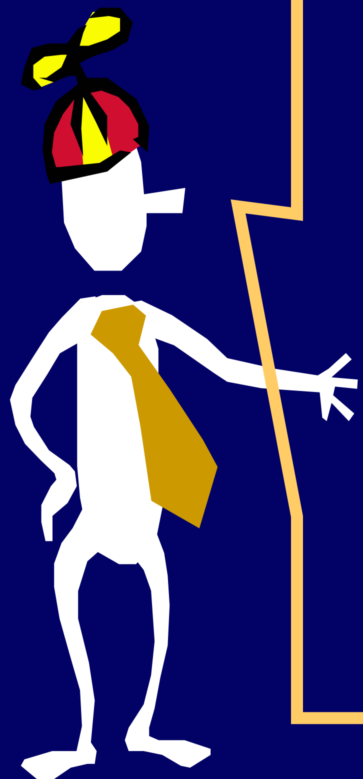
Zero_Ave := PREFIXSUM(<1>);

First_Ave := PREFIXSUM(Zero_Ave);

Second_Ave :=PREFIXSUM(First_Ave);

Output:=
RIGHT(Second_Ave) + Second_Ave

Second_Ave	=	<1, 3, 6, 10, 15,.
RIGHT(Second_Ave)	=	<0, 1, 3, 6, 10,.
Output	=	<1, 4, 9, 16, 25



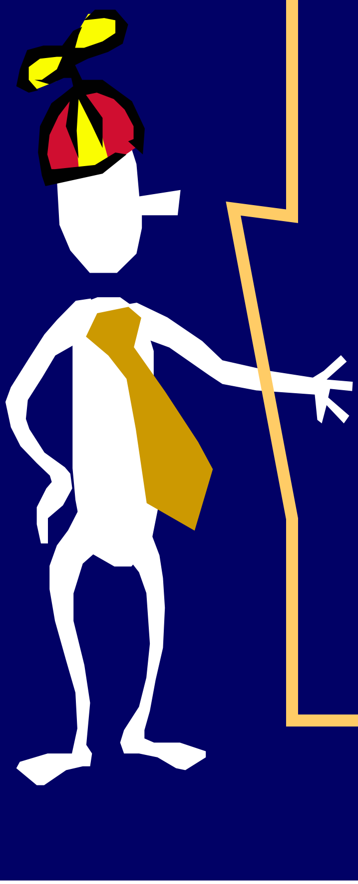
$(1+X)/(1-X)^3$

outputs $\langle 1, 4, 9, .. \rangle$

$X(1+X)/(1-X)^3$

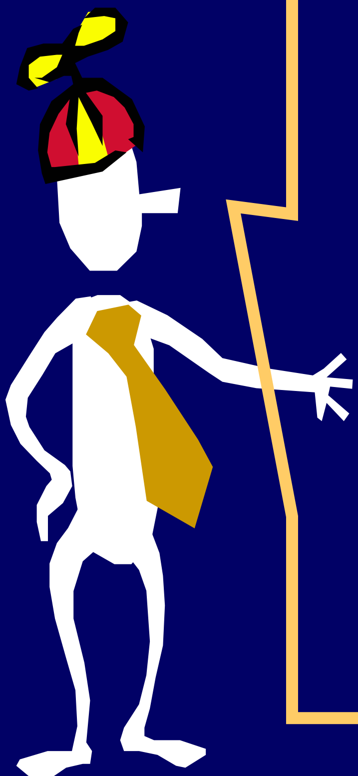
outputs $\langle 0, 1, 4, 9, .. \rangle$

The k^{th} entry is k^2



$$X(1+X)/(1-X)^3 = \sum k^2 X^k$$

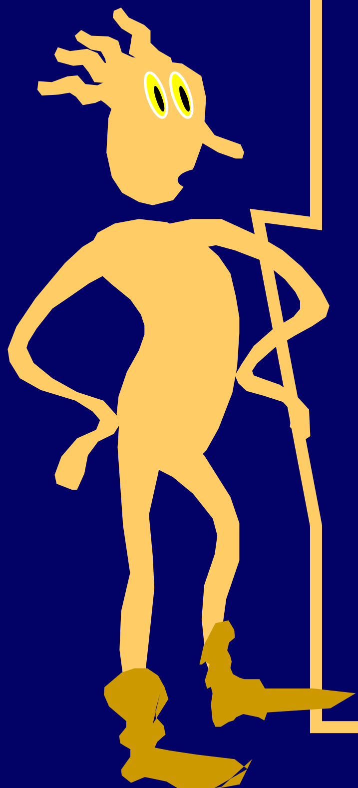
What does $X(1+X)/(1-X)^4$ do?



