Great Theoretical Ideas In Computer Science

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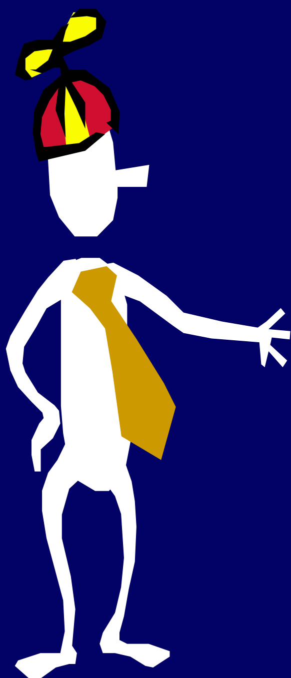
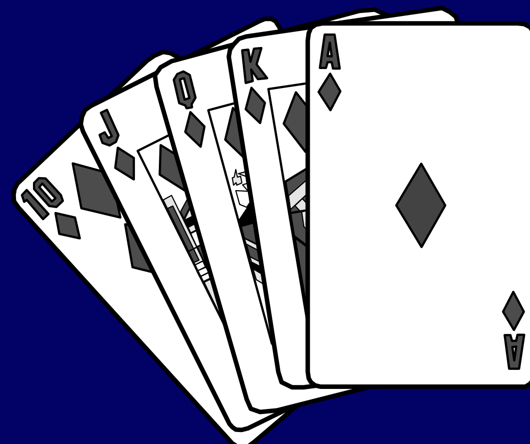
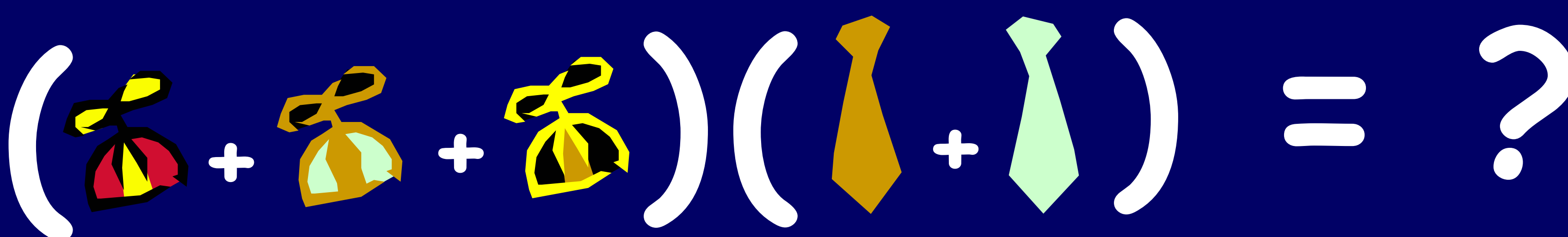
Counting II: Recurring Problems And Correspondences



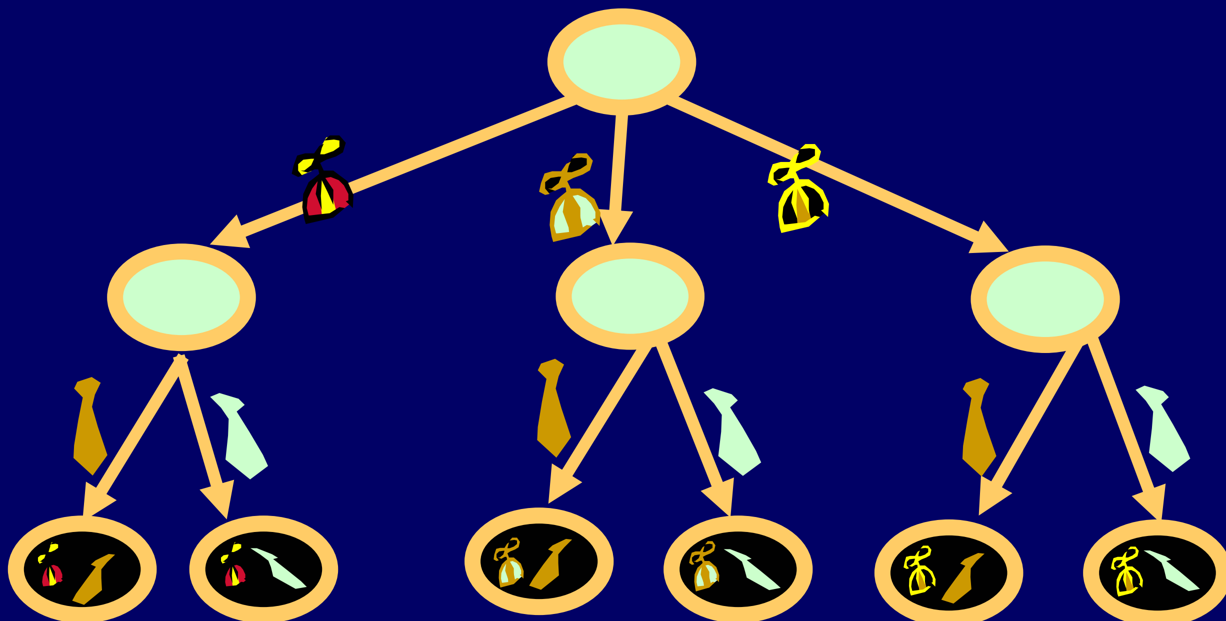




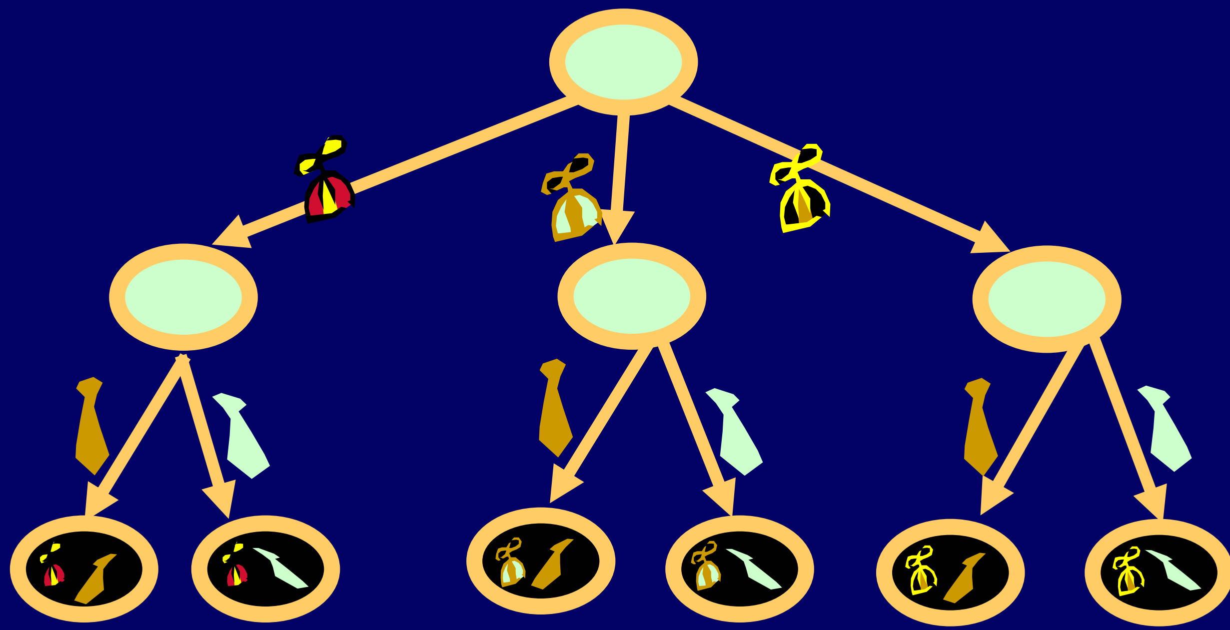
Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

Choice Tree



A <u>choice tree</u> is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf.



A choice tree provides a “choice tree representation” of a set S, if

1) Each leaf label is in S
2) No two leaf labels are the same

Product Rule

IF S has a choice tree representation with P_1 possibilities for the first choice, P_2 for the second, and so on,

THEN

there are $P_1 P_2 P_3 \ldots P_n$ objects in S

Proof: The leaves of the choice tree are in 1-1 onto correspondence with the elements of S.

Product Rule

Suppose that all objects of a type S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF

1) Each sequence of choices constructs an object of type S

AND

2) No two different sequences create the same object

THEN

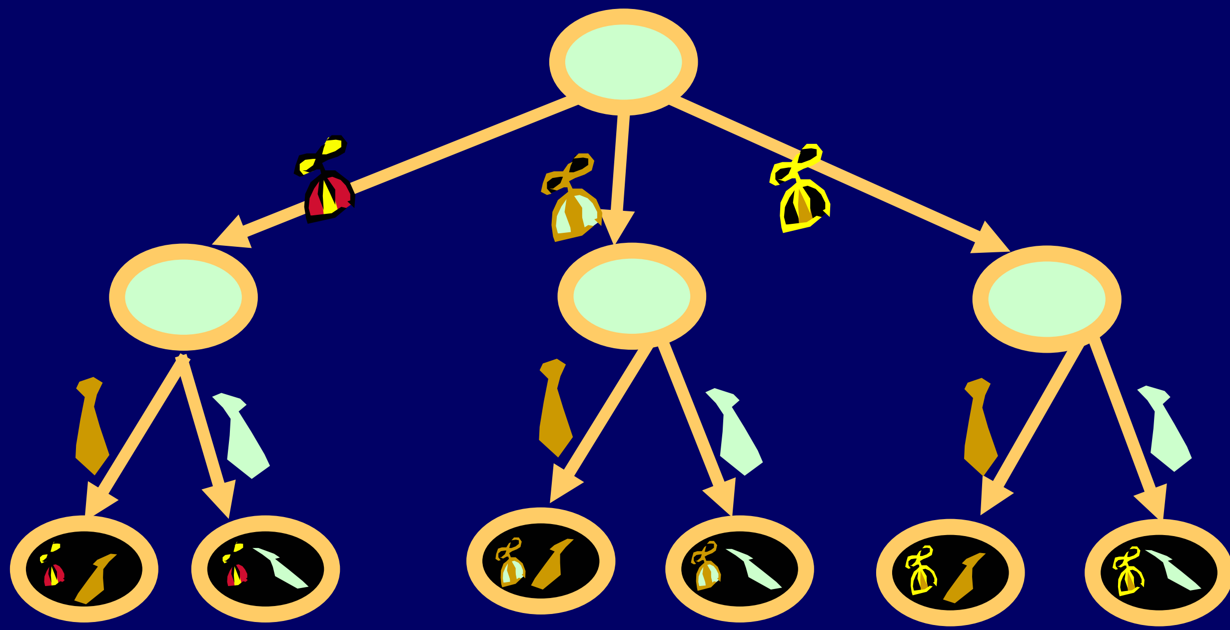
there are $P_1 P_2 P_3 \ldots P_n$ objects of type S.

