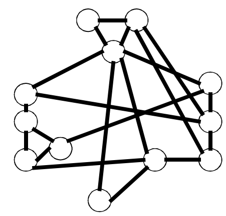
15-251

Great Theoretical Ideas in Computer Science

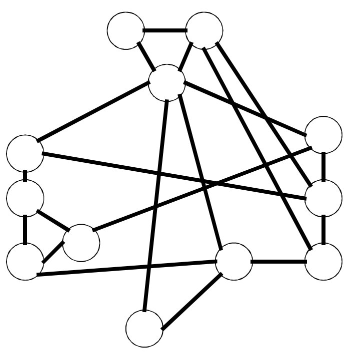
Complexity Theory:

Efficient Reductions Between Computational Problems

Lecture 27 (November 24, 2009)



A Graph Named "Gadget"



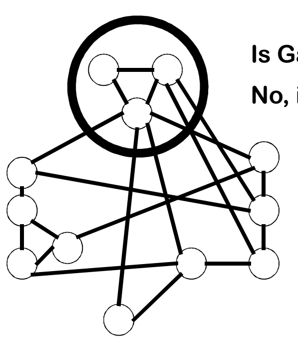
K-Coloring

We define a k-coloring of a graph:

- Each node gets colored with one color
- At most k different colors are used
- If two nodes have an edge between them they must have different colors

A graph is called k-colorable if and only if it has a k-coloring

A 2-CRAYOLA Question!



Is Gadget 2-colorable?

No, it contains a triangle

A 2-CRAYOLA Question!

Given a graph G, how can we decide if it is 2-colorable?

Answer: Enumerate all 2^n possible colorings to look for a valid 2-color

How can we efficiently decide if G is 2-colorable?

Theorem: G contains an odd cycle if and only if G is not 2-colorable

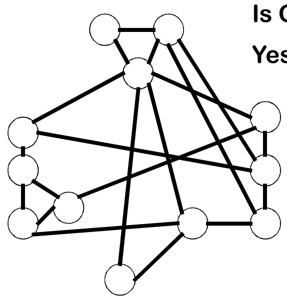
Alternate coloring algorithm:

- To 2-color a connected graph G, pick an arbitrary node v, and color it white
- Color all v's neighbors black
- Color all their uncolored neighbors white, and so on
- If the algorithm terminates without a color conflict, output the 2-coloring
- Else, output an odd cycle

A 2-CRAYOLA Question!

Theorem: G contains an odd cycle if and only if G is not 2-colorable

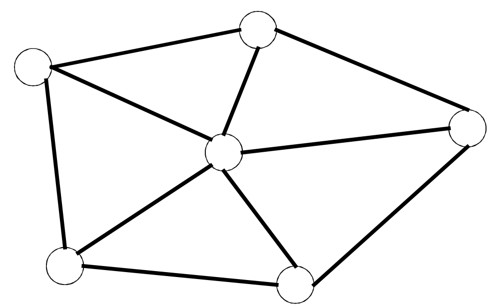
A 3-CRAYOLA Question!



Is Gadget 3-colorable?

Yes!

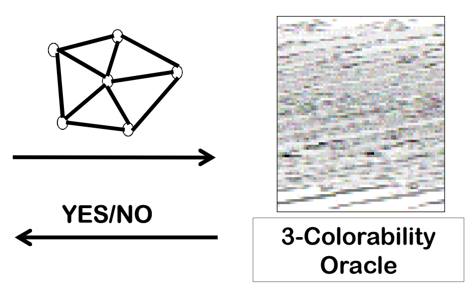
A 3-CRAYOLA Question!



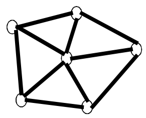
3-Coloring Is Decidable by Brute Force

Try out all 3^n colorings until you determine if G has a 3-coloring

A 3-CRAYOLA Oracle



Better 3-CRAYOLA Oracle



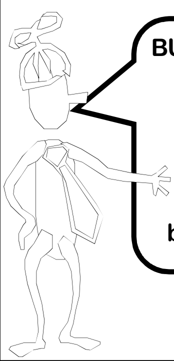
NO, or
YES here is how:
gives 3-coloring
of the nodes

3-Colorability
Search Oracle

3-Colorability
Search Oracle

3-Colorability
Decision Oracle

Christmas Present

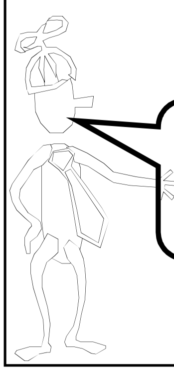


BUT I WANTED a SEARCH oracle for Christmas

I am really bummed out

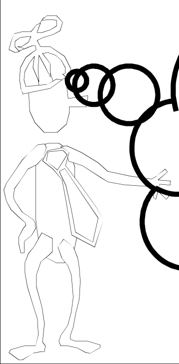
GIVEN:
3-Colorability
Decision Oracle

Christmas Present



How do I turn a mere decision oracle into a search oracle?