$X(1+X)/(1-X)^4$ expands to :

$$\sum S_k X^k$$

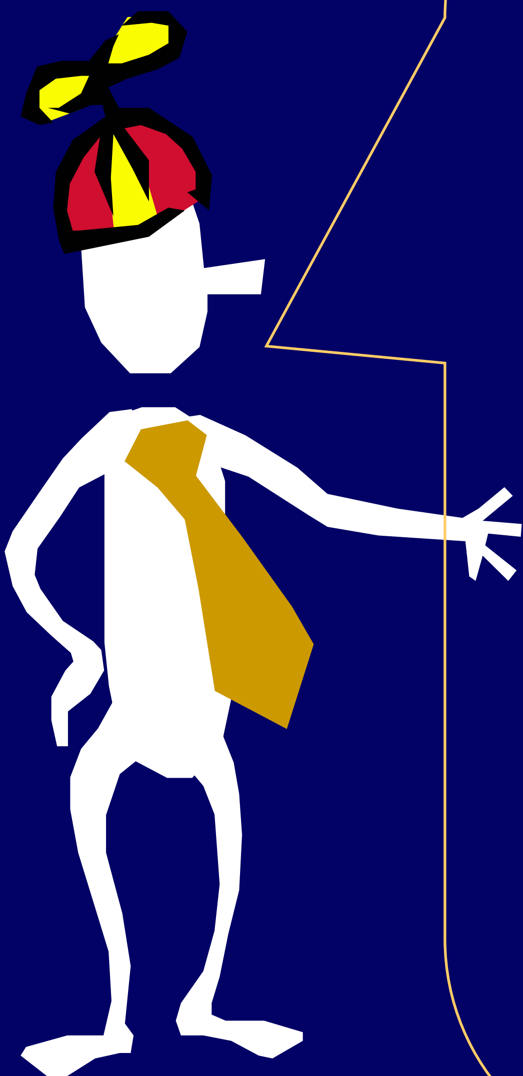
where S_k is the sum of the first k squares



Aha! Thus, if there is an alternative interpretation of the k^{th} coefficient of

$$X(1+X)/(1-X)^4$$

we would have a new way to get a formula for the sum of the first k squares.



Using pirates and gold we found that:

$$\frac{1}{(1-X)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} X^k$$

THUS:

$$\frac{1}{(1-X)^4} = \sum_{k=0}^{\infty} \binom{k+3}{3} X^k$$

Coefficient of X^k in $P_V = (X^2+X)(1-X)^{-4}$ is the sum of the first k squares:

$$\frac{X^2+X}{(1-X)^4} = (X^2+X)\sum_{k=0}^{\infty}\binom{k+3}{3}X^k$$

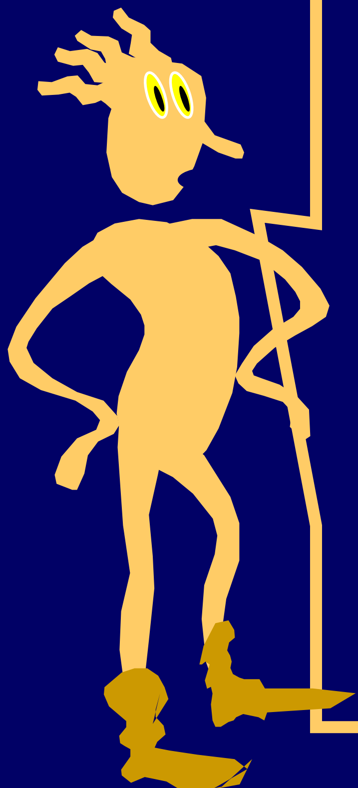
$$= \sum_{k=0}^{\infty}\left(\binom{k+2}{3}+\binom{k+1}{3}\right)X^k$$

