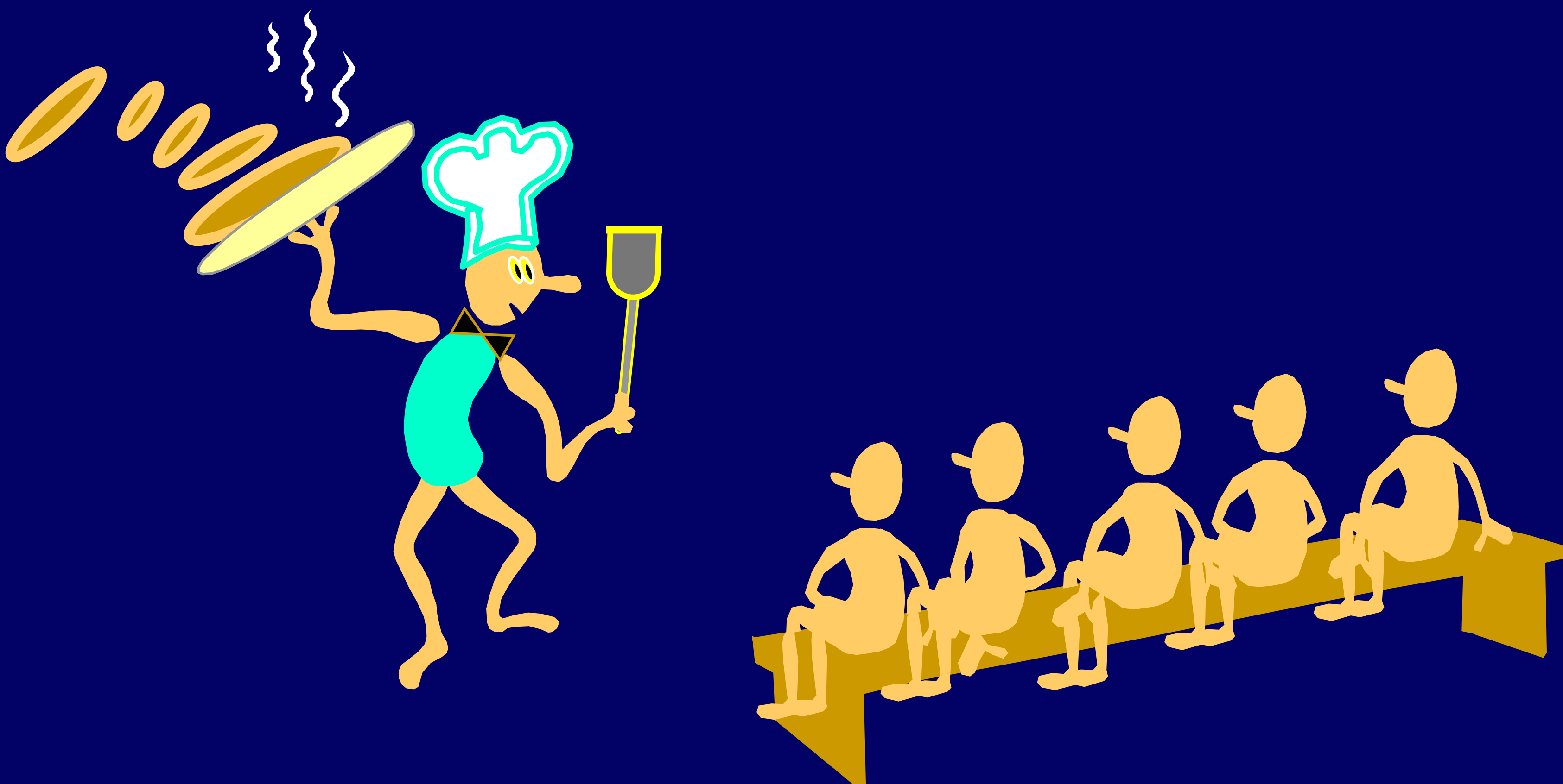
Great Theoretical Ideas In Computer Science

Steven Rudich | CS 15-251 | Spring 2004

Lecture 1 | Jan 13, 2004 | Carnegie Mellon University

# Pancakes With A Problem!



# Magic Trick At 3:00pm Sharp!

Be punctual.

Sit close-up: some of the tricks are hard to see from the back.

# Course Staff

Profs: Steven Rudich
Anupam Gupta

TAs:

| | |
|---|---|
| Yinmeng Zhang | Bella Voldman |
| Brendan Juba | Andrew Gilpin |
| Susmit Sarkar | Adam Wierman |


Please feel free
to ask questions!

# Course Document
# You must read this carefully.

1. Grading formula for the course.
   1. 40% homework
   2. 30% quizes
   3. 30% final
2. Seven points a day late penalty.
3. Collaboration/Cheating Policy
   1. You may NOT share written work.
   2. We reuse homework problems.

# My Low Vision and You.

I have a genetic retinal condition called Stargardt's disease. My central vision is going, one pixel at a time, to zero. I have working peripheral vision.

I can't recognize faces – so please introduce yourself to me every time!

I detect motion really well so please move your hand when you raise it in class.