GIVEN:
3-Colorability
Decision Oracle

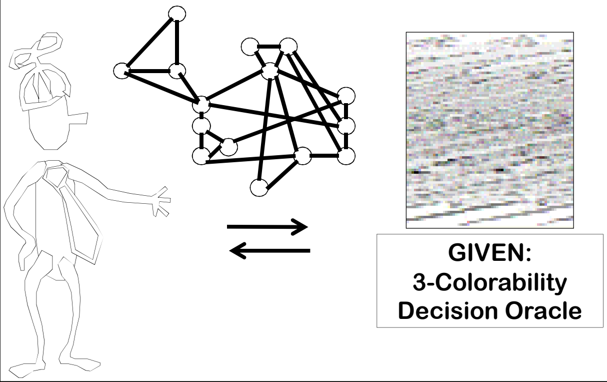


What if I gave the oracle partial colorings of G? For each partial coloring of G, I could pick an uncolored node and try different colors on it until the oracle says “YES”

Beanie’s Flawed Idea

Rats, the oracle does not take partial colorings….

Beanie's Fix

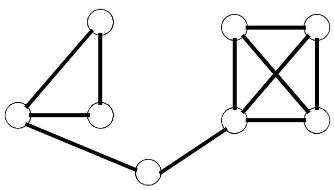


Let's now look at two other problems:

1. K-Clique
2. K-Independent Set

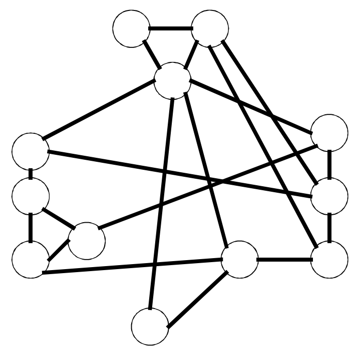
K-Cliques

A K-clique is a set of K nodes with all K(K-1)/2 possible edges between them



This graph contains a 4-clique

A Graph Named "Gadget"

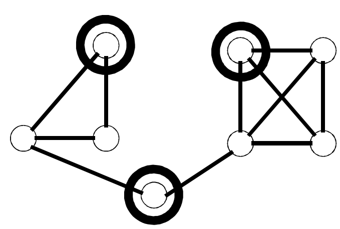


Given: (G, k)
Question: Does G contain a k-clique?

BRUTE FORCE: Try out all n choose k possible locations for the k clique

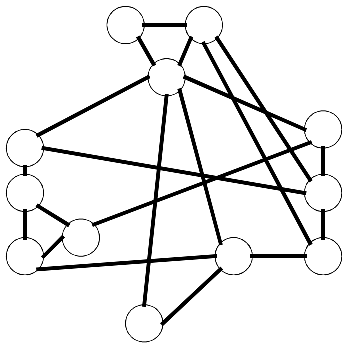
Independent Set

An independent set is a set of nodes with no edges between them



This graph contains an independent set of size 3

A Graph Named “Gadget”



Given: (G, k)
Question: Does G contain an independent set of size k?

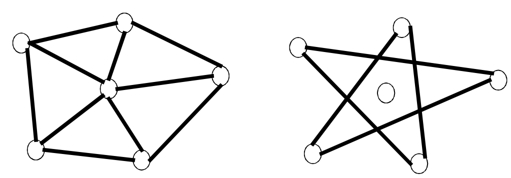
BRUTE FORCE: Try out all n choose k possible locations for the k independent set

Clique / Independent Set

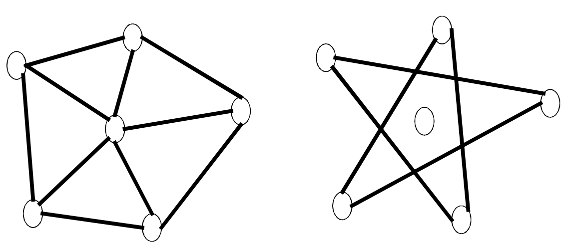
Two problems that are cosmetically different, but substantially the same

Complement of G

Given a graph G, let G*, the complement of G, be the graph obtained by the rule that two nodes in G* are connected if and only if the corresponding nodes of G are not connected