$$\frac{1}{(1-X)^4} = \sum_{k=0}^{\infty}\binom{k+3}{3}X^k$$

Vector programs -> Polynomials -> Closed form expression

$$\frac{X^2+X}{(1-X)^4} = \sum_{k=0}^{\infty}\left(\binom{k+2}{3}+\binom{k+1}{3}\right)X^k$$

$$\sum_{i=0}^{i=n} i^2 = \binom{n+2}{3}+\binom{n+1}{3}$$

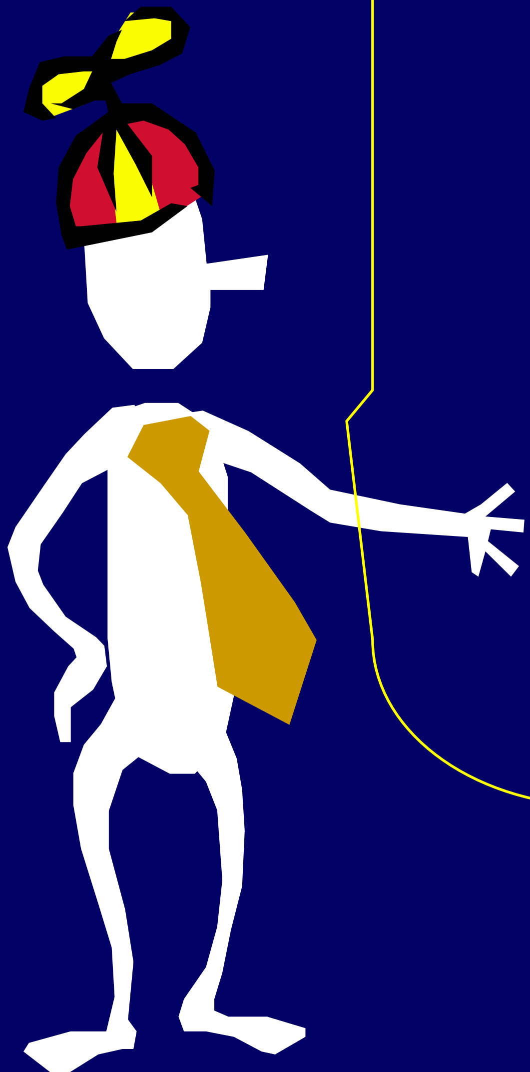


Expressions of the form

Finite Polynomial / Finite Polynomial

are called <u>Rational Polynomial Expressions</u>.

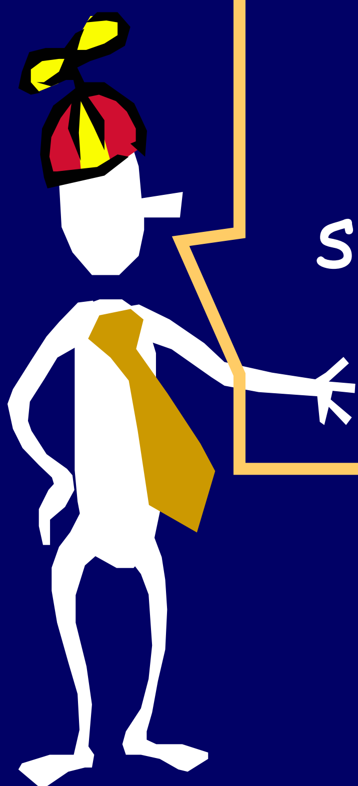
Clearly, these expressions have some deeper interpretation as a programming language.



What about this one?

$X/(1-X-X^2)$

$$\frac{x}{1-x-x^2} = F_0x^0 + F_1x^1 + F_2x^2 + F_3x^3 + \cdots = \sum_{i=0}^{\infty} F_i x^i$$



The action of dividing one polynomial by another can simulate a program to recursively compute Fibonacci numbers.

