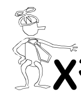
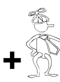
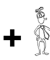
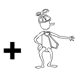
15-251

Great Theoretical Ideas in Computer Science

Polynomials, Lagrange, and Error-correction

Lecture 23 (November 10, 2009)

P(X) =  X^3 +  X^2 +  X^1 + 

Recall: Fields

Definition:

A field F is a set together with two binary operations + and ×, satisfying the following properties:

1. (F,+) is a commutative group
2. (F-{0},×) is a commutative group
3. The distributive law holds in F:
   $(a + b) \times c = (a \times c) + (b \times c)$

Recall: Polynomials in one variable over the reals

$P(x) = 3x^2 + 7x – 2$

$Q(x) = x^{123} – ½ x^{25} + 19 x^3 – 1$

$R(y) = 2y + \sqrt{2}$

$S(z) = z^2 – z - 1$

$T(x) = 0$

$W(x) = \pi$

Representing a polynomial

A degree-d polynomial is represented by its (d+1) coefficients:

$P(x) = c_d x^d + c_{d-1} x^{d-1} + \ldots + c_1 x^1 + c_0$

The d+1 numbers $c_d, c_{d-1}, \ldots, c_0$ are <u>coefficients</u>.

E.g. $P(x) = 3x^4 – 7x^2 + 12x – 19$

Degree(P) ?
Coefficients ?

Are we working over the reals?

We could work over any field
(set with addition, multiplication, division defined.)

E.g., we could work with the <u>rationals</u>, or the <u>reals</u>.

Or with $(Z_p,+,*)$, the integers mod prime p.

In this lecture, we will work with Z_p

the field $(Z_p, +_p, *_p)$

$(Z_p = \{0, 1, 2, \ldots, p\text{-}1\}, +)$
is a commutative group
$(Z_p^* = \{1, 2, 3, \ldots, p\text{-}1\} = Z_p \setminus \{0\}, *)$
is also a commutative group

Addition distributes over multiplication.

$\Bigg\}$ $(Z_p, +_p, *_p)$ is a field

Let $Z_p[x]$ denote the set of polynomials with variable x and coefficients from Z_p

$\mathbb{R}[x]$

Multiplying Polynomials

(say from $Z_{11}[x]$)

$(x^2+2x-1)(3x^3+7x)$

$= x^2(3x^2 + 7x) + 2x(3x^2 + 7x) – (3x^3 + 7x)$

$= 3x^5 + 6x^4 + 4x^3 + 14x^2 – 7x$

$= 3x^5 + 6x^4 + 4x^3 + 3x^2 + 4x$

Adding, Multiplying Polynomials

Let P(x), Q(x) be two polynomials.

P, Q degree d

The sum P(x)+Q(x) is also a polynomial. $\Rightarrow$ P+Q degree d
(i.e., polynomials are "closed under addition")

P×Q degree ≤ 2d.

Their product P(x)Q(x) is also a polynomial.
("closed under multiplication")

P(x)/Q(x) is not necessarily a polynomial.

$Z_p[x]$ is a commutative ring with identity

Let P(x), Q(x) be two polynomials.
The sum P(x)+Q(x) is also a polynomial.
(i.e., polynomials are "closed under addition")

Addition is associative
0 (the "zero" polynomial) is the additive identity
-P(x) is the additive inverse of P(x)
Also, addition is commutative

$(Z_p[x], +)$ is a commutative group

$Z_p[x]$ is a commutative ring with identity

Let P(x), Q(x) be two polynomials.
The sum P(x)*Q(x) is also a polynomial.
(i.e., polynomials are "closed under multiplication")

Multiplication is associative
1 (the "unit" polynomial) is the multiplicative identity
Multiplication is commutative
Finally, addition distributes over multiplication

$(Z_p[x], +, *)$ is a commutative ring with identity
(mult. inverses may not exist)

Evaluating a polynomial

Suppose:
$P(x) = c_d\, x^d + c_{d-1}\, x^{d-1} + \ldots + c_1\, x^1 + c_0$

E.g. $P(x) = 3x^4 – 7x^2 + 12x – 19$

$P(5) = 3×5^4 – 7×5^2 + 12×5 – 19$

$P(-1) = 3×(-1)^4 – 7×(-1)^2 + 12×(-1) – 19$

$P(0) = -19$

The roots of a polynomial

Suppose:
$P(x) = c_d x^d + c_{d-1} x^{d-1} + \ldots + c_1 x^1 + c_0$

Definition: r is a "root" of P(x) if P(r) = 0

E.g., $P(x) = 3x + 7$	root = -(7/3).
$P(x) = x^2 - 2x + 1$	roots = 1, 1
$P(x) = 3x^3 - 10x^2 + 10x - 2$	roots = 1/3, 1, 2.
$P(x) = 1$	no roots
$P(x) = 0$	roots everywhere