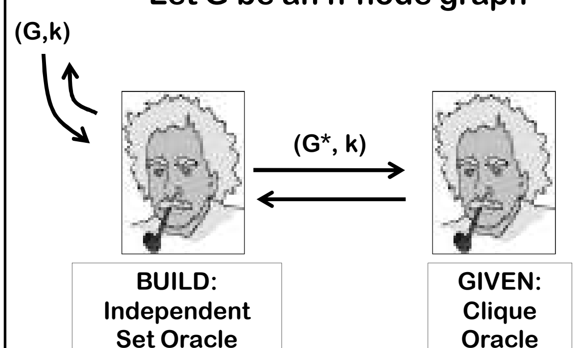


G G*

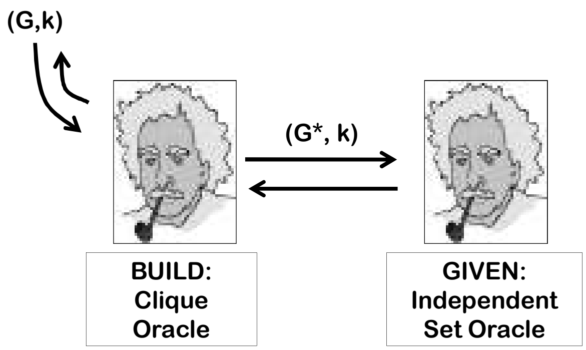


G has a k-clique ⇔ G* has an independent set of size k

Let G be an n-node graph



Let G be an n-node graph

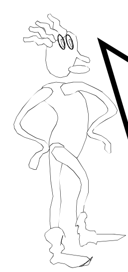


Clique / Independent Set

Two problems that are cosmetically different, but substantially the same

Thus, we can quickly reduce a clique problem to an independent set problem and vice versa

There is a fast method for one if and only if there is a fast method for the other

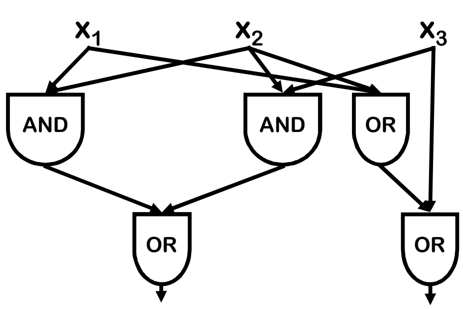


Let's now look at two other problems:

1. **Circuit Satisfiability**
2. **Graph 3-Colorability**

Combinatorial Circuits

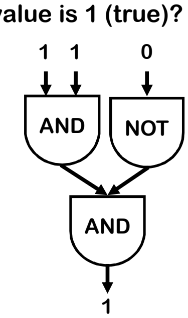
AND, OR, NOT, 0, 1 gates wired together with no feedback allowed



Circuit-Satisfiability

Given a circuit with n-inputs and one output, is there a way to assign 0-1 values to the input wires so that the output value is 1 (true)?

Yes, this circuit is satisfiable: 110

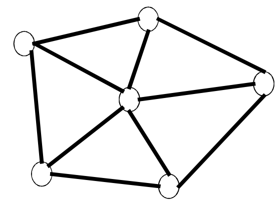


Circuit-Satisfiability

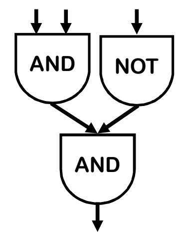
Given: A circuit with n-inputs and one output, is there a way to assign 0-1 values to the input wires so that the output value is 1 (true)?

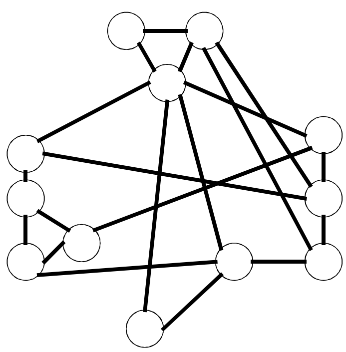
BRUTE FORCE: Try out all 2^n assignments

