①

Stable Marriage Problem

1	2	4	1	3	1	2	1	4	3
2	3	1	4	2	2	4	3	1	2
3	2	3	1	4	3	1	4	3	2
4	4	1	3	2	4	2	1	4	3

Men's Preferences Women's Preferences

Two stable matchings:

(1,4) (2,3) (3,2) (4,1)

(1,4) (2,1) (3,2) (4,3)

unstable:

(1,1) (2,3) (3,2) (4,4)

Blocking Pair ?

Thm: There always exists a stable marriage.

Prove Later

```
assign each person to be free ;
while some man m is free do
begin
    w := first woman on m's list to whom m has not yet proposed ;
    if w is free then
        assign m and w to be engaged {to each other}
    else
        if w prefers m to her fiancé m' then
            assign m and w to be engaged and m' to be free
        else
            w rejects m {and m remains free}
end ;
output the stable matching consisting of the n engaged pairs
```

Figure 1.3: Basic Gale-Shapley algorithm

Theorem 1.2.1 *For any given instance of the stable marriage problem, the Gale-Shapley algorithm terminates, and, on termination, the engaged pairs constitute a stable matching.*

Proof: ① The engagements always form a matching.

② Once a woman is engaged she remains engaged.

③ Each new engagement is to a better man for her.

A man cannot be rejected by all women. Because then all the women must be engaged.

Impossible since # men = # women

The algorithm must terminate because on each iteration a man progresses down his list.

$\Rightarrow$ at most n^2 iterations.

On termination the pairs form a perfect matching.

Proof Contd

Why is it stable?

Let M denote the matching.

If man m prefers w to $M(w)$ then w must have rejected m at some point.

$\Rightarrow$ the man she is paired with is better for her than m.

$\Rightarrow$ the pair (m, w) is not a blocking pair for M.

$\Rightarrow$ There is no blocking pair for M

$\Rightarrow$ M is stable.

QED

Theorem 1.2.2 *All possible executions of the Gale-Shapley algorithm (with the men as proposers) yield the same stable matching, and in this stable matching, each man has the best partner that he can have in any stable matching.*

Assume false
Assume stable matchings M, M'
s.t.

$$m \text{ prefers } w' = M'(m)$$
$$\text{to } \quad w = M(m)$$

w' must have rejected m
she rejected him for m'.
i.e. w' prefers m' to m

Suppose WLOG that this is the first time running the alg constructing M that this happens.

[ie: a Woman rejects a stable partner]

$\Rightarrow$ m' can have no stable partner he prefers to w'. (Because that partner must have rejected him earlier)

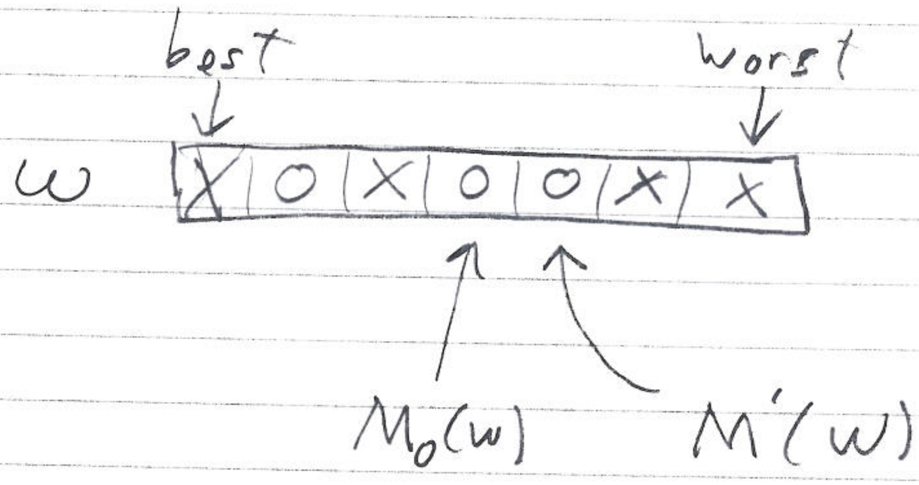
$\Rightarrow$ m' prefers w' to his partner in M'

$\Rightarrow$ M' is blocked by $(m' w')$

QED

Theorem 1.2.3 *In the man-optimal stable matching, each woman has the worst partner that she can have in any stable matching.*

Proof Suppose not. Let M_0 be the man-optimal stable matching, and suppose there is a stable matching M' and a woman w such that w prefers $m = p_{M_0}(w)$ to $m' = p_{M'}(w)$. But then (m, w) blocks M' unless m prefers $p_{M'}(m)$ to $w = p_{M_0}(m)$, in contradiction of the fact that m has no stable partner better than his partner in M_0. □



could m prefer $M'(m)$ to $M_0(m)$?
No

$\Rightarrow$ m prefers $M_0(m)$ to $M'(m)$

$\Rightarrow$ (m, w) blocks M'.