Linear Polynomials

$P(x) = ax + b$
One root: $P(x) = ax + b = 0$ $\Rightarrow x = -b/a$

E.g., $P(x) = 7x - 9$ in $Z_{11}[x]$
root $= (-(-9)/7) = 9 * 7^{-1}$
$= 9 * 8 = 72$
$= 6 \pmod{11}$.

Check: $P(6) = 7*6 - 9 = 42 - 9 = 33 = 0 \pmod{11}$

The Single Most Important Theorem About Low-degree Polynomials

A non-zero degree-d polynomial P(x) has at most d roots.

This fact has many applications…

An application: Theorem

Given pairs $(a_1, b_1), \ldots, (a_{d+1}, b_{d+1})$ of values there is at most one degree-d polynomial P(x) such that:
$P(a_k) = b_k$ for all k

P(1) = 7
P(2) = 17
P(5) = 37

when we say "degree-d", we mean degree at most d.

we'll always assume $a_i \neq a_k$ for $i \neq k$

An application: Theorem

Given pairs $(a_1, b_1), \ldots, (a_{d+1}, b_{d+1})$ of values there is at most one degree-d polynomial P(x) such that:
$P(a_k) = b_k$ for all k

Let's prove the contrapositive

Assume P(x) and Q(x) have degree at most d

Suppose $a_1, a_2, \ldots, a_{d+1}$ are d+1 points such that $P(a_k) = Q(a_k)$ for all k = 1,2,…,d+1

Then P(x) = Q(x) for all values of x ← we claim

Proof: Define R(x) = P(x) – Q(x)

R(x) has degree (at most) d

R(x) has d+1 roots, so it must be the zero polynomial □

Theorem:

Given pairs $(a_1, b_1), \ldots, (a_{d+1}, b_{d+1})$ of values
there is at most one
degree-d polynomial P(x)
such that:
$P(a_k) = b_k$ for all k

do there exist d+1 pairs
for which there are
no such polynomials??

Revised Theorem:

Given pairs $(a_1, b_1), \ldots, (a_{d+1}, b_{d+1})$ of values
there is exactly one
degree-d polynomial P(x)
such that:
$P(a_k) = b_k$ for all k

degree at most d

The algorithm to construct P(x)
is called Lagrange Interpolation

Two different representations

$P(x) = c_d x^d + c_{d-1} x^{d-1} + \ldots + c_1 x^1 + c_0$
can be represented either by

a) its d+1 coefficients
$c_d, c_{d-1}, \ldots, c_2, c_1, c_0$

b) Its value at any d+1 points
$P(a_1), P(a_2), \ldots, P(a_d), P(a_{d+1})$
(e.g., P(1), P(2), …, P(d+1).)

Converting Between The Two Representations

Coefficients to Evaluation:

Evaluate P(x) at d+1 points

Evaluation to Coefficients:

Use Lagrange Interpolation

Now for some Lagrange Interpolation

Given pairs $(a_1, b_1), \ldots, (a_{d+1}, b_{d+1})$ of values
there is exactly one
degree-d polynomial P(x)
such that:
$P(a_k) = b_k$ for all k

Special case

Can we get a polynomial h such that

$h(a_1) = 1$
$h(a_2) = 0$
$h(a_3) = 0$
…
$h(a_{d+1}) = 0$

construction by example

want a quadratic h with h(3) = 1, h(1)=0, h(6)=0

(say, in $Z_{11}[x]$)

Let's first get the roots in place:

$h(x) = (x-1)(x-6)$

Are we done? No! We wanted h(3) = 1

But $h(3) = (3-1)(3-6) = -6$

So let's fix that!

$h(x) = (-6)^{-1} (x-1)(x-6)$

$= 9 (x-1)(x-6)$

$9 * (-6) =_{11} 1$

done!