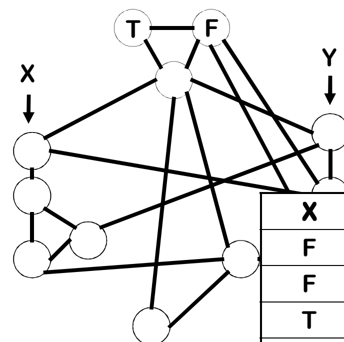
3-Colorability



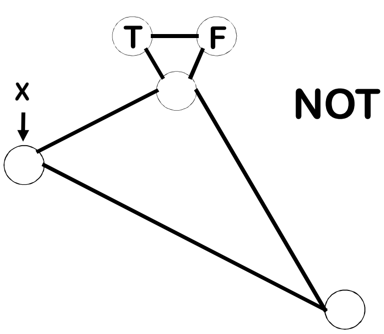
Circuit Satisfiability



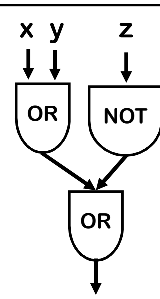


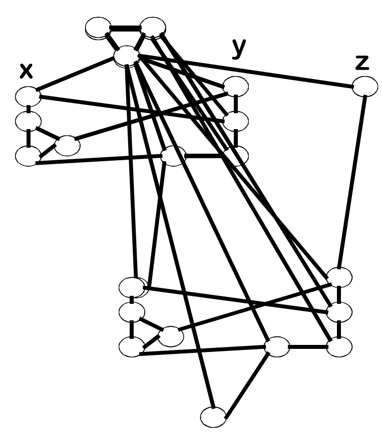


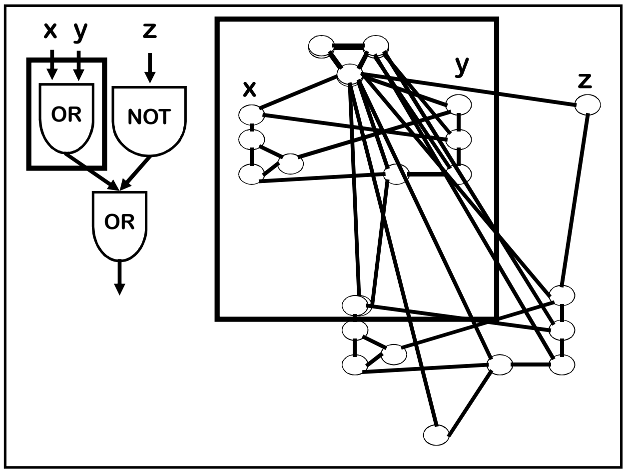
X	Y	OR
F	F	F
F	T	T
T	F	T
T	T	T

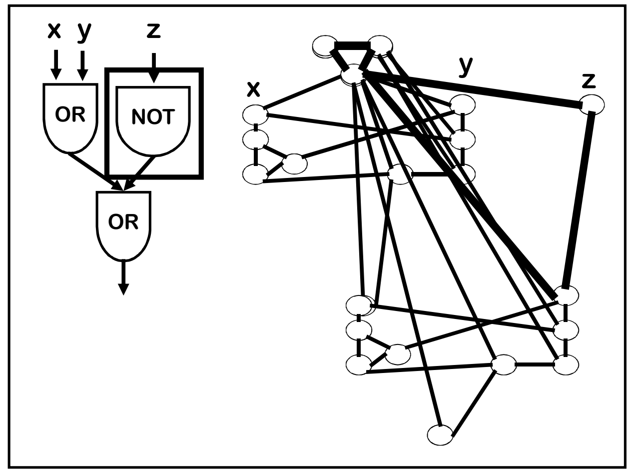


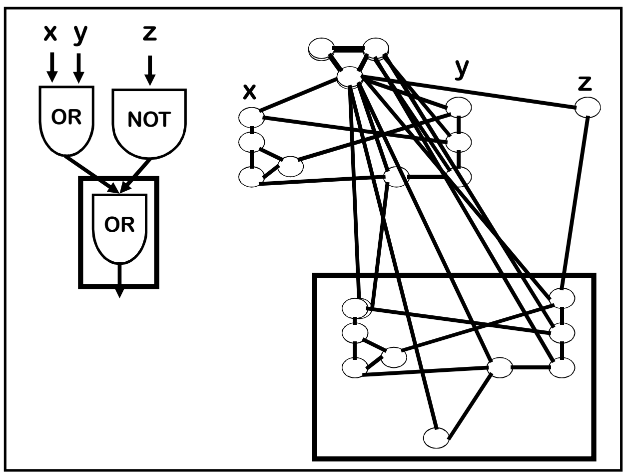
NOT gate!