Condition 2 of the product rule:

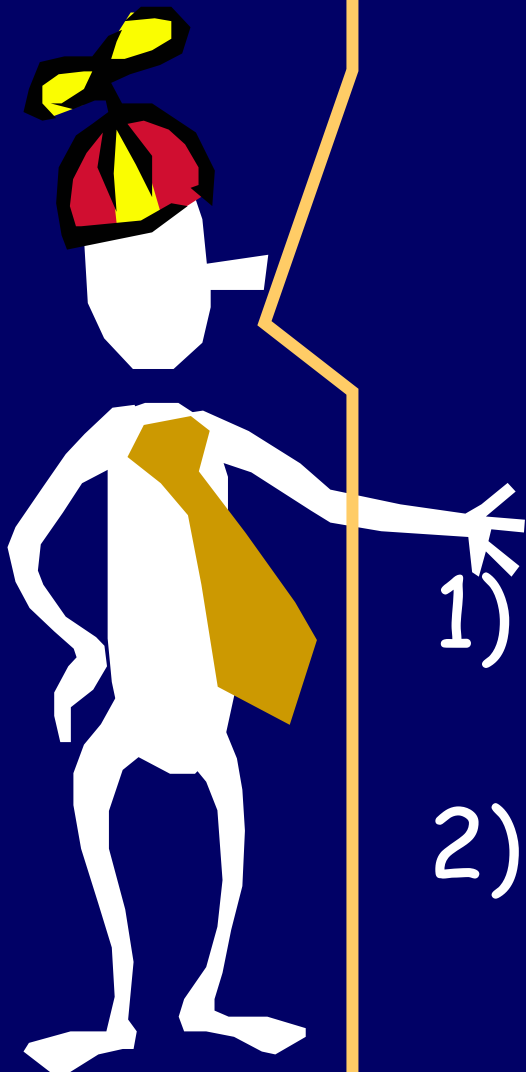
No two leaves have the same label.

Equivalently,

No object can be created in two different ways.



Reversibility Check: Given an arbitrary object, can we reverse engineer the choices that created it?



The two big mistakes people make in associating a choice tree with a set S are:

1) Creating objects not in S

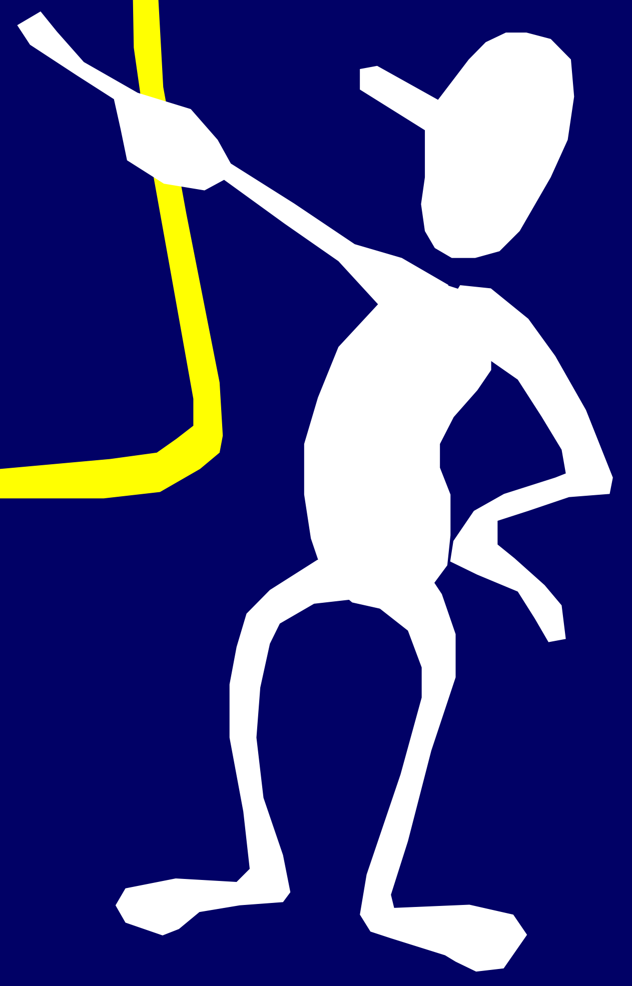
2) Creating the same object two different ways

DEFENSIVE THINKING:

Am I creating objects of the right type?

Can I reverse engineer my choice sequence from any given object?

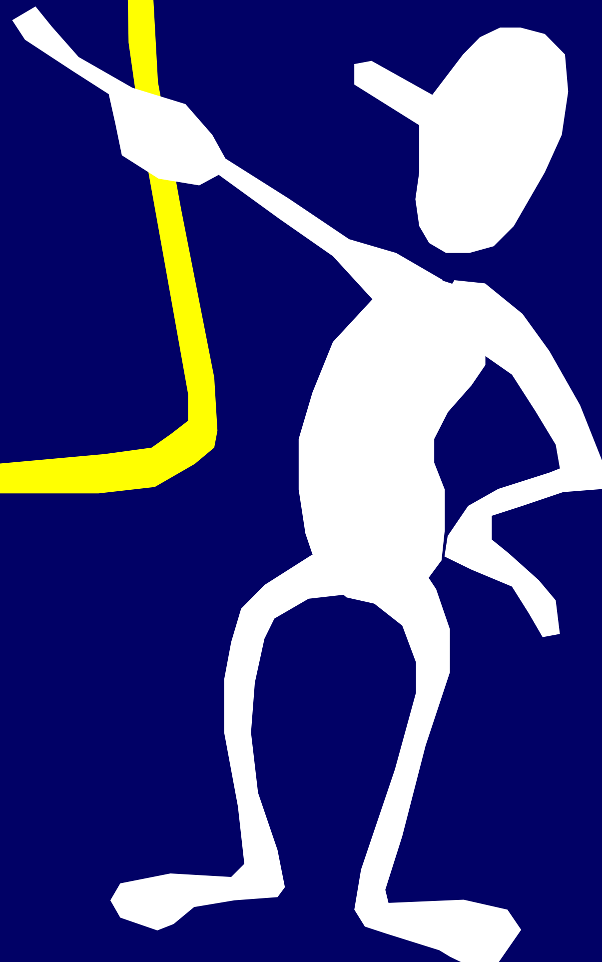
The number of subsets of an n-element set is 2^n.



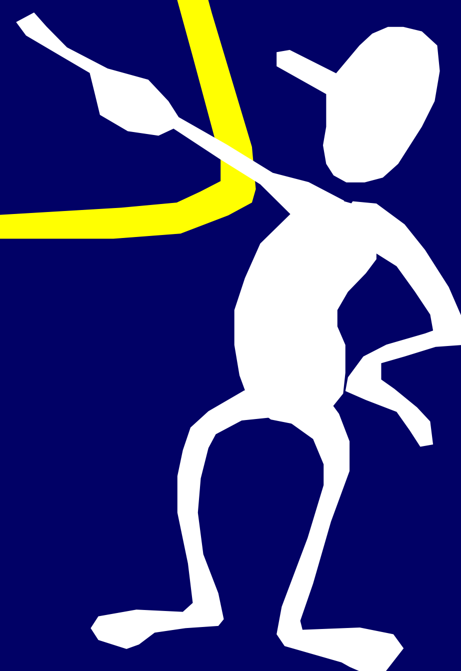
The number of permutations of n distinct objects is $n!$

The number of subsets of size r that can be formed from an n-element set is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

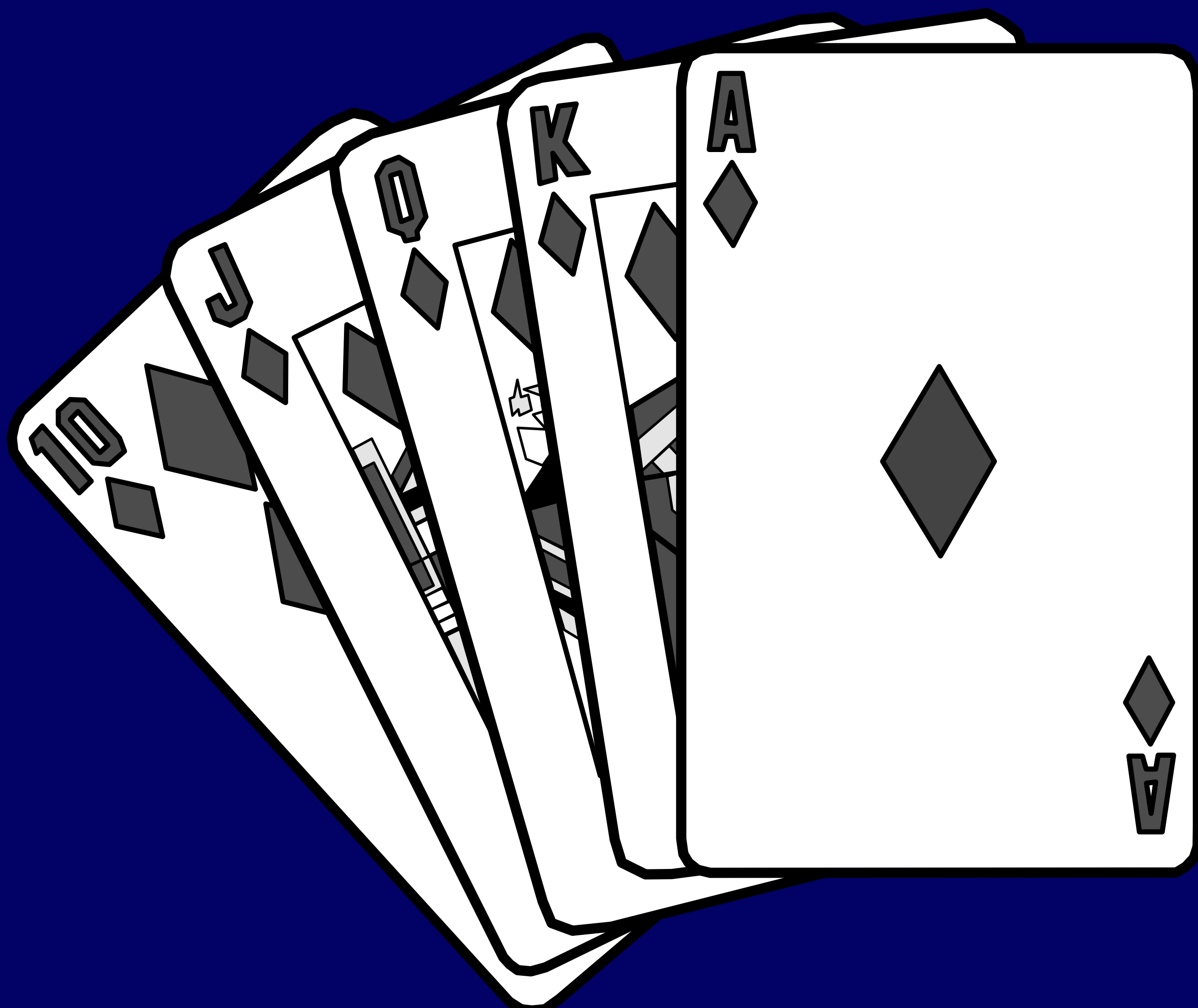


Sometimes it is easiest to count something by counting its opposite.



Let's use our principles to extend our reasoning to different types of objects.