Vector Program I/O

Example:

INPUT $\vec{I}$; /* not allowed to alter $\vec{I}$ */

$\vec{V} := \vec{I} + 1$;
Loop n times:
 $\vec{V} := \vec{V} + \mathrm{RIGHT}(\vec{V}) + \vec{I}$;

OUTPUT $\vec{V}$

Vector Recurrence Relations

Let P be a vector program that takes input.

A <u>vector relation</u> is any statement of the form:

$$\vec{V} = P(\vec{V})$$

If there is a unique $\vec{V}$ satisfying the relation, then $\vec{V}$ is said to be <u>defined by the relation</u> $\vec{V} = P(\vec{V})$.

Fibonacci Numbers

Recurrence Relation Definition:

$$F_0 = 0, \quad F_1 = 1,$$

$$F_n = F_{n-1} + F_{n-2}, n > 1$$

Vector Recurrence Relation Definition:

$\vec{F}$ = RIGHT($\vec{F}$ + <1>) + RIGHT(RIGHT($\vec{F}$))

$$\boxed{\vec{F} = \text{RIGHT}(\vec{F} + \langle 1 \rangle) + \text{RIGHT}(\text{RIGHT}(\vec{F}))}$$

$\vec{F} = a_0, a_1, a_2, a_3, a_4, \ldots$

$\text{RIGHT}(\vec{F} + \langle 1 \rangle) = 0, a_0 + 1, a_1, a_2, a_3,$

$\text{RIGHT}(\text{RIGHT}(\vec{F}))$

$= 0, 0, a_0, a_1, a_2, a_3, .$

$$\boxed{F^{\rightarrow} = \mathrm{RIGHT}(\,F^{\rightarrow} + 1\,) + \mathrm{RIGHT}(\,\mathrm{RIGHT}(\,F^{\rightarrow}\,)\,)}$$

$$F = a_0 + a_1 X + a_2 X^2 + a_3 X^3 +$$

$$\mathrm{RIGHT}(F + 1) = (F+1)\,X$$

$$\mathrm{RIGHT}(\,\mathrm{RIGHT}(\,F\,)\,) = F\,X^2$$

$$\boxed{F = (F + 1)\,X + F\,X^2}$$

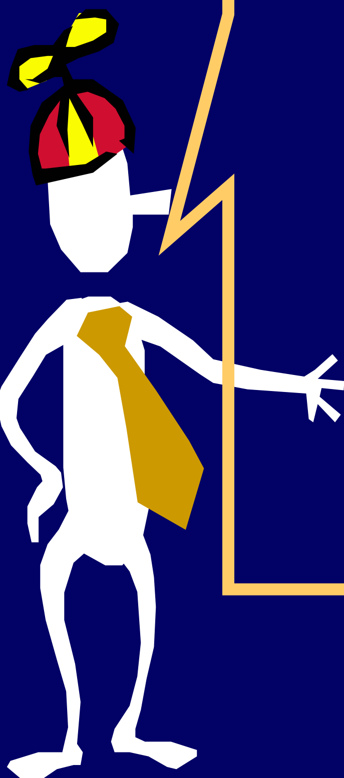
$$F = \quad a_0 + a_1 X + a_2 X^2 + a_3 X^3 + \ldots$$

$\mathrm{RIGHT}(F + 1) = (F+1)\,X$

$\mathrm{RIGHT}(\,\mathrm{RIGHT}(\,F\,)\,)$

$$= \quad F\,X^2$$

$$F = F X + X + F X^2$$



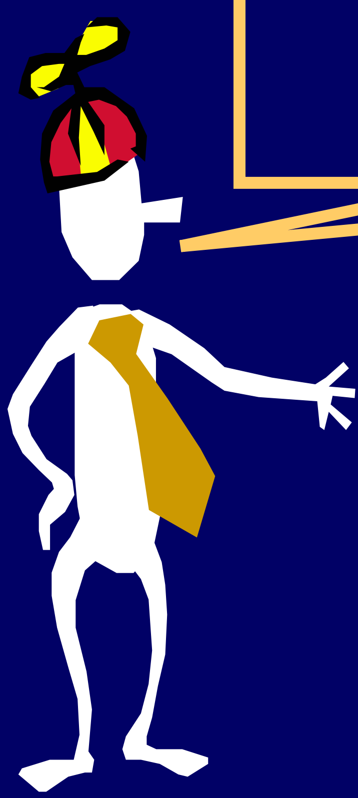
Solve for F.