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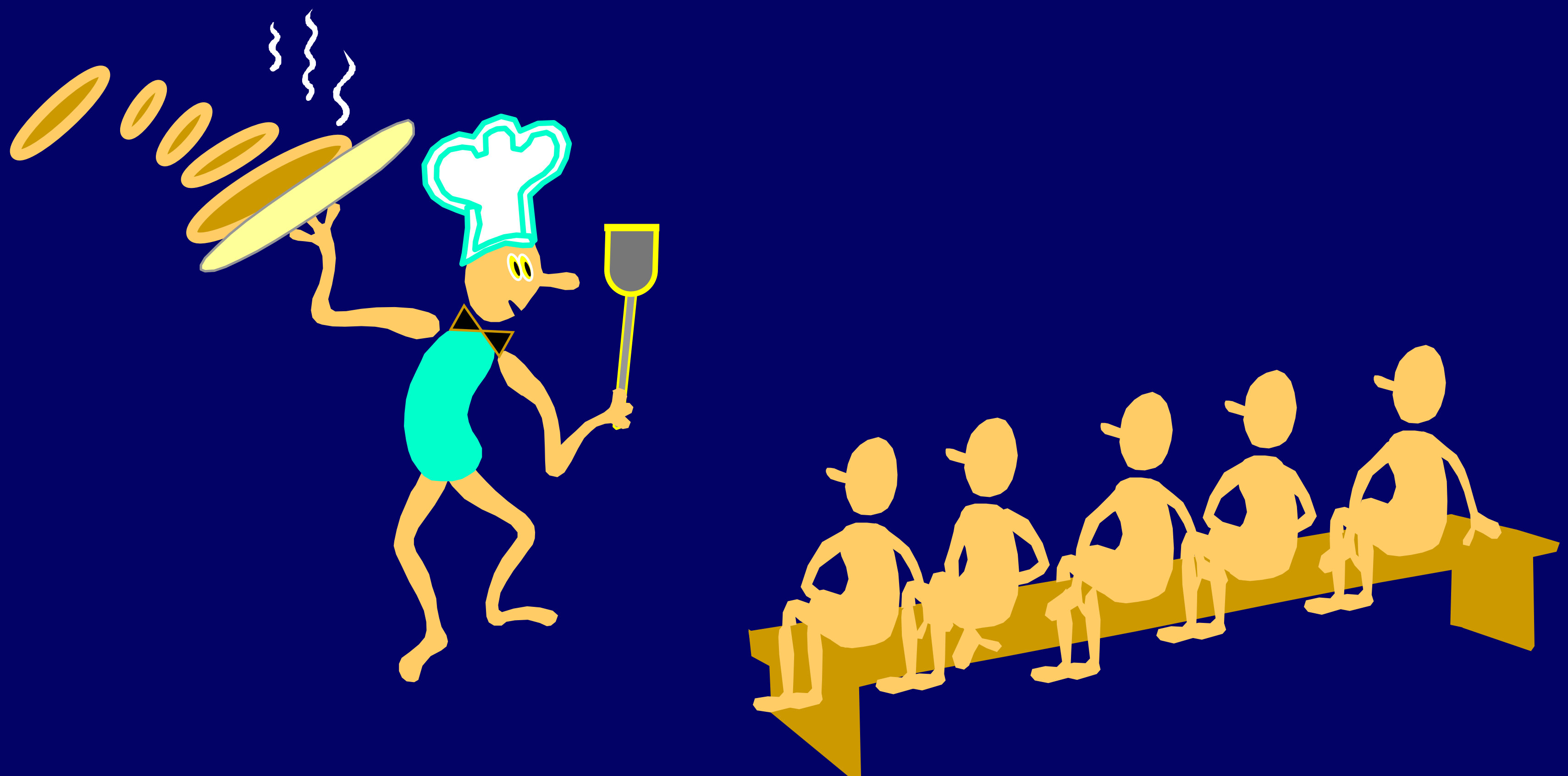
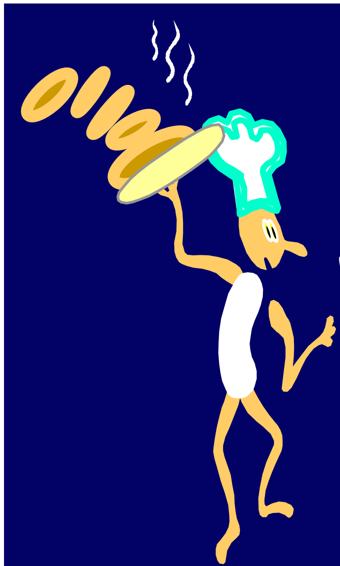
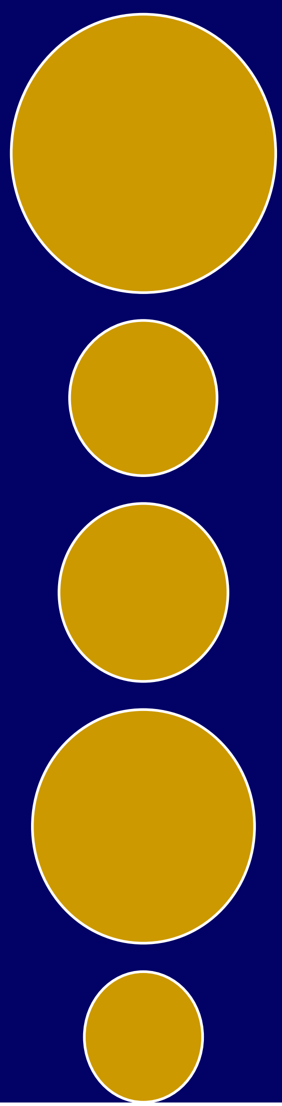
# Pancakes With A Problem!



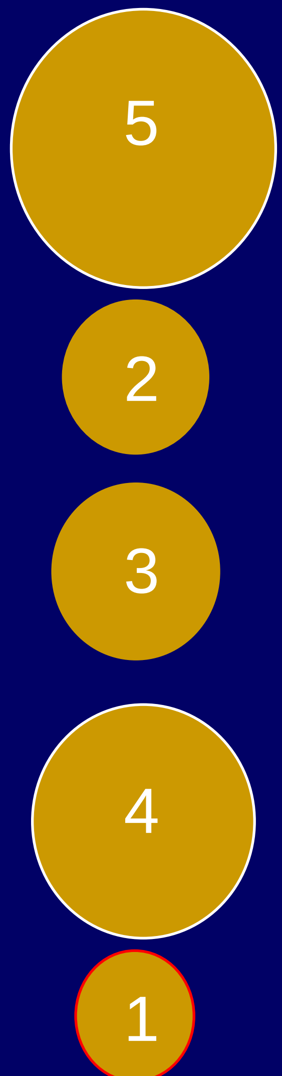


The chef at our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest at the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary.

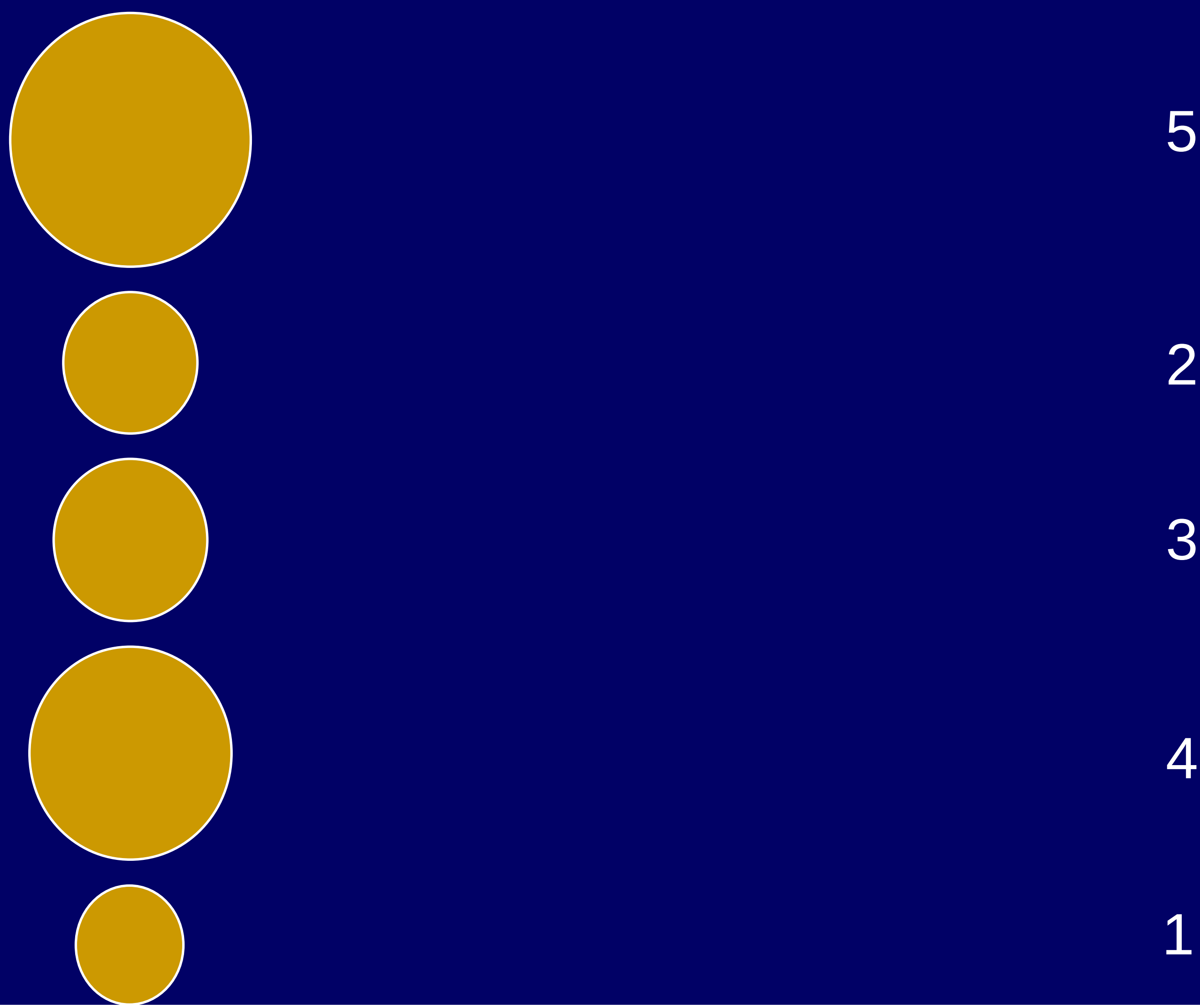
# Developing A Notation:
# Turning pancakes into numbers