Counting Poker Hands…



52 Card Deck
5 card hands

4 possible suits:

- ♥ ♦ ♣ ♠

13 possible ranks:

- 2,3,4,5,6,7,8,9,10,J,Q,K,A

52 Card Deck
5 card hands

4 possible suits:

- ♥ ♦ ♣ ♠

13 possible ranks:

- 2,3,4,5,6,7,8,9,10,J,Q,K,A

<u>Pair</u>: set of two cards of the same rank

<u>Straight</u>: 5 cards of consecutive rank

<u>Flush</u>: set of 5 cards with the same suit

Ranked Poker Hands

Straight Flush
- A straight and a flush

4 of a kind
- 4 cards of the same rank

Full House
- 3 of one kind and 2 of another

Flush
- A flush, but not a straight

Straight
- A straight, but not a flush

3 of a kind
- 3 of the same rank, but not a full house or 4 of a kind

2 Pair
- 2 pairs, but not 4 of a kind or a full house

A Pair

Straight Flush

9 choices for rank of lowest card at the start of the straight.

4 possible suits for the flush.

$9 \times 4 = 36$

$$\frac{36}{\binom{52}{5}} = \frac{36}{2598960} = 1 \text{ in } 72{,}193.33..$$

4 Of A Kind

13 choices of rank.

48 choices for remaining card.

$$13 \times 48 = 624$$

$$\frac{624}{2598960} = 1 \text{ in } 4165$$

Flush

4 choices of suit.

$\binom{13}{5}$ choices of set of 5 ranks.

$= 5148$

$- 36$ Straight Flushes

$= 5112$

$$\frac{5112}{2598960} = 1 \text{ in } 508.4$$

Straight

9 choices of lowest rank in the straight.

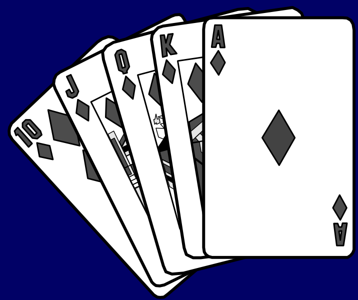
4^5 choices of suits to each card in sequence.

=9216

- 36 Straight Flushes

= 9180

$$\frac{9180}{2598960} = 1 \text{ in } 283.11$$



Storing Poker Hands
How many bits per hand?

I want to store a 5 card poker hand using the smallest number of bits (space efficient).

Naïve scheme: 2 bits for suit,
4 bits for a rank,
and hence 6 bits per card

Total: 30 bits per hand

How can I do better?


How can we store a poker hand without storing its order?

Order the 2,598,560 Poker hands lexicographically [or in any fixed manner]

To store a hand all I need is to store its index of size $\lceil \log_2(2{,}598{,}560) \rceil = 22$ bits.

Hand 0000000000000000000000
Hand 0000000000000000000001
Hand 0000000000000000000010

.

.

.

22 Bits Is OPTIMAL

$2^{21} < 2{,}598{,}560$

There are more poker hands than there are 21 bit strings. Hence, you can't have a string for each hand.

An n-element set can be stored so that each element uses $\lceil \log_2(n) \rceil$ bits.

Furthermore, any representation of the set will have some strings of that length.

Information Counting Principle:

If each element of a set can be represented using k bits, the size of the set is bounded by 2^k

How many ways to rearrange the letters in the word "SYSTEMS"?

SYSTEMS

1) 7 places to put the Y, 6 places to put the T, 5 places to put the E, 4 places to put the M, and the S's are forced.

   $7 \times 6 \times 5 \times 4 = 840$

2) $\binom{7}{3}$ choices of positions for the S's

   4 choices for the Y

   3 choices for the T

   2 choices for the E

   1 choice for the M

$$\frac{7!}{3!4!} \times 4 \times 3 \times 2 \times 1 = \frac{7!}{3!} = 840$$

SYSTEMS

3) Let's pretend that the S's are distinct:

$S_1YS_2TEMS_3$

There are 7! permutations of $S_1YS_2TEMS_3$

- But when we stop pretending we see that we have counted each arrangement of SYSTEMS 3! times, once for each of 3! rearrangements of $S_1S_2S_3$.

$$\frac{7!}{3!} = 840$$

Arrange n symbols r_1 of type 1, r_2 of type 2, …, r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{r_k}{r_k}$$

$$=\frac{n!}{r_1!(n-r_1)!}\frac{(n-r_1)!}{r_2!(n-r_1-r_2)!}\frac{(n-r_1-r_2)!}{r_3!(n-r_1-r_2-r_3)!}\cdots 1$$

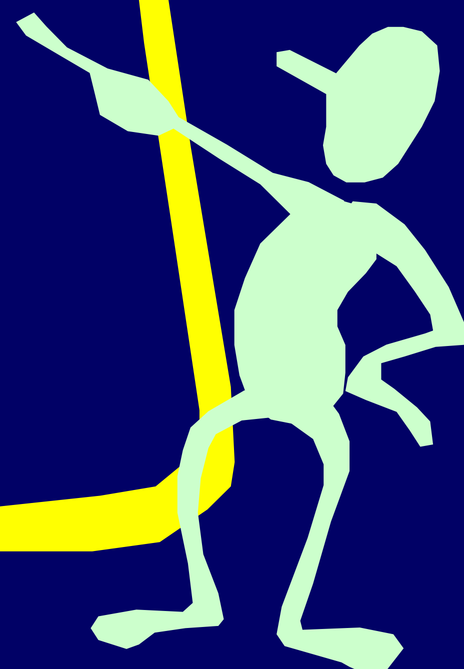
$$=\frac{n!}{r_1!r_2!r_3!\ldots r_k!}$$

CARNEGIEMELLON

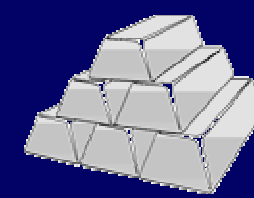
$$\frac{14!}{2!3!2!} = 3{,}632{,}428{,}800$$

The number of ways to arrange n symbols with r_1 of type 1, r_2 of type 2, …, r_k of type k is:

$$\frac{n!}{r_1!\,r_2!\,r_3!\ldots r_k!}$$



5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?



Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGGG/

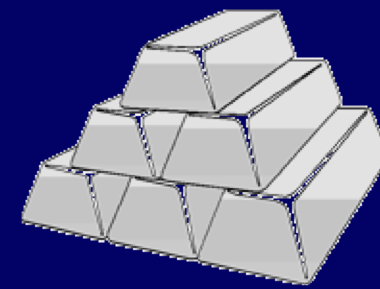
represents the following division among the pirates: 2, 1, 0, 17, 0

In general, the i^{th} pirate gets the number of G's after / i-1 and before / i. This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s.

How many different ways to divide up the loot?

Sequences with 20 G’s and 4 /’s

$$\binom{24}{4}$$



How many different ways can n distinct pirates divide k identical, indivisible bars of gold?

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1\ , x_2, x_3, x_4, x_5 \geq 0$$

Think of X_k as being the number of gold bars that are allotted to pirate k.

$$\binom{24}{4}$$

How many integer solutions to the following equations?

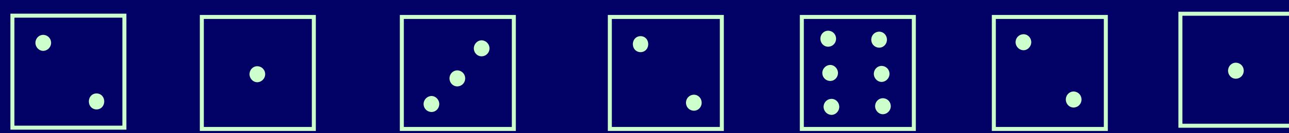
$$x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_{n-1}, x_n \geq 0$$

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Identical/Distinct Dice

Suppose that we roll seven dice.



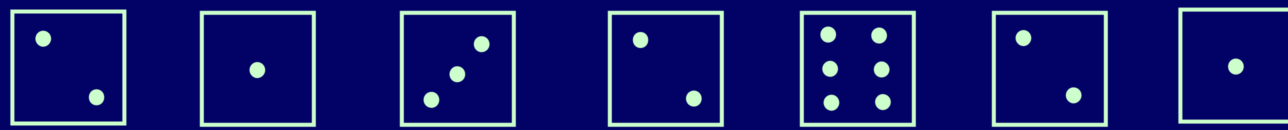
How many different outcomes are there, if order matters?

$$6^7$$

What if order doesn't matter?
(E.g., Yahtzee)

$$\binom{12}{7}$$

7 Identical Dice



How many different outcomes?

Corresponds to 6 pirates
and 7 bars of gold!

Let X_k be the number of dice showing k.
The k^{th} pirate gets X_k gold bars.

$$\binom{6+7-1}{7}$$

Multisets

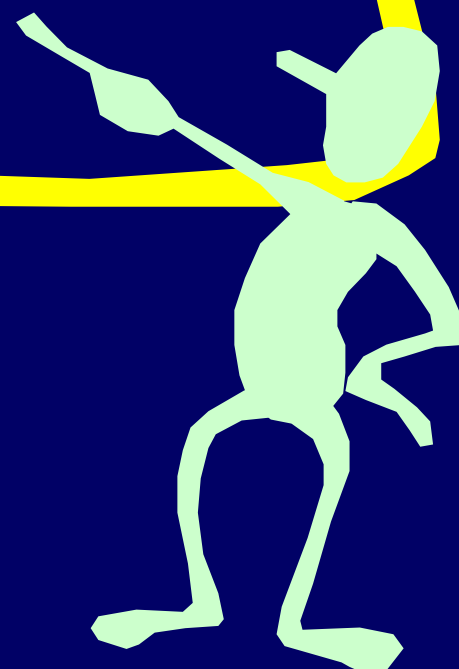
A multiset is a set of elements, each of which has a multiplicity. The size of the multiset is the sum of the multiplicities of all the elements.

Example:

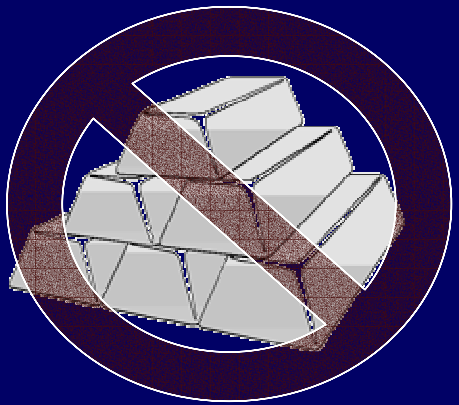
$\{X, Y, Z\}$ $m(X)=0$ $m(Y)=3$, $m(Z)=2$

Unary visualization: $\{Y, Y, Y, Z, Z\}$

There are $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ ways to choose a multiset of size k from n types of elements



Back to the pirates



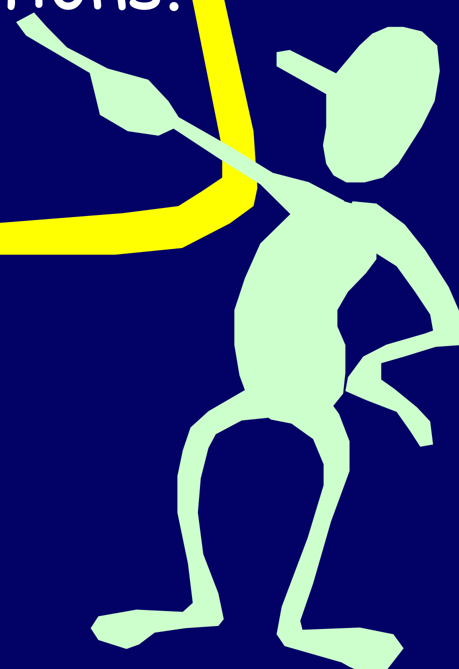
How many ways are there of choosing 20 pirates from a set of 5, with repetitions allowed?