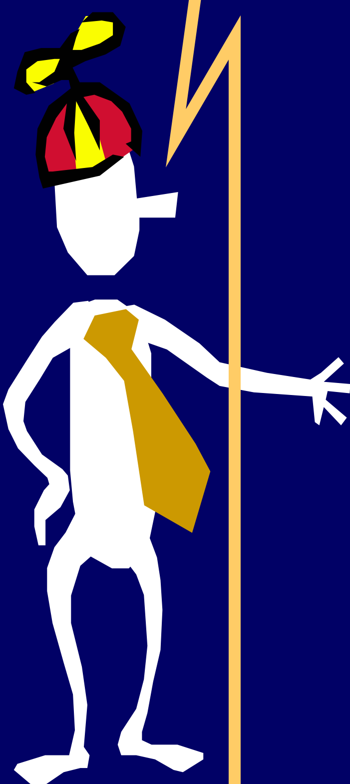
$$F - FX - FX^2 = X$$

$$F(1-X-X^2) = X$$

$$F = X/(1-X-X^2)$$

What is the Power Series Expansion of $x / (1-x-x^2)$?





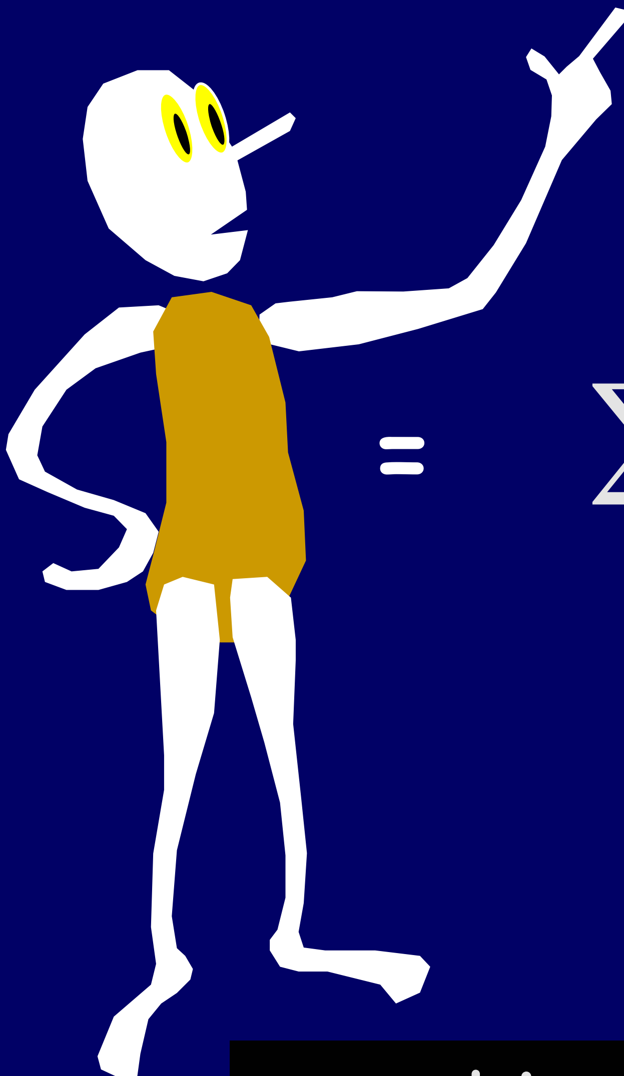
Since the bottom is quadratic we can factor it.

$$X / (1-X-X^2) =$$

$$X/(1-\phi X)(1-(-\phi)^{-1}X)$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$