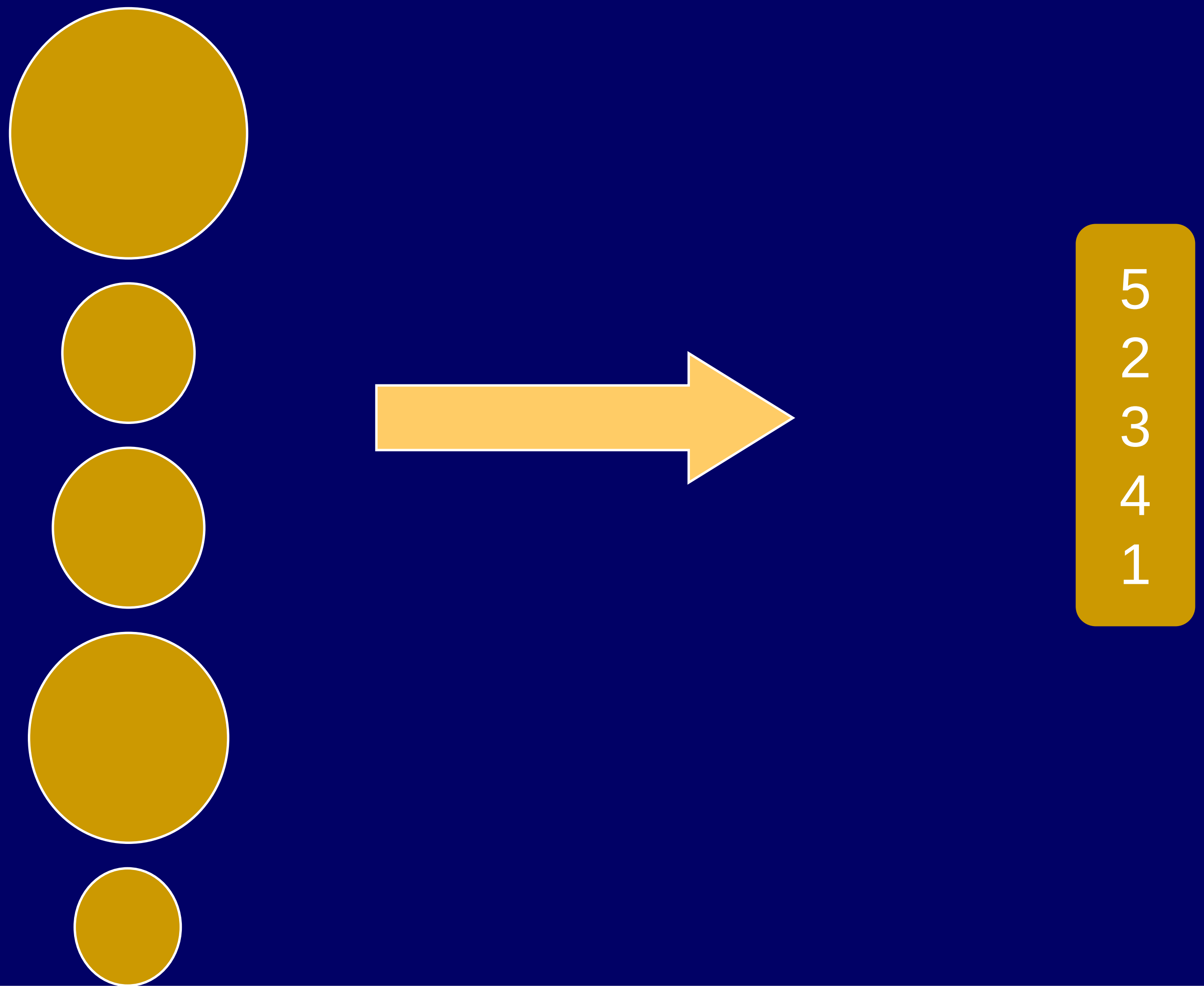
# Developing A Notation: Turning pancakes into numbers



# Developing A Notation: Turning pancakes into numbers

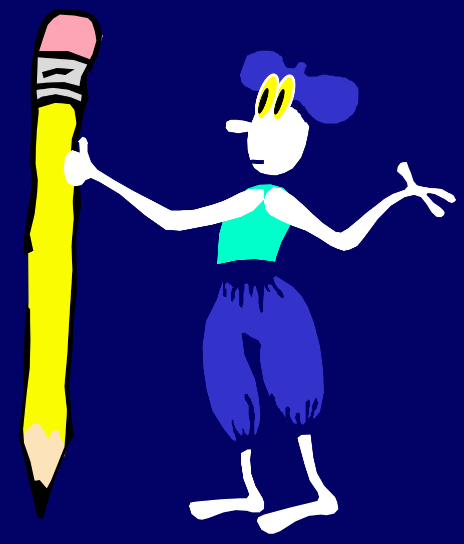


# Developing A Notation: Turning pancakes into numbers

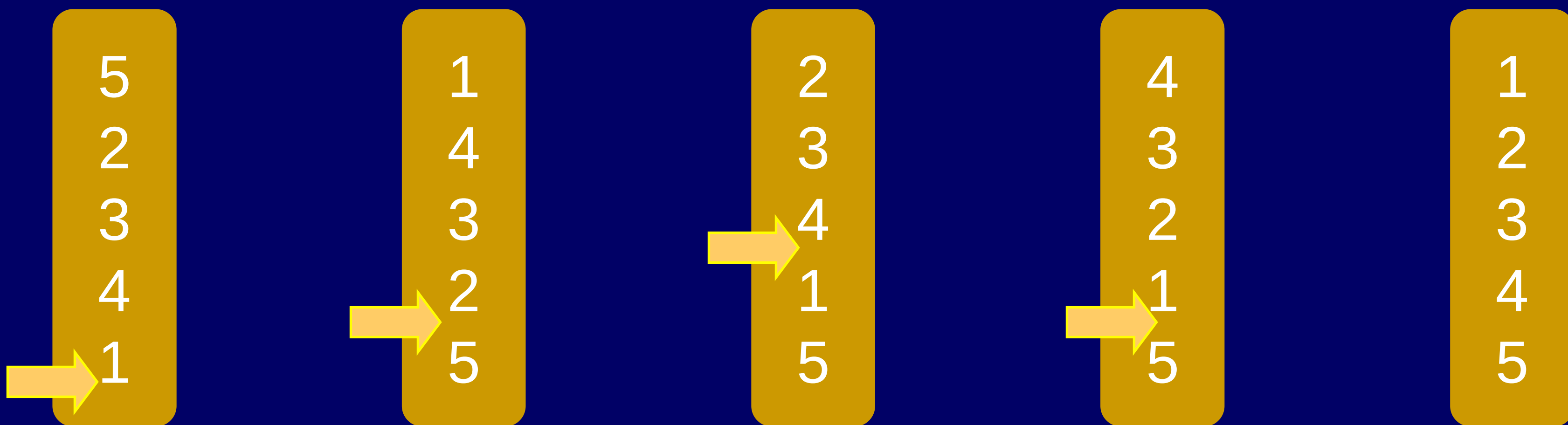


How do we sort this stack?
How many flips do we need?

5
2
3
4
1



# 4 Flips Are Sufficient



# Algebraic Representation

X = The smallest number of flips required to sort:

5
2
3
4
1

$? \leq X \leq ?$

Lower Bound

Upper Bound

# Algebraic Representation

X = The smallest number of flips required to sort:

5
2
3
4
1

$? \leq X \leq 4$

Upper Bound

Lower Bound

# 4 Flips Are Necessary

| | | |
|---|---|---|
| 5 | 1 | 4 |
| 2 | 4 | 1 |
| 3 | 3 | 3 |
| 4 | 2 | 2 |
| 1 | 5 | 5 |

Flip 1 has to put 5 on bottom

Flip 2 must bring 4 to top.