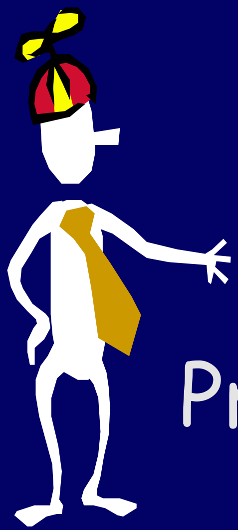
$$\binom{5+20-1}{20} = \binom{24}{20} = \binom{24}{4}$$

$$x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_{n-1}, x_n \geq 0$$

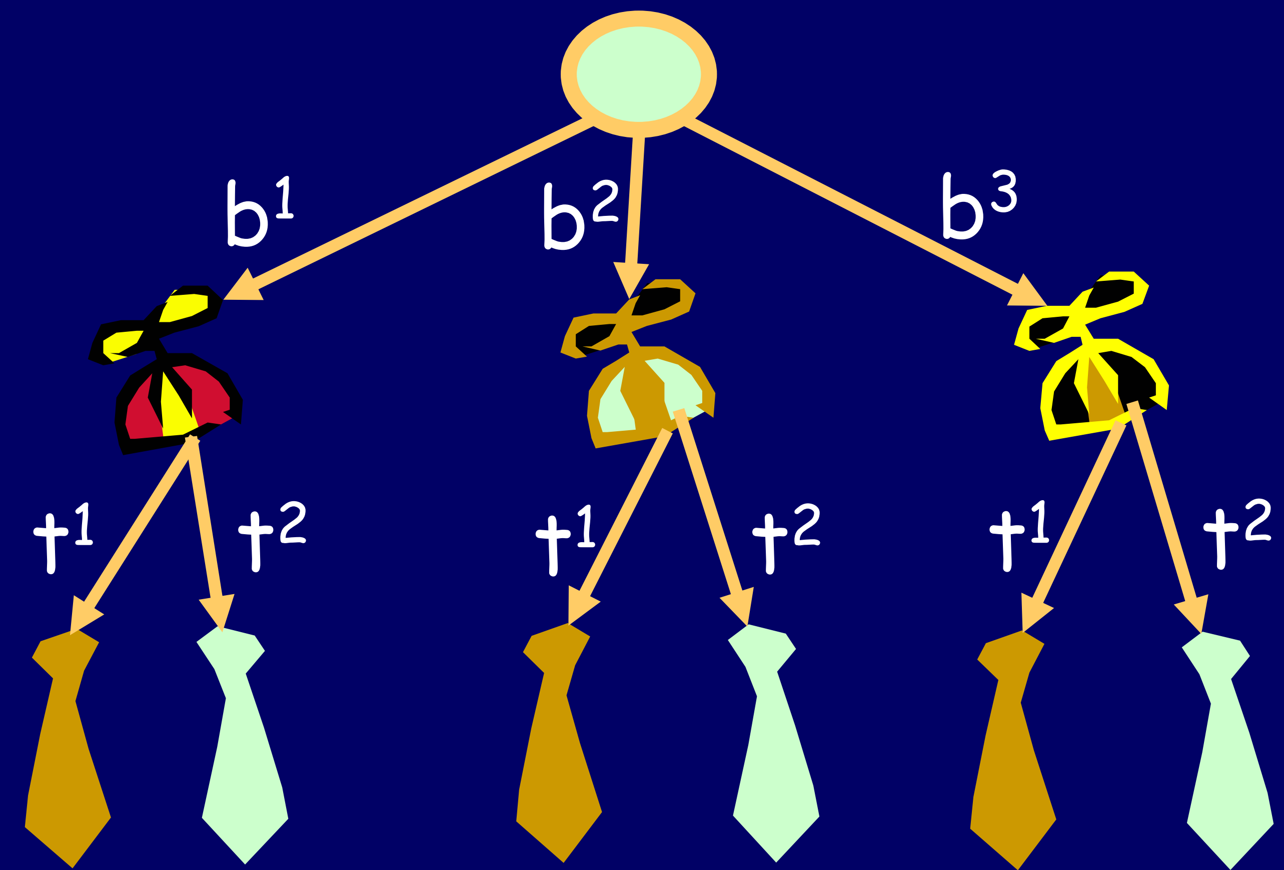
has $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ integer solutions.