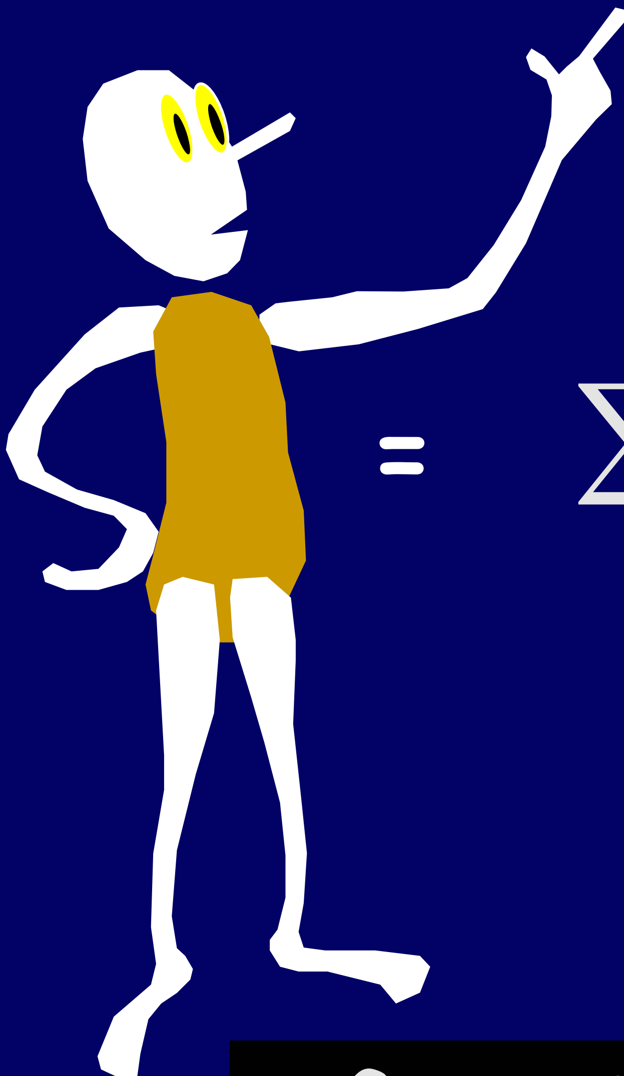
"The Golden Ratio"



$$\frac{X}{(1 - \phi X)(1- (-\phi^{-1}X)} = \sum_{n=0..\infty} \; ? \; X^n$$

Linear factors on the bottom

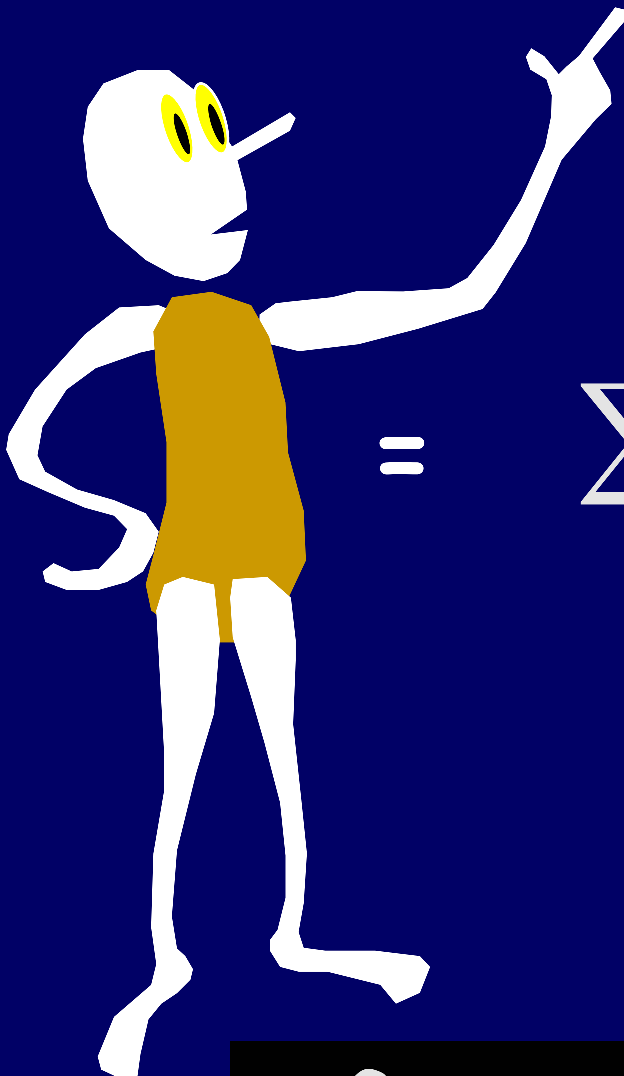
$$\left(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..\right)\left(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..\right) =$$