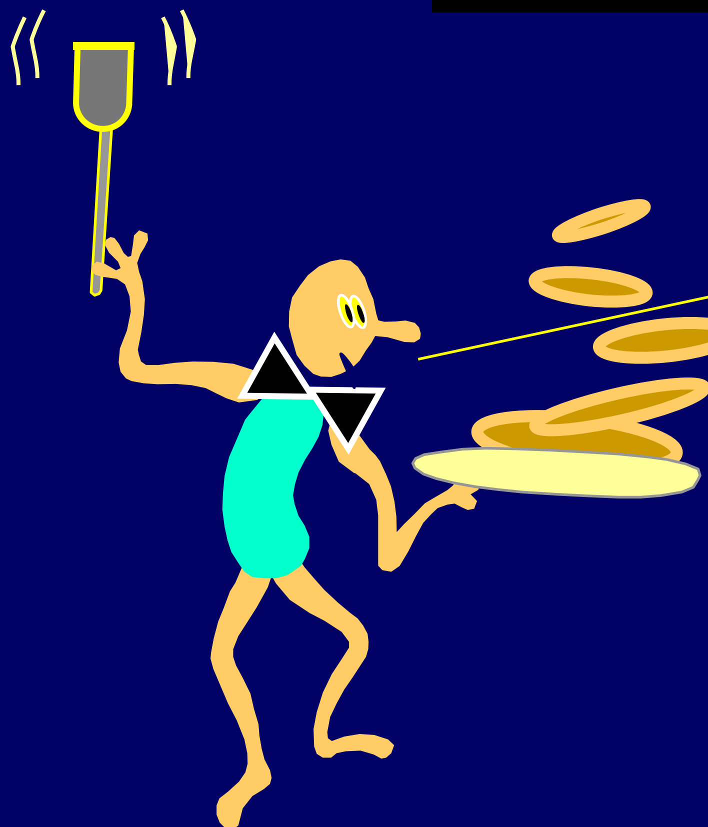
$$? \le X \le 4$$

Lower
Bound

$$4 \leq X \leq 4$$

Lower Bound

Upper Bound



X = 4

# 5th Pancake Number

$P_5 =$ The number of flips required to sort the worst case stack of 5 pancakes.

$$? \leq P_5 \leq ?$$

Upper Bound

Lower Bound

# $5^{th}$ Pancake Number

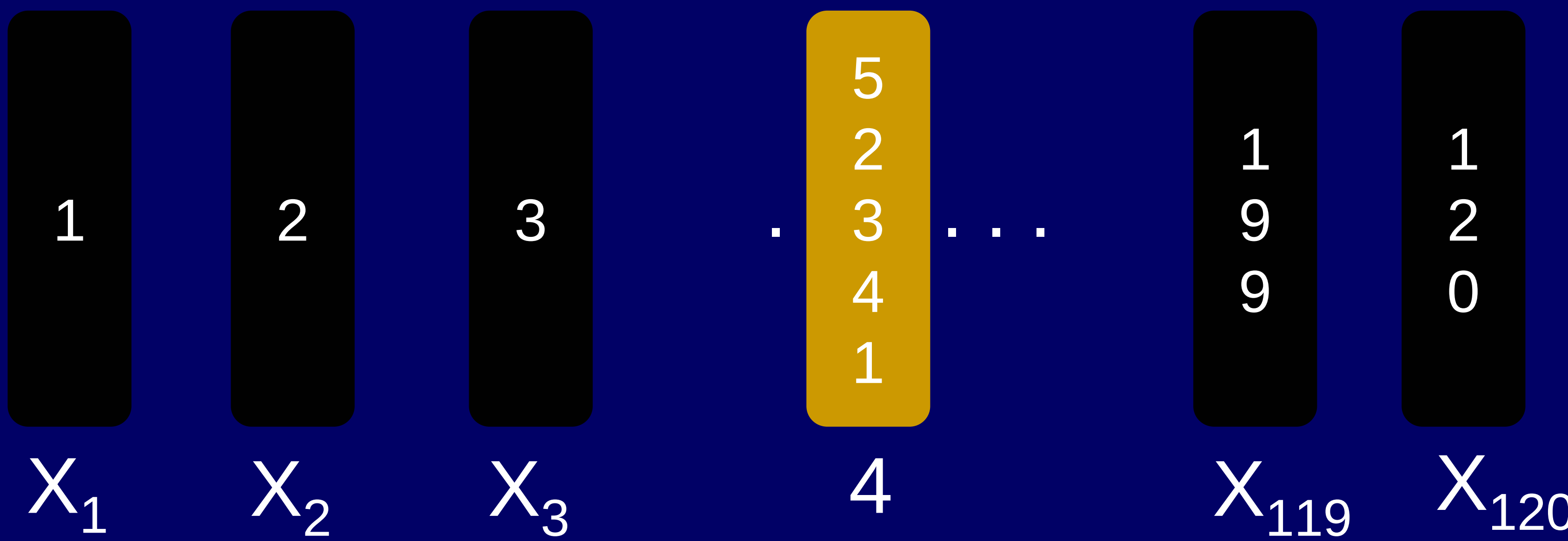
$P_5 =$ The number of flips required to sort the worst case stack of 5 pancakes.

$$4 \leq P_5 \leq ?$$

Upper Bound

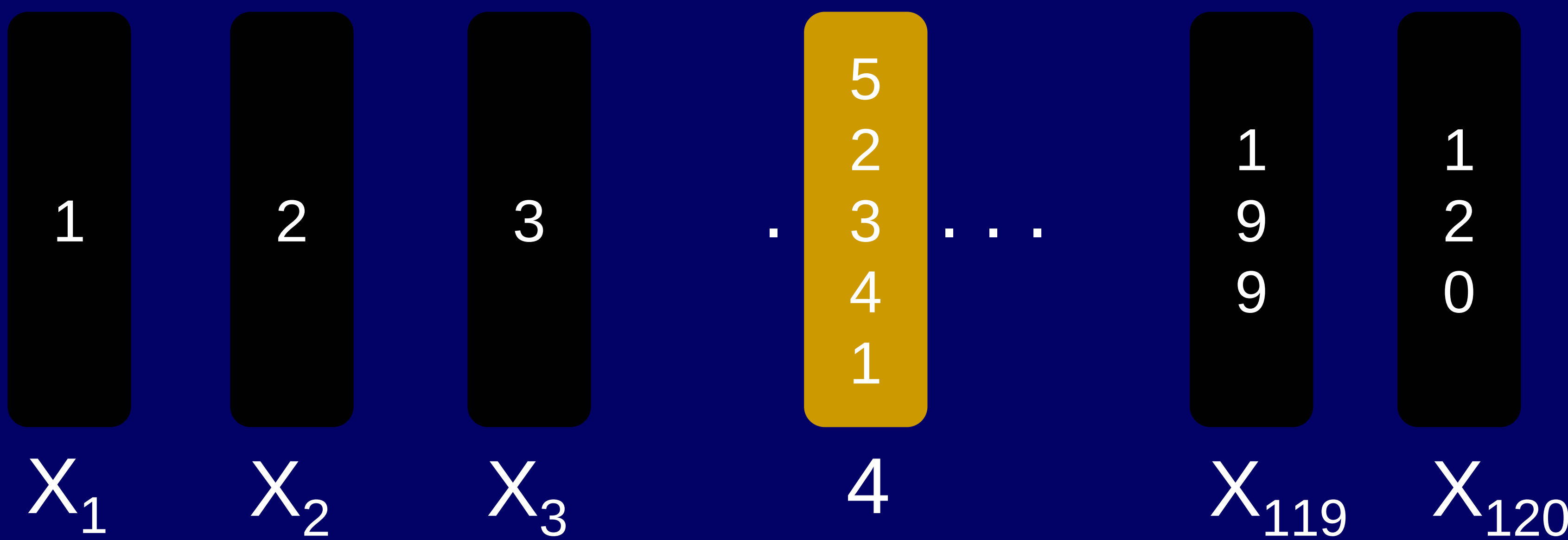
Lower Bound

# The 5th Pancake Number: The MAX of the X's



$P_5 =$

MAX over $s \in$ stacks of 5 of MIN # of flips to sort $s$



$P_n =$

MAX over $s \in$ stacks of n pancakes of MIN # of flips to sort s

$P_n =$

The number of flips required to sort the worst-case stack of n pancakes.