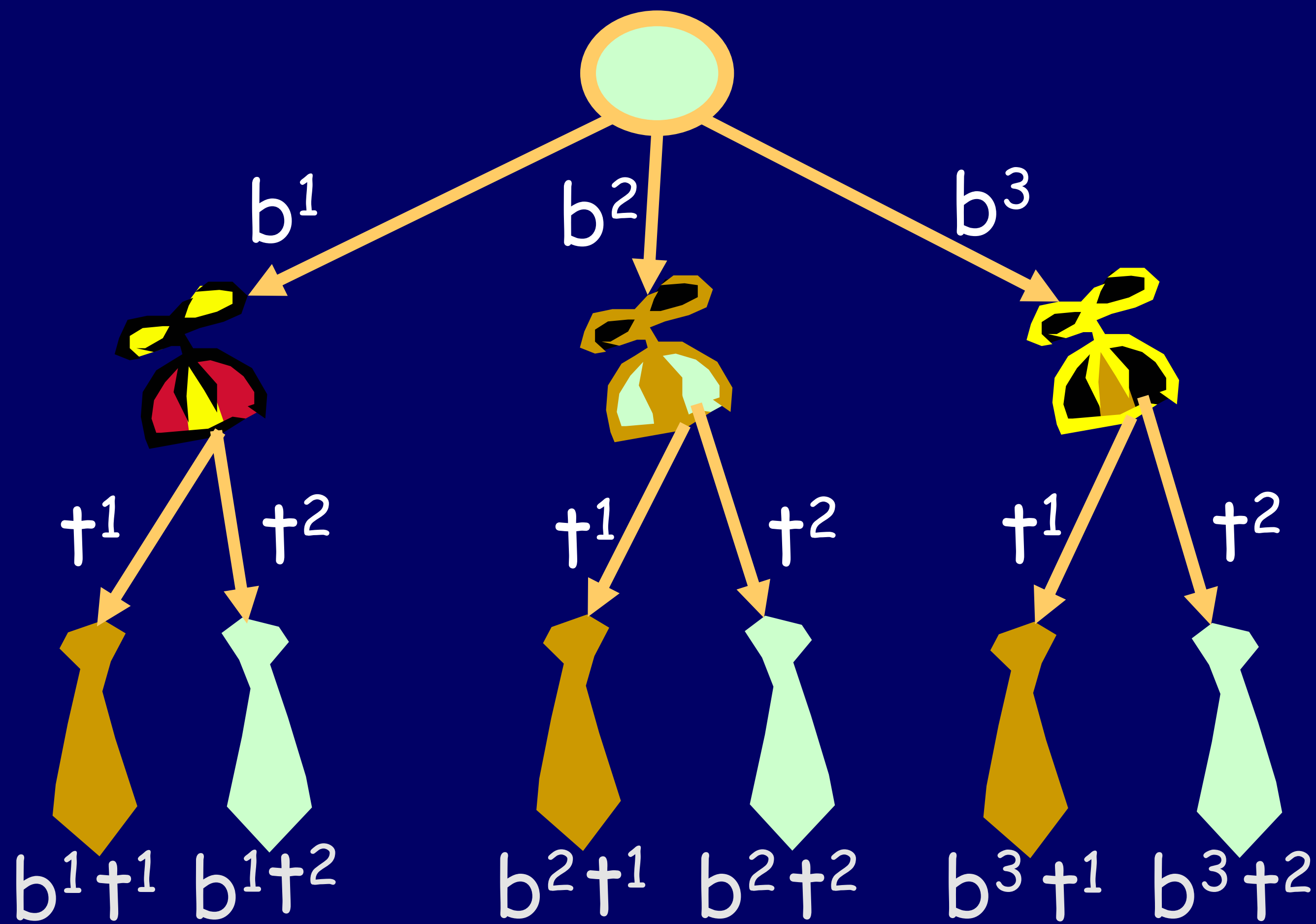


POLYNOMIALS EXPRESS CHOICES AND OUTCOMES

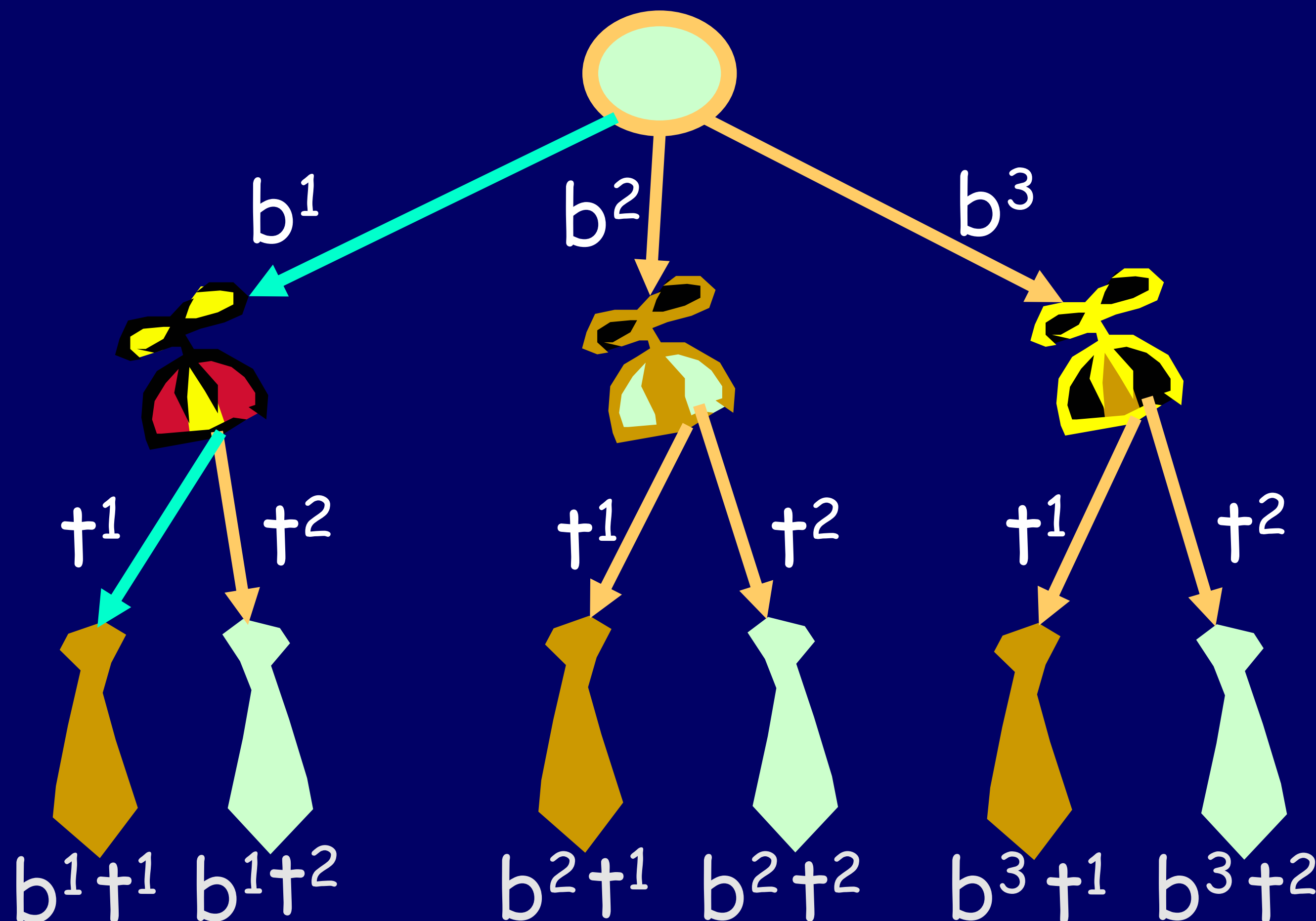
Products of Sum = Sums of Products



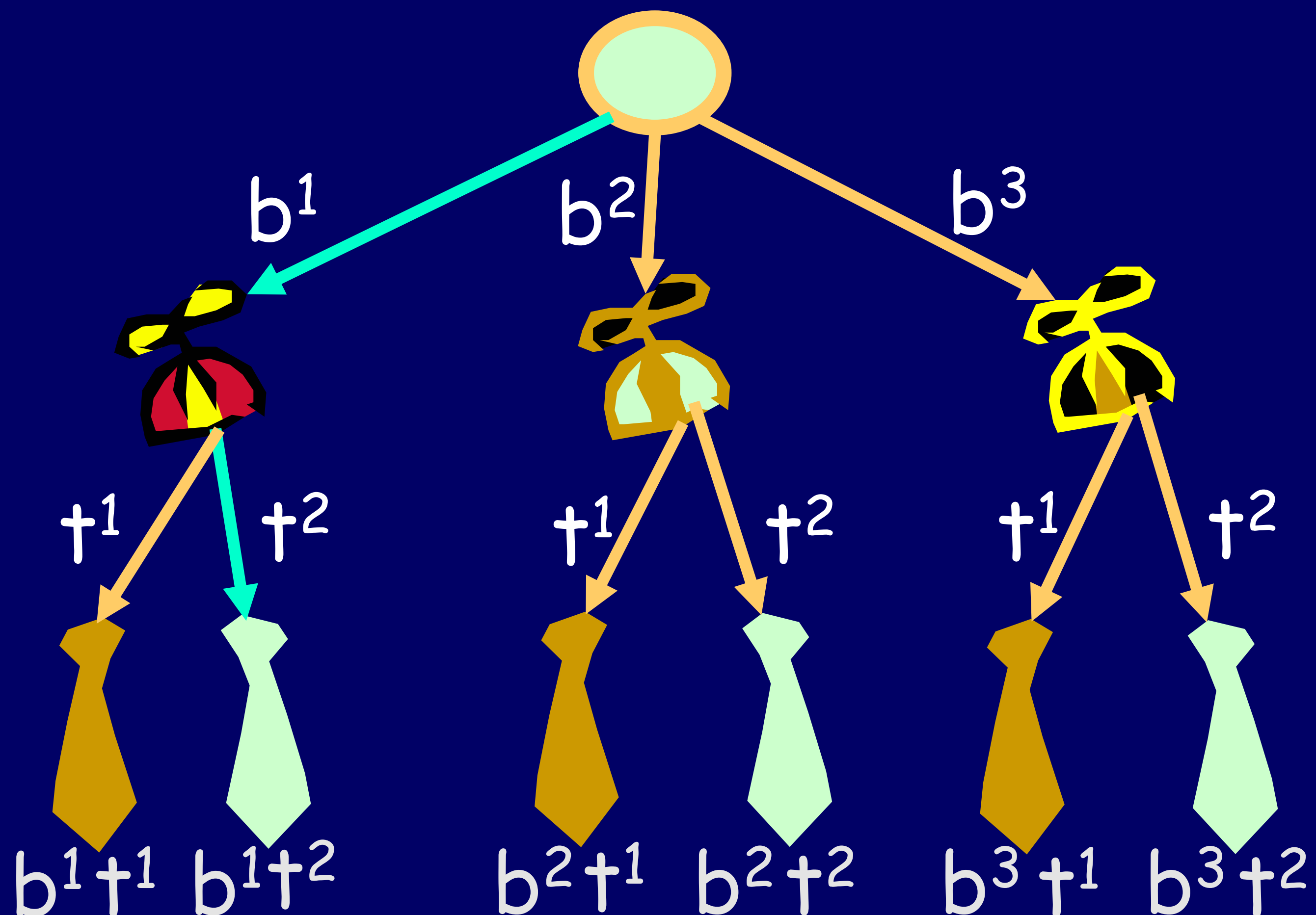
b1
b2
b3
t1
t2
t1
t2
t1
t2



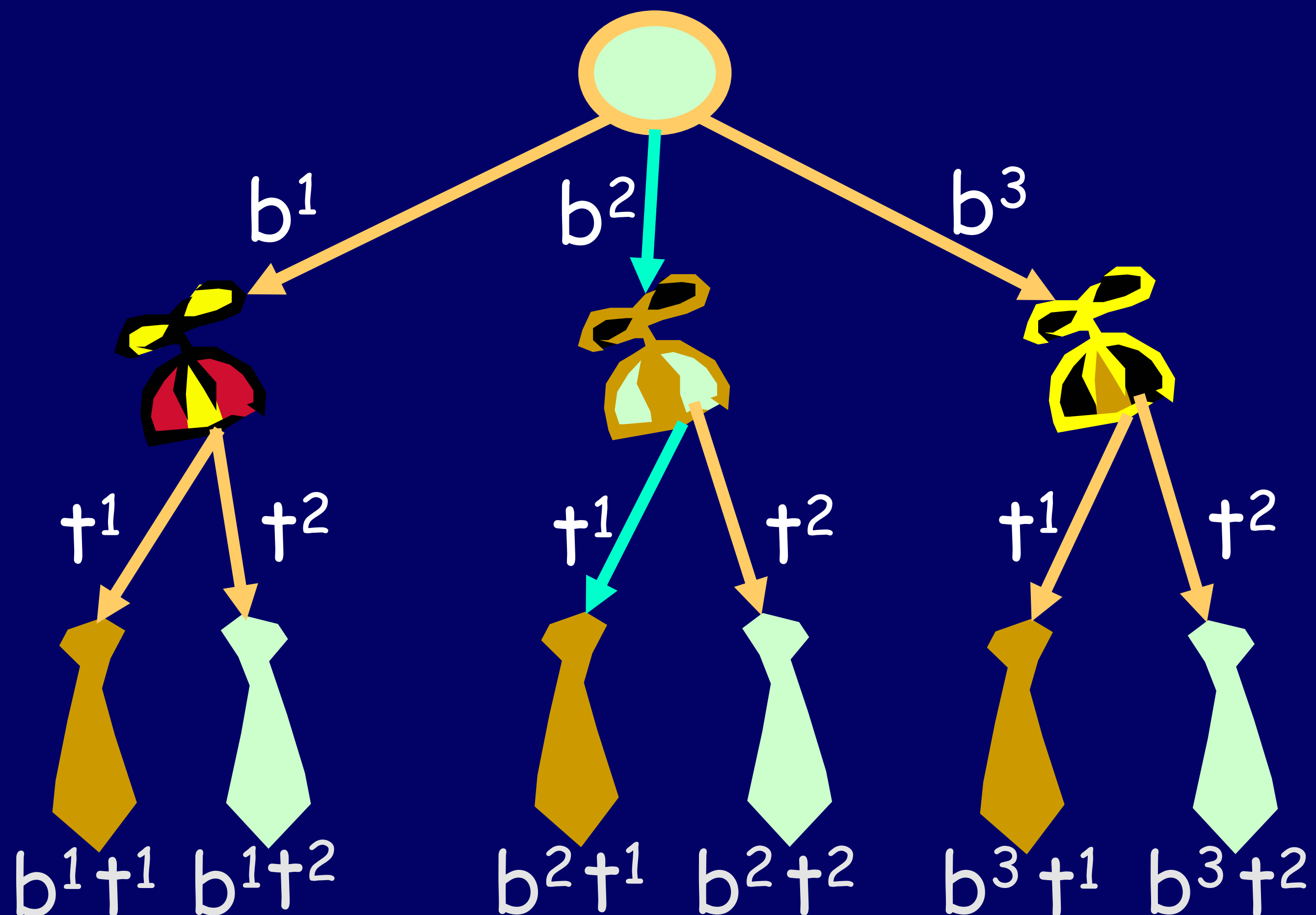
$(b^1 + b^2 + b^3)(t^1 + t^2) =$



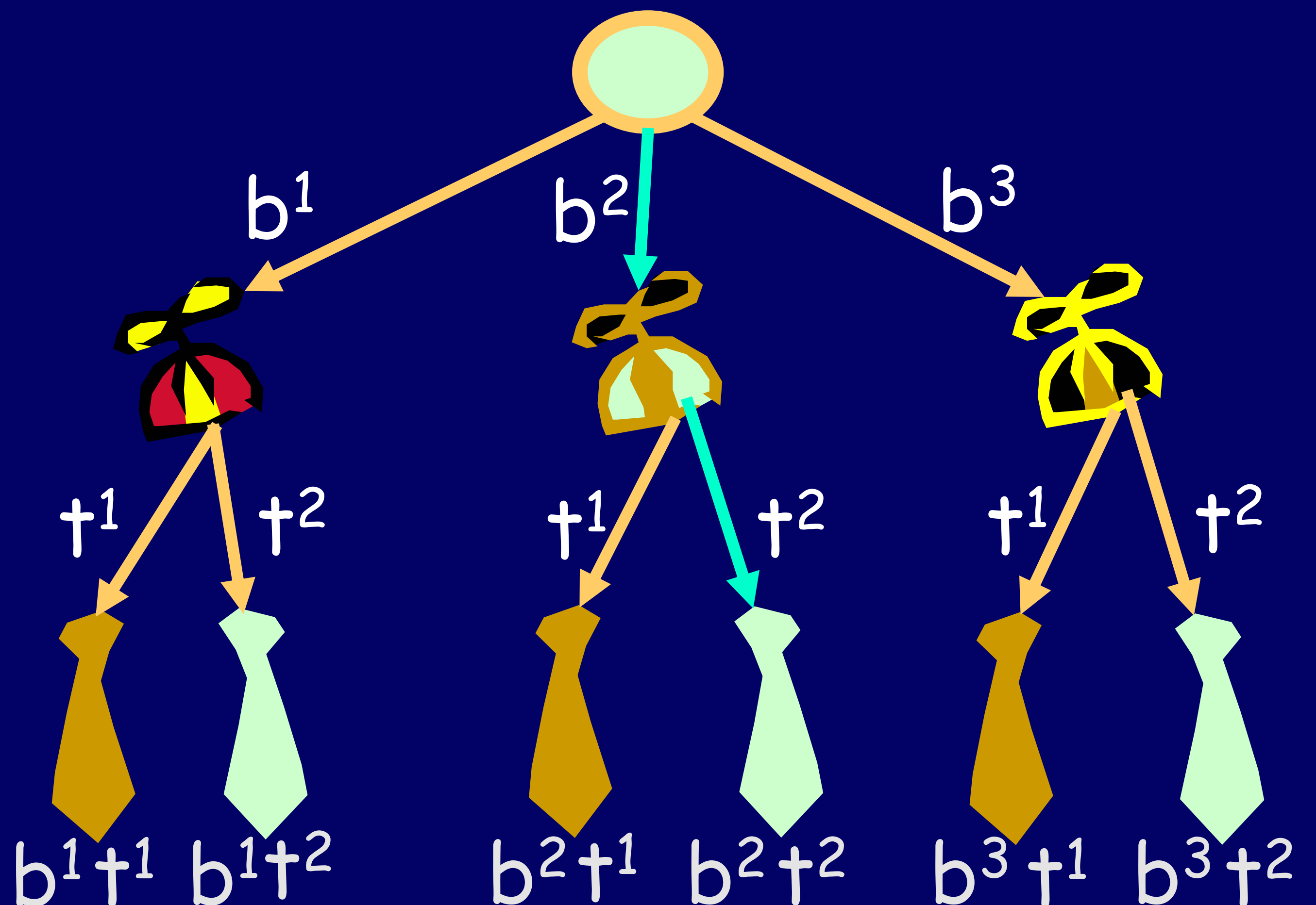
$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 +$$