$$= \frac{1}{(1 - aX)(1-bX)}$$

$$= \sum_{n=0..\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

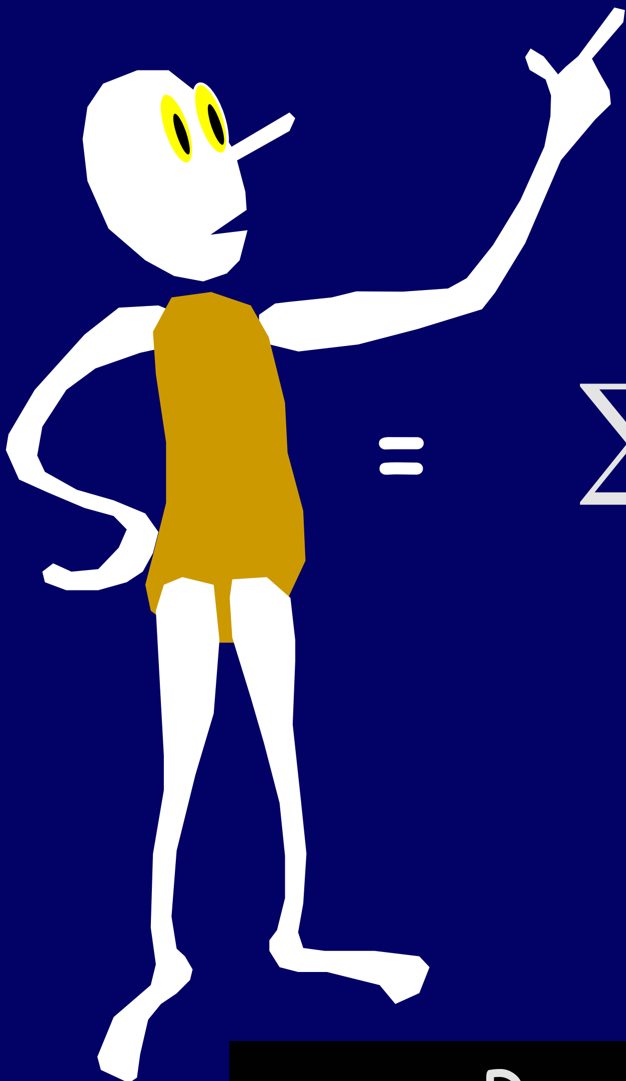
Geometric Series (Quadratic Form)



$$\frac{1}{(1 - \phi X)(1- (-\phi^{-1}X)}$$

$$= \sum_{n=0..\infty} \frac{\phi^{n+1} - (-\phi^{-1})^{n+1}}{\sqrt{5}} X^n$$

Geometric Series (Quadratic Form)



$$\frac{x}{(1 - \phi X)(1- (-\phi^{-1}X)} = \sum_{n=0..\infty} \frac{\phi^{n+1} - (-\phi^{-1})^{n+1}}{\sqrt{5}} X^{n+1}$$

Power Series Expansion of F

$$\frac{x}{1-x-x^2} = F_0x^0 + F_1x^1 + F_2x^2 + F_3x^3 + \cdots = \sum_{i=0}^{\infty} F_i x^i$$

$$\frac{x}{1-x-x^2} = \sum_{i=0}^{\infty} \frac{1}{\sqrt{5}}\left(\phi^i - \left(-\frac{1}{\phi}\right)^i\right)x^i$$

$$\frac{x}{1-x-x^2} = F_0x^0 + F_1x^1 + F_2x^2 + F_3x^3 + \cdots = \sum_{i=0}^{\infty} F_i x^i$$

The i^{th} Fibonacci number is:

$$\frac{1}{\sqrt{5}}\left(\phi^i - \left(-\frac{1}{\phi}\right)^i\right)$$

References

Applied Combinatorics, by Alan Tucker

Generatingfunctionology, Herbert Wilf