$P_n =$

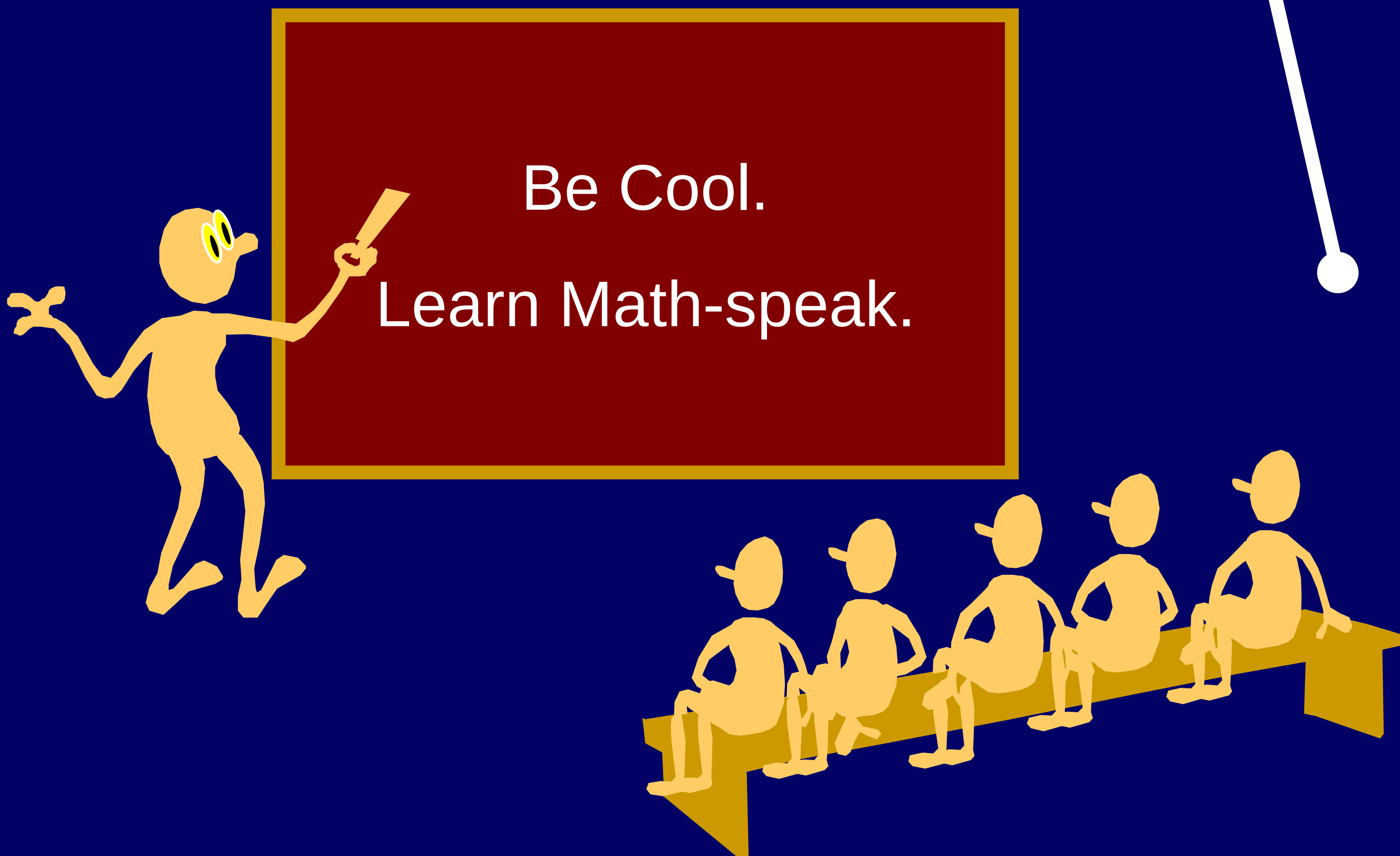
MAX over $s \in$ stacks of n pancakes of MIN # of flips to sort s

$P_n =$

The number of flips required to sort a worst-case stack of n pancakes.

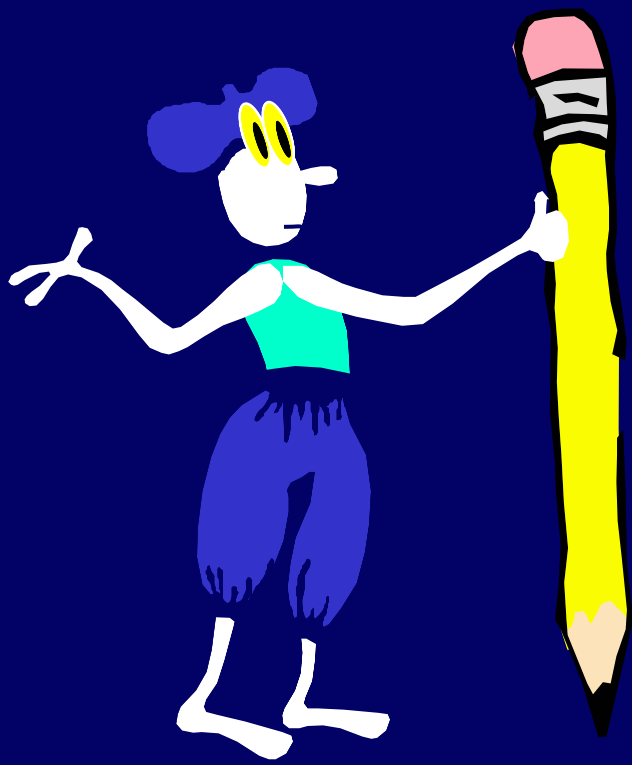
$P_n =$

The number of flips required to sort a worst-case stack of n pancakes.



# What is $P_n$ for small n?

Can you do n= 0,1,2, 3 ?



# Initial Values Of $P_n$.

| | | | | |
|---|---|---|---|---|
| $n$ | 0 | 1 | 2 | 3 |
| $P_n$ | 0 | 0 | 1 | 3 |

# $P_3 = 3$

1
3
2 requires 3 Flips, hence $P_3 \geq 3$.

ANY stack of 3 can be done by getting the big one to the bottom ($\leq$ 2 flips), and then using $\leq$ 1 extra flip to handle the top two. Hence, $P_3 \leq 3$.

# $n^{th}$ Pancake Number

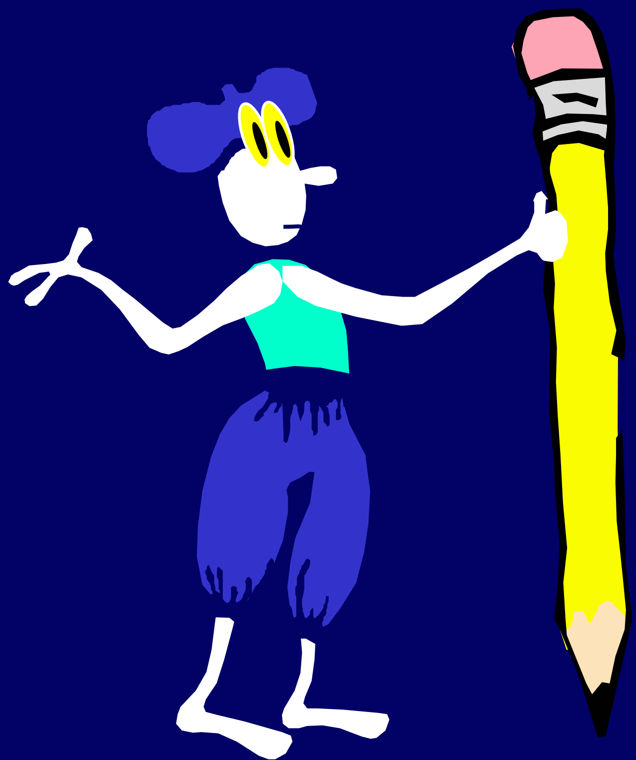
$P_n$ = The number of flips required to sort a worst case stack of n pancakes.

$$? \leq P_n \leq ?$$

Lower Bound

Upper Bound

# $? \le P_n \le ?$



Take a few minutes to try and prove bounds on $P_n$, for $n>3$.