$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 +$

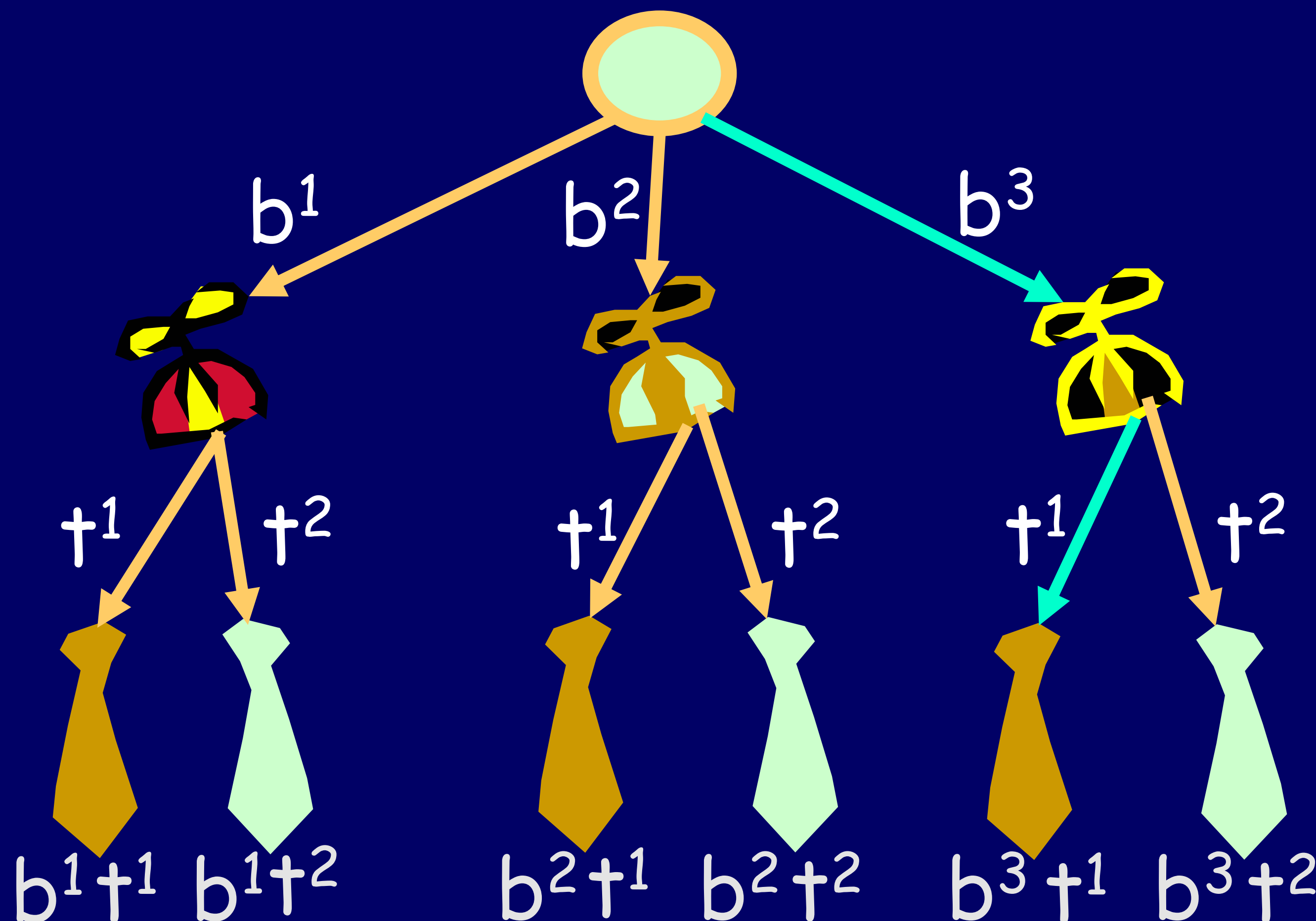


$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 +$$

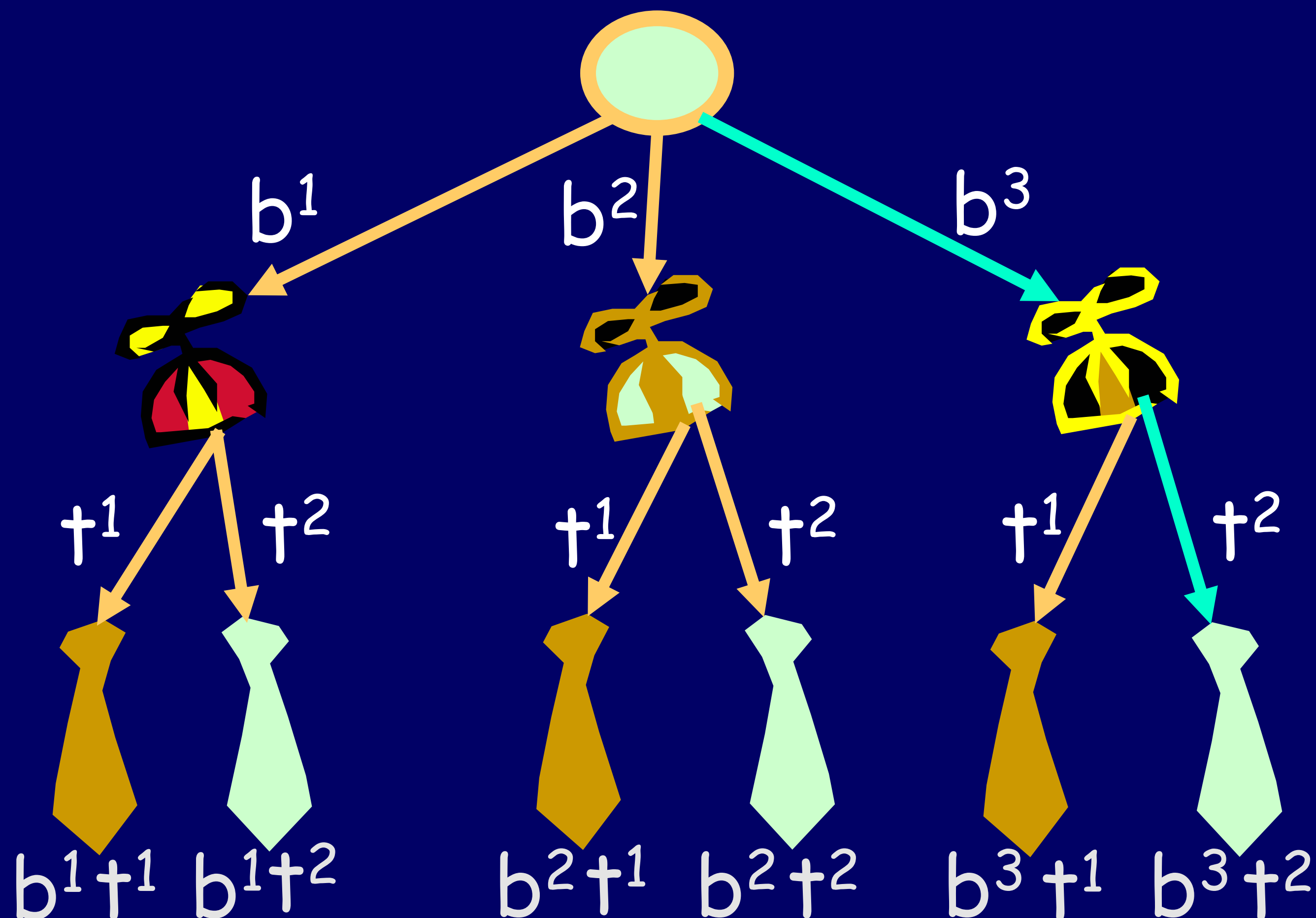


$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$$

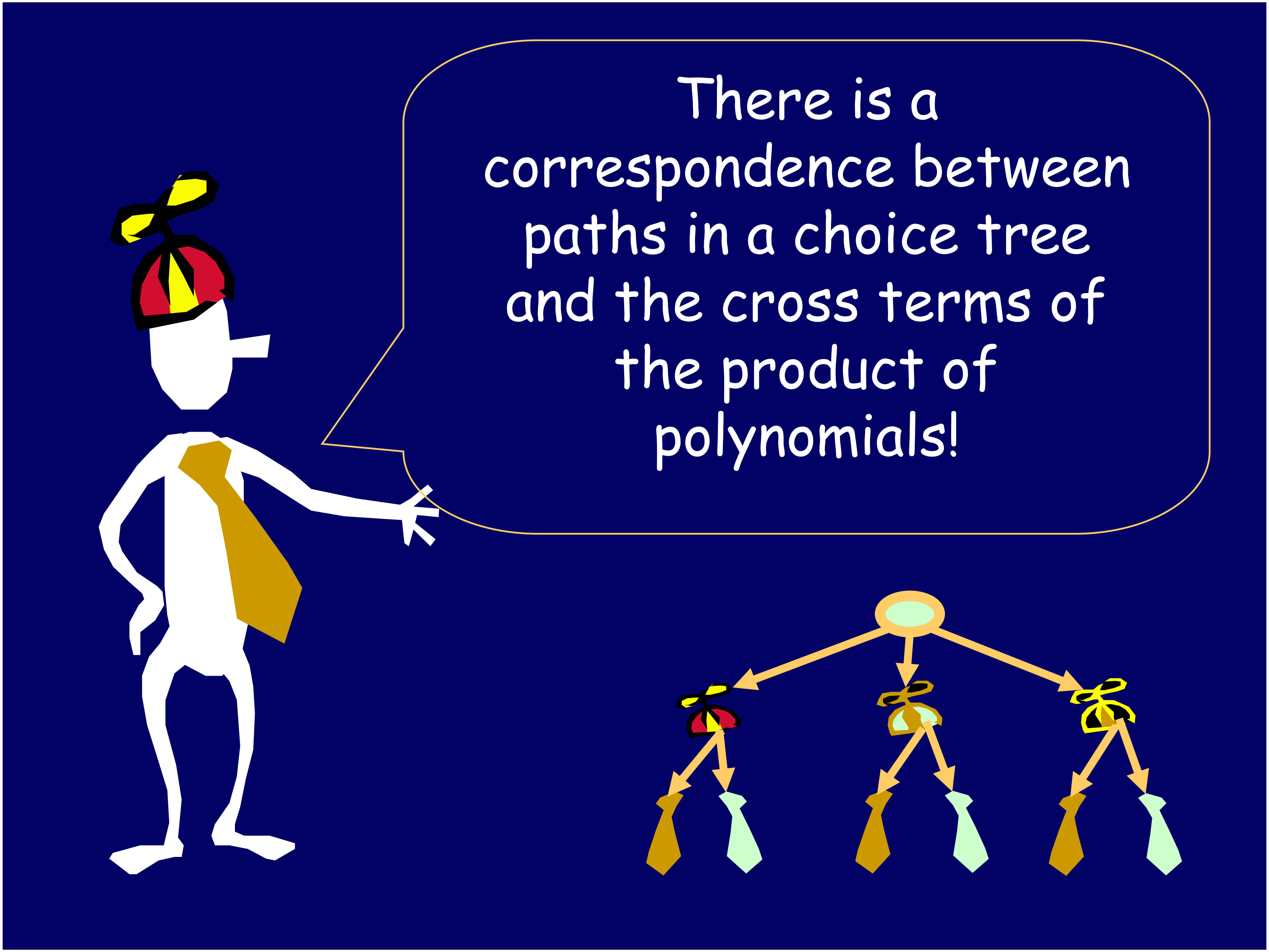
$$+$$



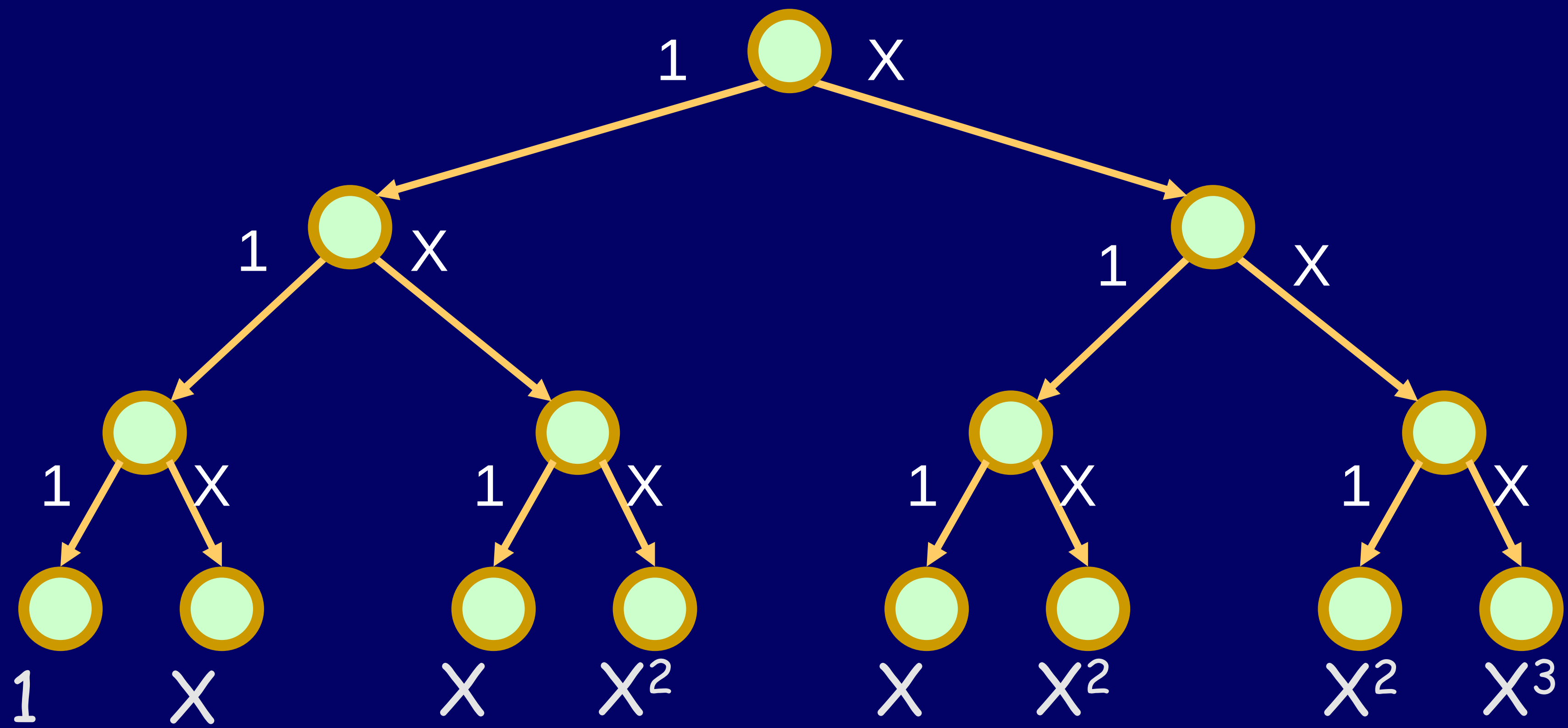
$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$$
$$+ b^3t^1 +$$



$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2 + b^3t^1 + b^3t^2$$


There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!

Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

What is a closed form expression for c_k?