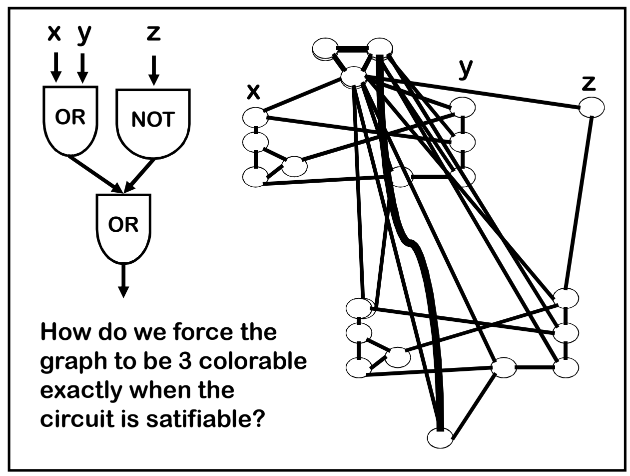


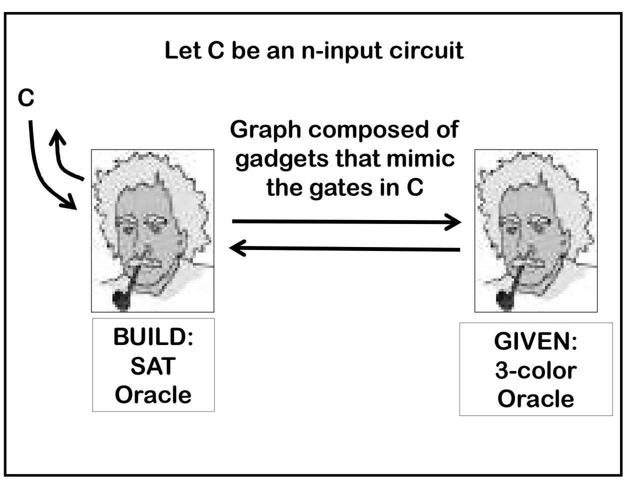




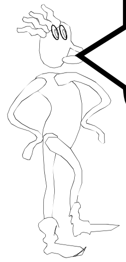






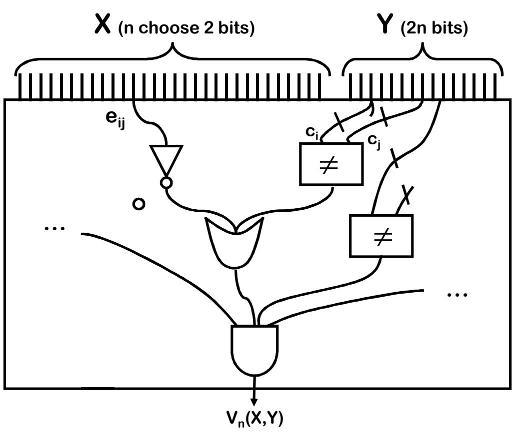


You can quickly transform a method to decide 3-coloring into a method to decide circuit satifiability!



Given an oracle for circuit SAT, how can you quickly solve 3-colorability?

Can you make a circuit that takes a description of a graph and a node coloring, and checks if it is a valid 3-coloring?



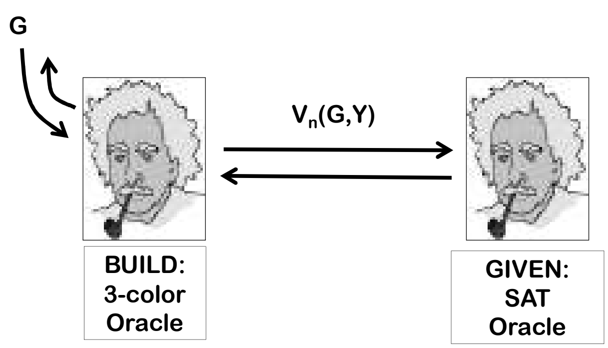
$V_n(X,Y)$

Let V_n be a circuit that takes an n-node graph X and an assignment of colors to nodes Y, and **verifies** that Y is a valid 3 coloring of X. I.e., $V_n(X,Y) = 1$ iff Y is a 3 coloring of X

X is expressed as an n choose 2 bit sequence. Y is expressed as a 2n bit sequence

Given n, we can construct V_n in time $O(n^2)$

Let G be an n-node graph



Circuit-SAT / 3-Colorability

Two problems that are cosmetically different, but substantially the same

Circuit-SAT / 3-Colorability

Clique / Independent Set

Given an oracle for circuit SAT, how can you quickly solve k-clique?

Circuit-SAT / 3-Colorability

Clique / Independent Set

Four problems that are cosmetically different, but substantially the same

FACT: No one knows a way to solve any of the 4 problems that is fast on all instances

Summary

Many problems that appear different on the surface can be efficiently reduced to each other, revealing a deeper similarity