# Bring To Top Method

Bring biggest to top. Place it on bottom. Bring next largest to top. Place second from bottom. And so on...

# Upper Bound On $P_n$:

## *Bring To Top Method For n Pancakes*

If n=1, no work - we are done.

Otherwise, flip pancake n to top and then flip it to position n.

Now use:

Bring To Top Method
For n-1 Pancakes

Total Cost: at most $2(n-1) = 2n - 2$ flips.

# Better Upper Bound On $P_n$: *Bring To Top Method For n Pancakes*

If n=2, at most one flip, we are done.

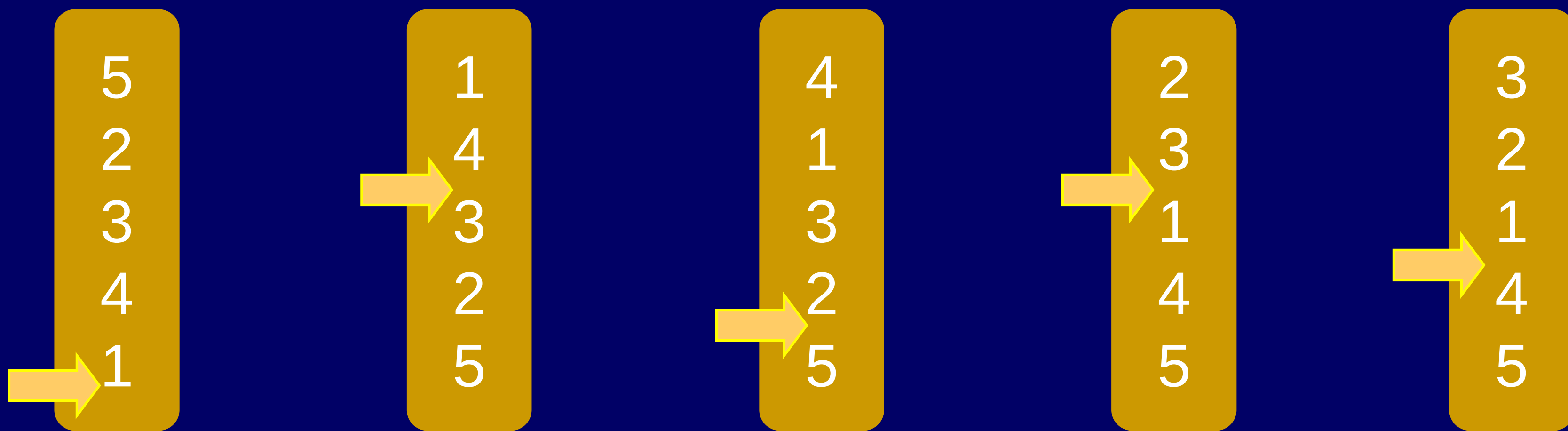
Otherwise, flip pancake n to top and then flip it to position n.

Now use:

Bring To Top Method
For n-1 Pancakes

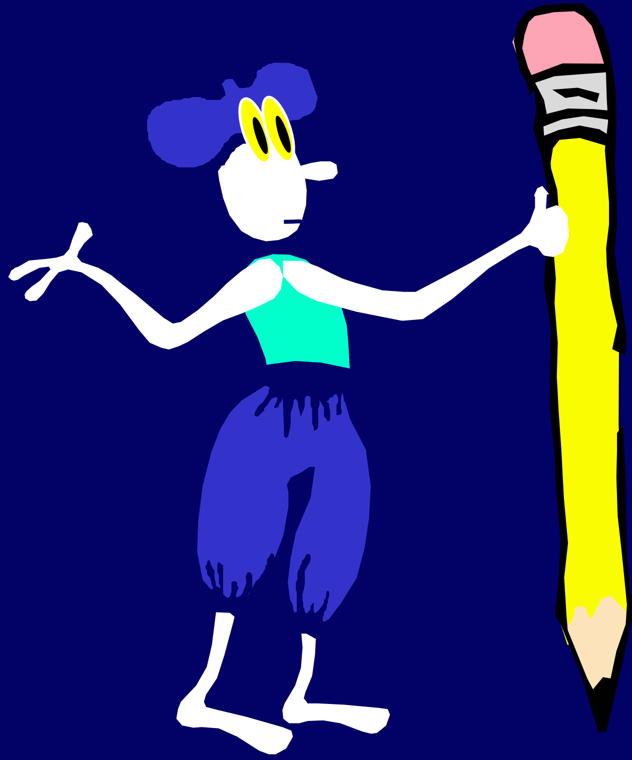
Total Cost: at most 2(n-2) + 1 = 2n –3 flips.

# Bring to top not always optimal for a particular stack




5 flips, but can be done in 4 flips

$$? \leq P_n \leq 2n - 3$$



What bounds can you prove on $P_n$?

# Breaking Apart Argument

Suppose a stack S contains a pair of adjacent pancakes that will not be adjacent in the sorted stack. Any sequence of flips that sorts stack S must involve one flip that inserts the spatula between that pair and breaks them apart.

# Breaking Apart Argument

Suppose a stack S contains a pair of adjacent pancakes that will not be adjacent in the sorted stack. Any sequence of flips that sorts stack S must involve one flip that inserts the spatula between that pair and breaks them apart. Furthermore, this same principle is true of the "pair" formed by the bottom pancake of S and the plate.

# $n \leq P_n$

Suppose n is even. S contains n pairs that will need to be broken apart during any sequence that sorts stack S.

S

| |
|---|
| 2 |
| 4 |
| 6 |
| 8 |
| . |
| . |
| n |
| 1 |
| 3 |
| 5 |
| 7 |
| . |
| . |
| n-1 |

# $n \le P_n$

S


Suppose n is even. S contains n pairs that will need to be broken apart during any sequence that sorts stack S.

Detail: This construction only works when n>2

# $n \leq P_n$

Suppose n is odd. S contains n pairs that will need to be broken apart during any sequence that sorts stack S.

S

| S |
| --- |
| 1 |
| 3 |
| 5 |
| 7 |
| . |
| . |
| n |
| 2 |
| 4 |
| 6 |
| 8 |
| . |
| . |
| n-1 |

# $n \leq P_n$

S

1
3
2

Suppose n is odd. S contains n pairs that will need to be broken apart during any sequence that sorts stack S.

Detail: This construction only works when n>3

$$n \le P_n \le 2n - 3$$
$$\text{for } n \ge 3$$

Bring To Top is within a factor of two of optimal!

$$n \le P_n \le 2n - 3$$

So starting from ANY stack we can get to the sorted stack using no more than $P_n$ flips.

From ANY stack to sorted stack in $\leq P_n$.

From sorted stack to ANY stack in $\leq P_n$ ?

Reverse the sequences we use to sort.

From ANY stack to sorted stack in $\leq P_n$.

From sorted stack to ANY stack in $\leq P_n$.

Hence,

From ANY stack to ANY stack in $\leq 2P_n$.

From ANY stack to ANY stack in $\leq 2P_n$.

Can you find
a faster way
than $2P_n$ flips
to go from
ANY to ANY?