$$(1+X)^n = c_0 + c_1X + c_2X^2 + \ldots + c_nX^n$$

What is a closed for expression for c_n?

$$(1+X)^n$$

$$= \overbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}^{\text{n times}}$$

Before combining like terms, when we multiply things out we get 2^n cross terms, i.e., paths in the choice tree. C_k, the coefficient of X^k, is the number of paths with exactly k X's.

$$c_k = \binom{n}{k}$$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

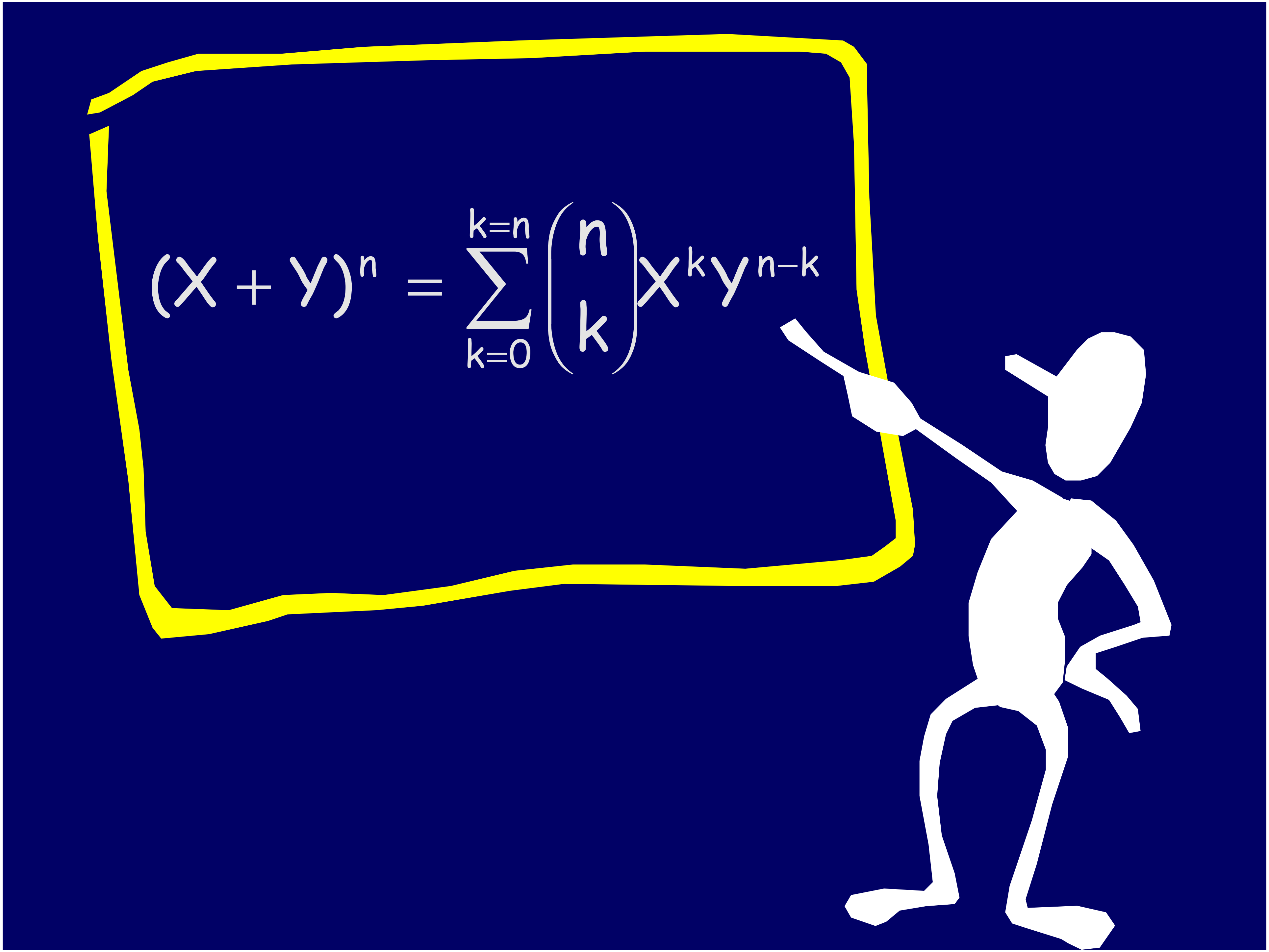
The Binomial Formula

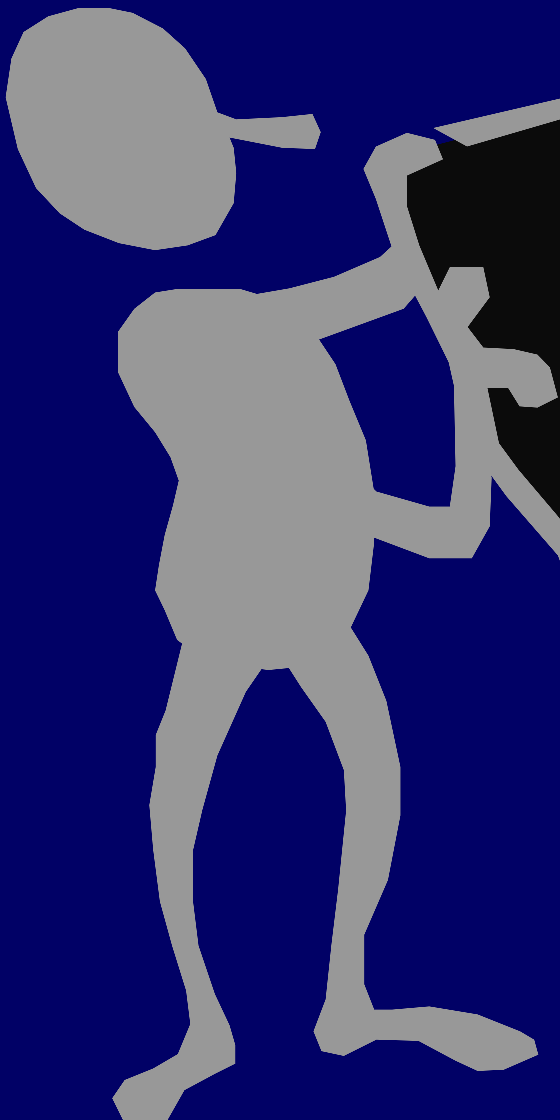
$(1+X)^0 =$	1
$(1+X)^1 =$	$1 + 1X$
$(1+X)^2 =$	$1 + 2X + 1X^2$
$(1+X)^3 =$	$1 + 3X + 3X^2 + 1X^3$
$(1+X)^4 =$	$1 + 4X + 6X^2 + 4X^3 + 1X^4$