# From ANY Stack S to ANY stack T in $\leq P_n$

Rename the pancakes in S to be 1,2,3,..,n. Rewrite T using the new naming scheme that you used for S. T will be some list: $\pi(1),\pi(2),..,\pi(n)$. The sequence of flips that brings the sorted stack to $\pi(1),\pi(2),..,\pi(n)$ will bring S to T.

| S: | T: |
|---|---|
| 4,3,5,1,2 | 5,2,4,3,1 |
| 1,2,3,4,5 | 3,5,1,2,4 |

# The Known Pancake Numbers

| $n$ | $P_n$ |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 3 | 3 |
| 4 | 4 |
| 5 | 5 |
| 6 | **7** |
| 7 | **8** |
| 8 | **9** |
| 9 | **10** |
| 10 | **11** |
| 11 | **13** |
| 12 | **14** |
| 13 | **15** |

# $P_{14}$ Is Unknown

14! Orderings of 14 pancakes.

14! = 87,178,291,200

# Is This Really Computer Science?

Posed in *Amer. Math. Monthly* 82 (1) (1975),
"Harry Dweighter" a.k.a. Jacob Goodman

$(17/16)n \le P_n \le (5n+5)/3$

Bill Gates & Christos Papadimitriou: Bounds For Sorting By Prefix Reversal. *Discrete Mathematics*, vol 27, pp 47-57, 1979.

$$(15/14)n \le P_n \le (5n+5)/3$$

H. Heydari & H. I. Sudborough: On the Diameter of he Pancake Network. *Journal of Algorithms,* vol 25, pp 67-94, 1997.

# Permutation

Any particular ordering of all n elements of an n element set S, is called a <u>permutation</u> on the set S.

Example: S = {1, 2, 3, 4, 5}

Example permutation: 5 3 2 4 1

120 possible permutations on S

# Permutation

Any particular ordering of all n elements of an n element set S, is called a <u>permutation</u> on the set S.

Each different stack of n pancakes is one of the permutations on [1..n].

# Representing A Permutation

We have many choices of how to specify a permutation on S. Here are two methods:

1) We list a sequence of all the elements of [1..n], each one written exactly once.
   Ex: 6 4 5 2 1 3

2) We give a function $\pi$ on S such that $\pi$ (1) $\pi(2)$ $\pi(3)$ .. $\pi(n)$ is a sequence that lists [1..n], each one exactly once.
   Ex:
   $\pi(1)=6$ $\pi(2)=4$ $\pi(3) = 5$ $\pi(4) = 2$ $\pi(4) = 1$ $\pi(6) =3$

# A Permutation is a NOUN

An ordering of a stack of pancakes is a permutation.

# A Permutation is a NOUN.
# A permutation can also be a VERB.

An ordering S of a stack of pancakes is a permutation.

We can permute S to obtain a new stack S'.

Permute also means to rearrange so as to obtain a permutation of the original.

# Permute A Permutation.

I start with a permutation S of pancakes. I continue to use a flip operation to permute my current permutation, so as to obtain the sorted permutation.

There are $n! = 1*2*3*4*...*n$ permutations on n elements.

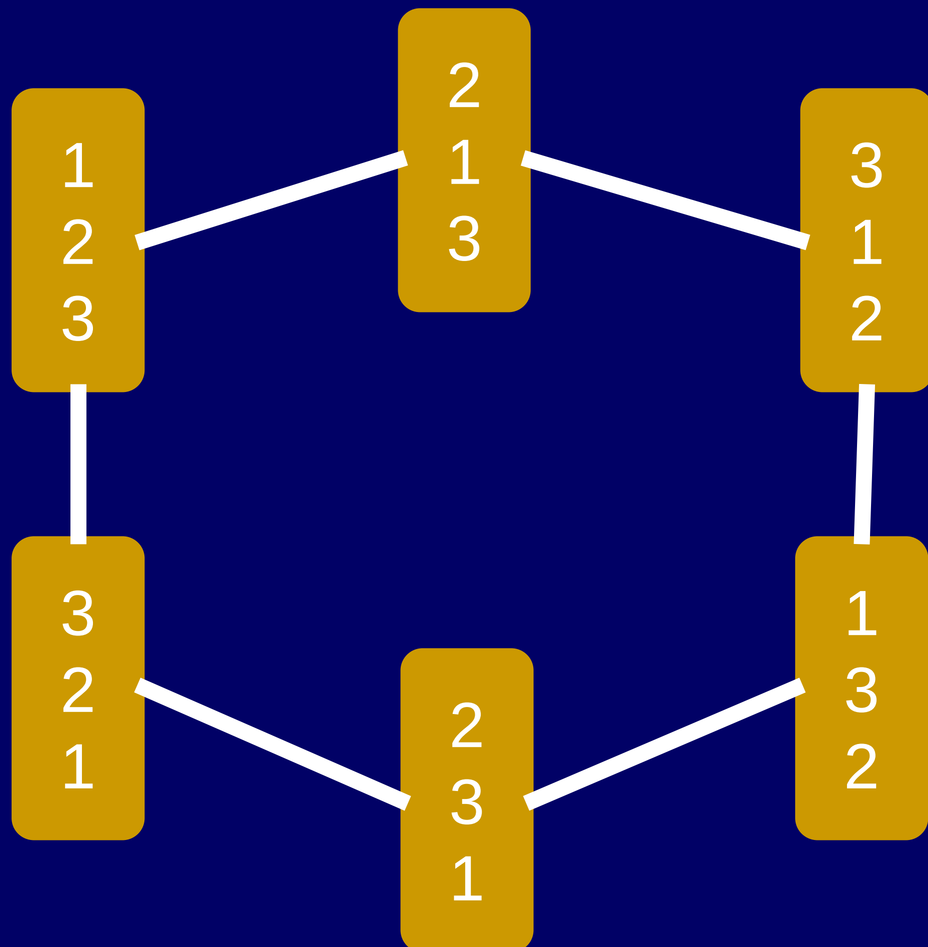
Proof in the first counting lecture.

# Pancake Network: Definition For n! Nodes

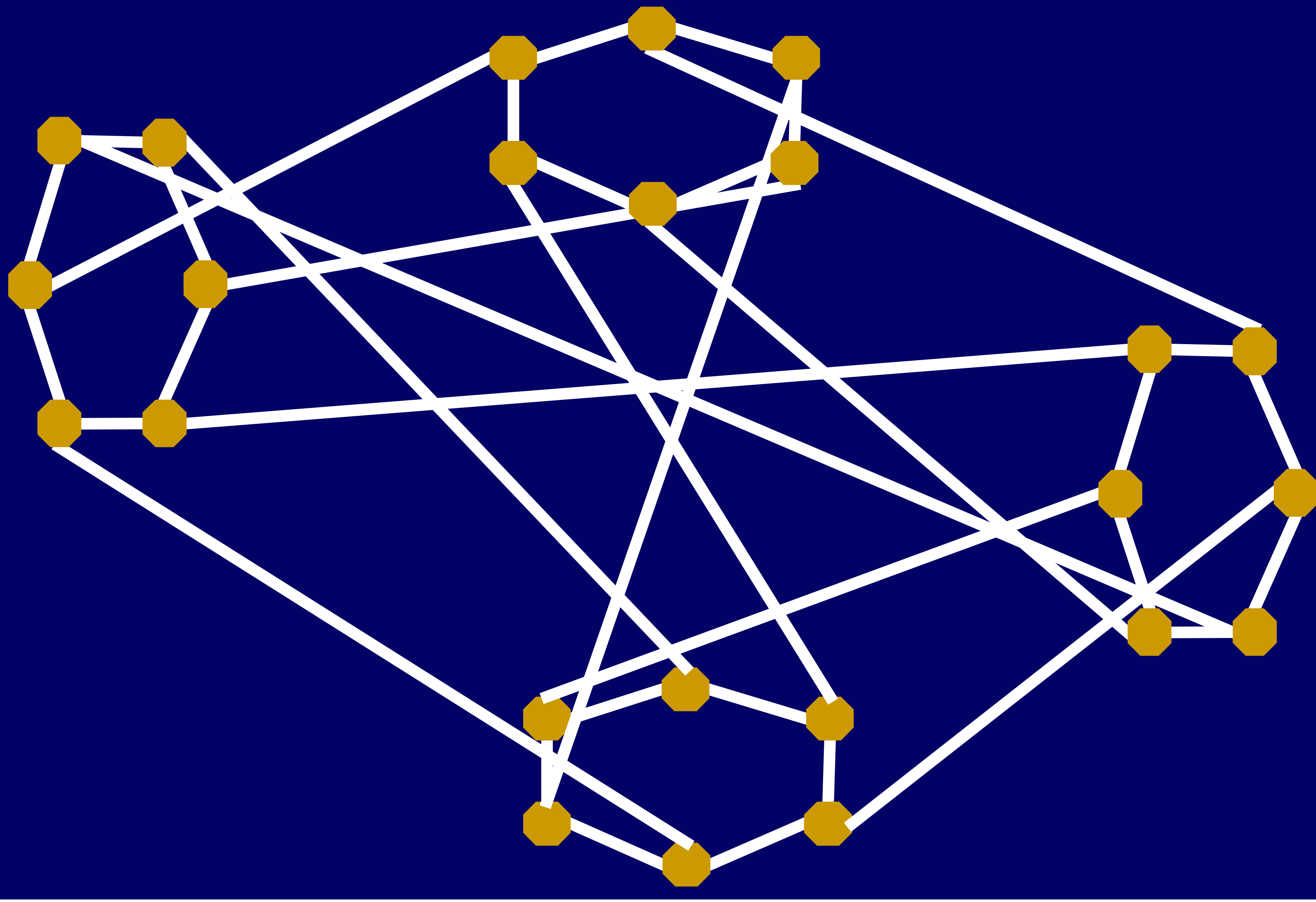
For each node, assign it the name of one of the n! stacks of n pancakes.