The Binomial Formula

$$(x+y)^n$$

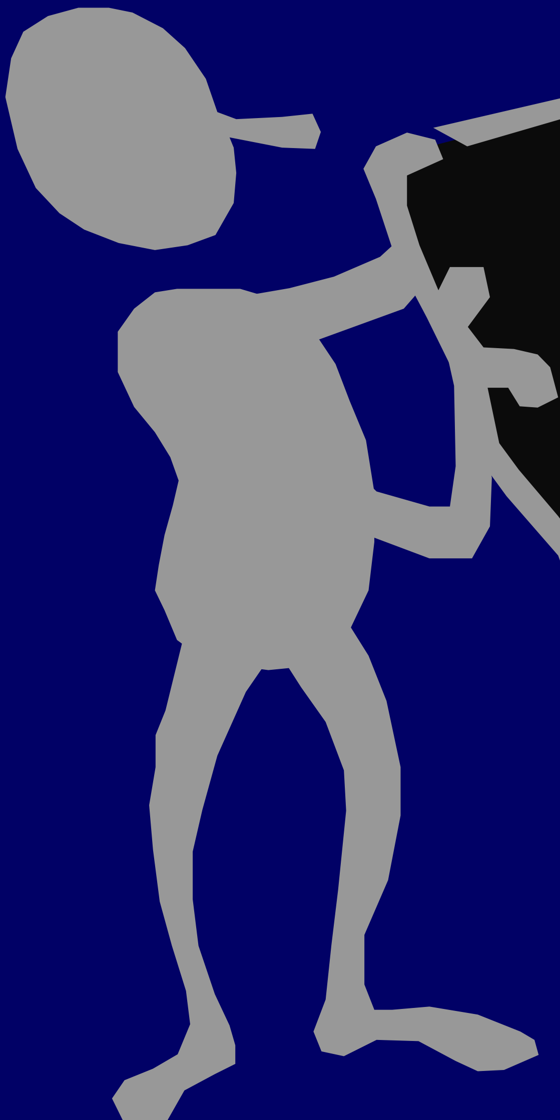
$$= \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \ldots + \binom{n}{k}x^ky^{n-k} + \ldots + \binom{n}{n}x^ny^0$$


$$(X+Y)^n = \sum_{k=0}^{k=n} \binom{n}{k} X^k Y^{n-k}$$



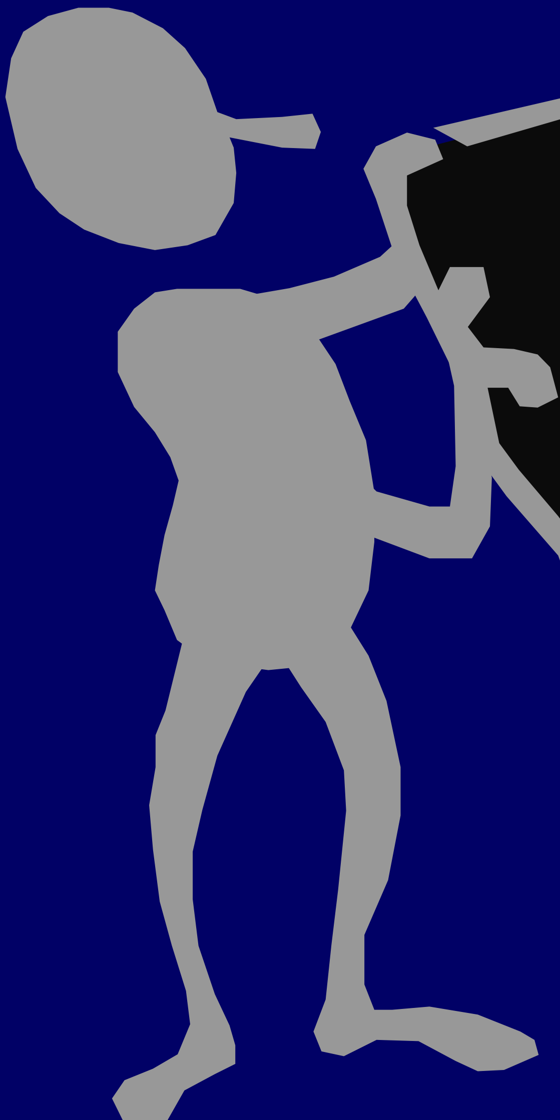
What is the coefficient of EMSTY in the expansion of $(E + M + S + T + Y)^5$?

$5!$



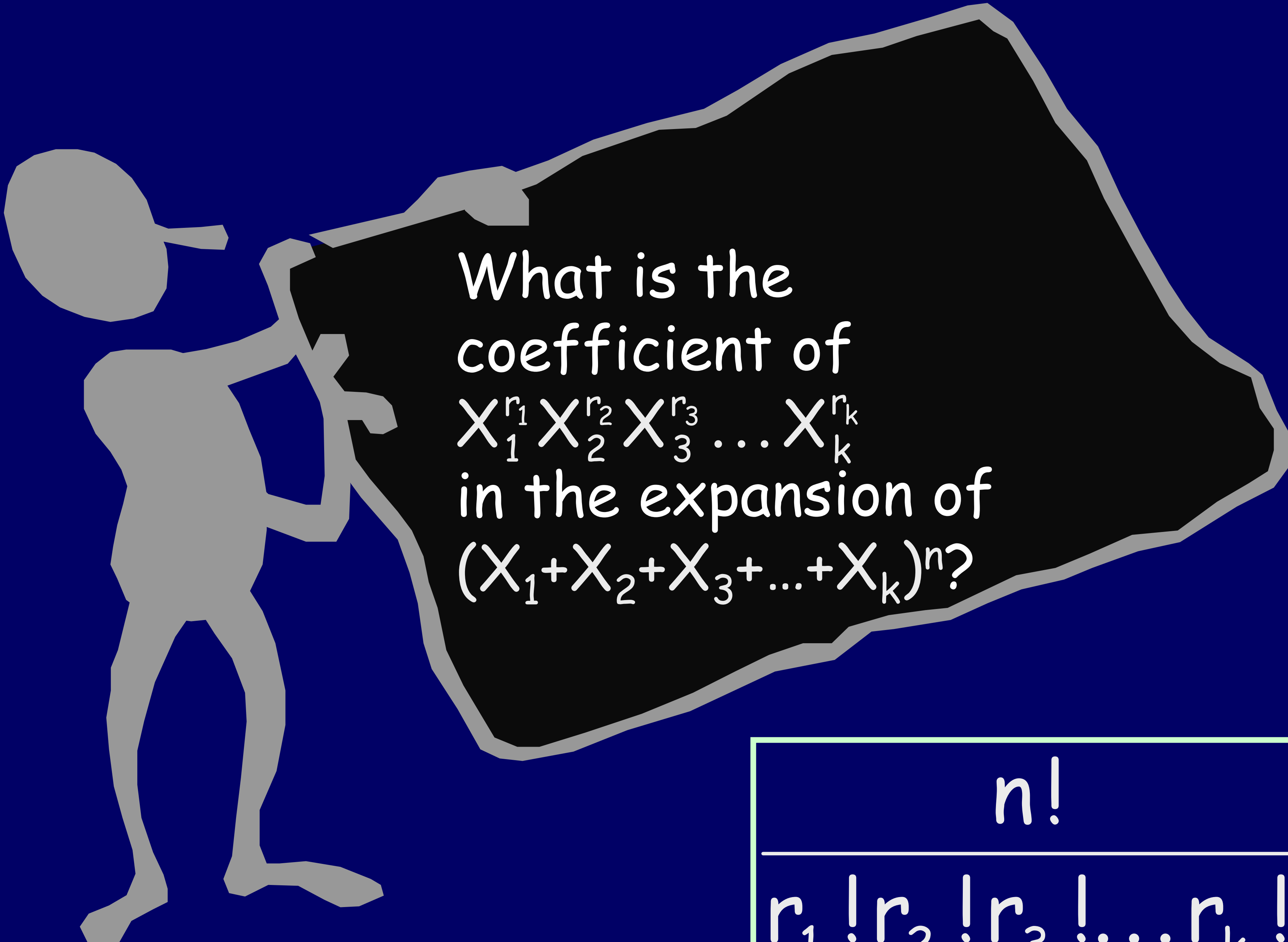
What is the coefficient of EMS^3TY in the expansion of $(E + M + S + T + Y)^7$?

The number of ways to rearrange the letters in the word SYSTEMS.



What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?

The number of ways to rearrange the letters in the word BANANA.



What is the coefficient of

$$X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}$$

in the expansion of

$$(X_1+X_2+X_3+\ldots+X_k)^n?$$

$$\boxed{\frac{n!}{r_1!\,r_2!\,r_3!\ldots r_k!}}$$

Multinomial Coefficients

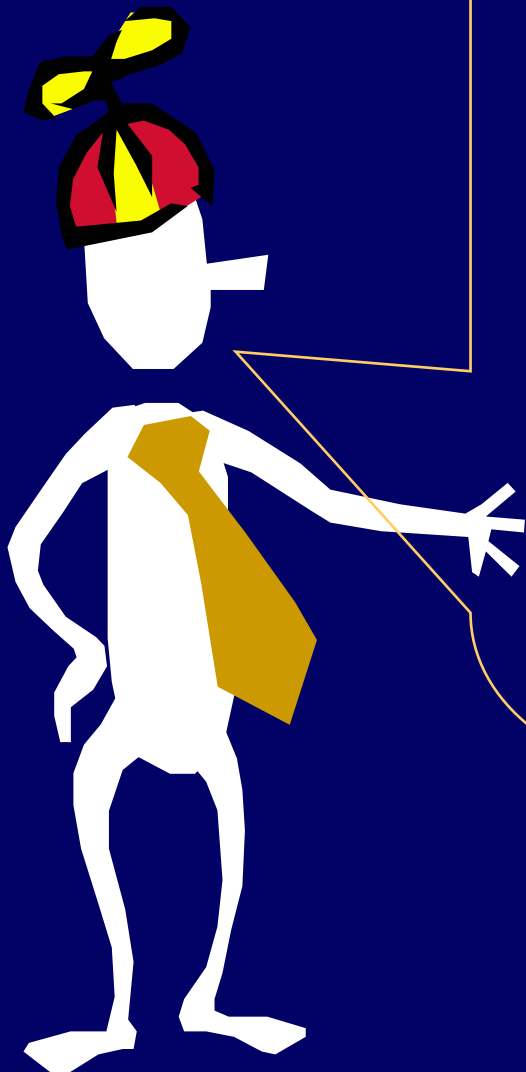
$$\binom{n}{r_1; r_2; \ldots; r_k} \equiv \begin{cases} 0 \text{ if } r_1 + r_2 + \ldots + r_k \neq n \\ \dfrac{n!}{r_1! r_2! \ldots r_k!} \end{cases}$$

$$\binom{n}{k;\ n-k} = \binom{n}{k}$$

The Multinomial Formula

$$(X_1+X_2+\ldots+X_k)^n$$

$$= \sum_{\substack{r_1,r_2,\ldots,r_k \\ \sum r_i=n}} \binom{n}{r_1;r_2;\ldots;r_k} X_1^{r_1}X_2^{r_2}X_3^{r_3}\ldots X_k^{r_k}$$



There is much, much more to be said about how polynomials encode counting questions!

References

Applied Combinatorics, by Alan Tucker