Put a wire between two nodes if they are one flip apart.

# Network For n=3

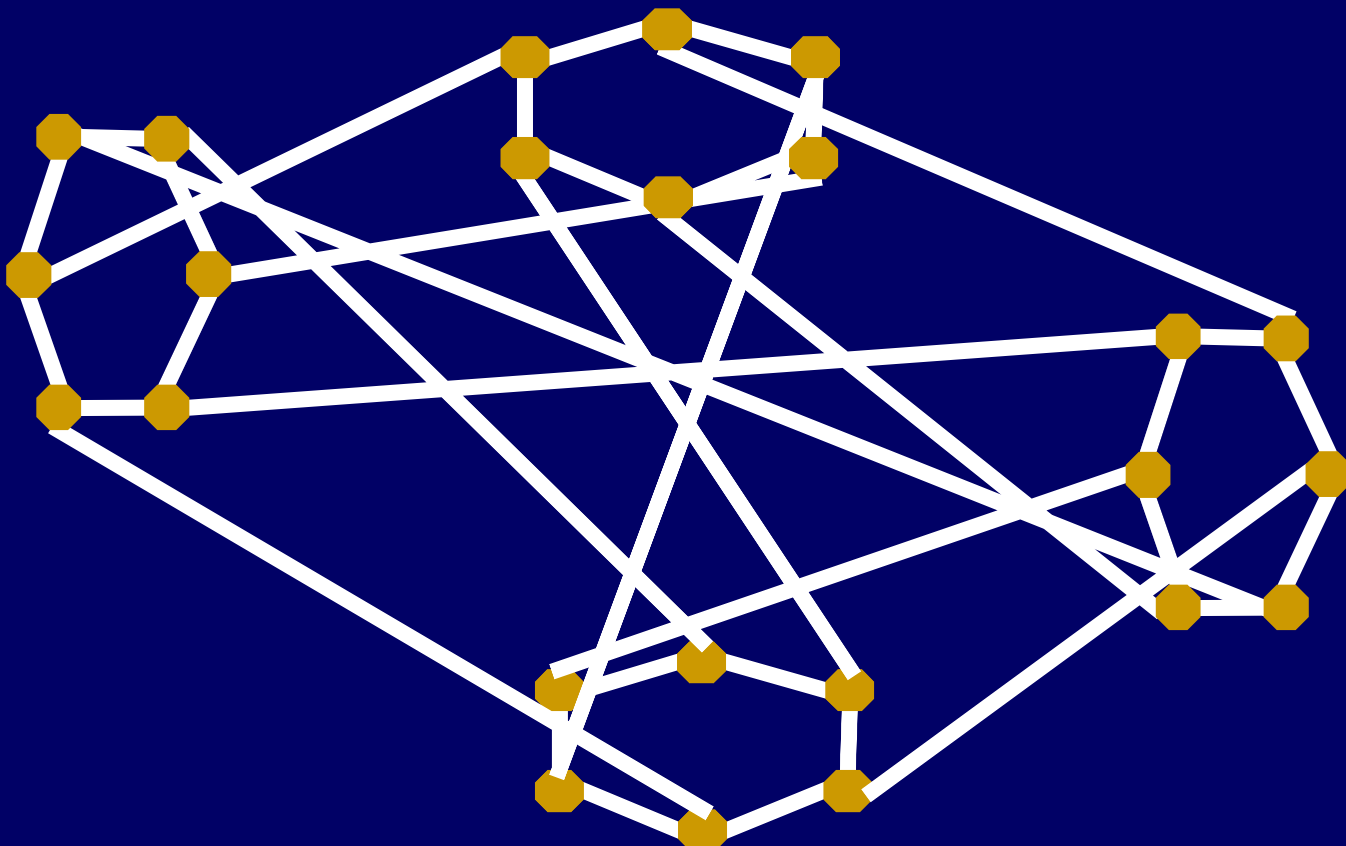


# Network For n=4



# Pancake Network: Message Routing Delay

What is the maximum distance between two nodes in the network?



$P_n$

# Pancake Network: Reliability

If up to n-2 nodes get hit by lightning the network remains connected, even though each node is connected to only n-1 other nodes.

The Pancake Network is optimally reliable for its number of edges and nodes.

# Mutation Distance


# One "Simple" Problem

A host of problems and applications at the frontiers of science

Study Bee

You must read the course document carefully.

You must hand-in the signed cheating policy page.

Study Bee

Definitions of:

nth pancake number

lower bound

upper bound

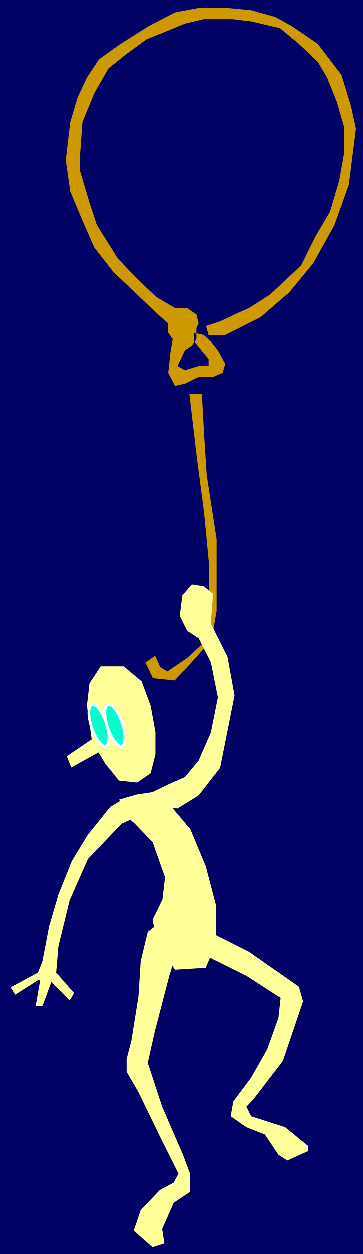
permutation

Proof of:

ANY to ANY in $\leq P_n$

# High Level Point

This lecture is a microcosm of mathematical modeling and optimization.



# References

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