Special case

So once we have degree-d poly $h_1(x)$:
$h_1(a_1) = 1$
$h_1(a_t) = 0$ for all t = 2,…,d+1

"switch" polynomial #1

Special case

So once we have degree-d poly $h_1(x)$:
$h_1(a_1) = 1$
$h_1(a_t) = 0$ for all t = 2,…,d+1

Then we can get degree-d poly $H_1(x)$:
$H_1(a_1) = b_1$
$H_1(a_t) = 0$ for all t = 2,…,d+1

Just set $H_1(x) = b_1 * h_1(x)$

Special case

So once we have degree-d poly $h_1(x)$:
$h_1(a_1) = 1$
$h_1(a_t) = 0$ for all t = 2,…,d+1

Using same idea, get degree-d poly $H_k(x)$:
$H_k(a_k) = b_k$
$H_k(a_t) = 0$ for all $t \neq k$

Finally, $P(x) = \sum_k H_k(x)$

formally, the constructions

k-th "Switch" polynomial

$g_k(x) = (x-a_1)(x-a_2)\ldots(x-a_{k-1})(x-a_{k+1})\ldots(x-a_{d+1})$

Degree of $g_k(x)$ is: d

$g_k(x)$ has d roots: $a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_{d+1}$

$g_k(a_k) = (a_k-a_1)(a_k-a_2)\ldots(a_k-a_{k-1})(a_k-a_{k+1})\ldots(a_k-a_{d+1})$

For all $i \neq k$, $g_k(a_i) = 0$

k-th "Switch" polynomial

$g_k(x) = (x-a_1)(x-a_2)\ldots(x-a_{k-1})(x-a_{k+1})\ldots(x-a_{d+1})$

$$h_k(x) = \frac{(x-a_1)(x-a_2)\ldots(x-a_{k-1})(x-a_{k+1})\ldots(x-a_{d+1})}{(a_k-a_1)(a_k-a_2)\ldots(a_k-a_{k-1})(a_k-a_{k+1})\ldots(a_k-a_{d+1})}$$

$h_k(a_k) = 1$

For all $i \neq k$, $h_k(a_i) = 0$

The Lagrange Polynomial

$$h_k(x) = \frac{(x-a_1)(x-a_2)\ldots(x-a_{k-1})(x-a_{k+1})\ldots(x-a_{d+1})}{(a_k-a_1)(a_k-a_2)\ldots(a_k-a_{k-1})(a_k-a_{k+1})\ldots(a_k-a_{d+1})}$$

$P(x) = b_1\, h_1(x) + b_2\, h_2(x) + \ldots + b_{d+1}\, h_{d+1}(x)$

$P(x)$ is the unique polynomial of degree d such that $P(a_1) = b_1$, $P(a_2) = b_2$, …, $P(a_{d+1}) = b_{d+1}$

Example

Input: (5,1), (6,2), (7,9)

Want quadratic in $Z_{11}[x]$

Switch polynomials:

$h_1(x) = (x-6)(x-7)/(5-6)(5-7) = ½\, (x-6)(x-7)$

$h_2(x) = (x-5)(x-7)/(6-5)(6-7) = -\, (x-5)(x-7)$

$h_3(x) = (x-5)(x-6)/(7-5)(7-6) = ½\, (x-5)(x-6)$

$P(x) = 1 \times h_1(x) + 2 \times h_2(x) + 9 \times h_3(x)$

$= (3x^2 - 32x + 86)$

$= (3x^2 + x + 9)$ in $Z_{11}[x]$

the Chinese Remainder Theorem uses very similar ideas in its proof

Revised Theorem:

Given pairs (a_1, b_1), …, (a_{d+1}, b_{d+1}) of values there is exactly one degree-d polynomial $P(x)$ such that: $P(a_k) = b_k$ for all k

The algorithm to construct P(x) is called Lagrange Interpolation

An Application:

Error Correcting Codes

Error Correcting Codes

Messages as sequences of numbers in Z_{29}:

HELLO 8 5 12 12 15

I want to send a sequence of d+1 numbers

Suppose your mailer may corrupt any k among all the numbers I send.

How should I send the numbers to you?

In Particular

Suppose I just send over the numbers
8 5 12 12 15
and you get
8 9 0 12 15

say k=2 errors

How do you correct errors?

How do you even <u>detect</u> errors?

A Simpler Case: Erasures

Suppose I just send over the numbers
8 5 12 12 15
and you get
8 * * 12 15

say k=2 erasures

H**LO

(Numbers are either correct or changed to *)

What can you do to correct errors?

A Simple Solution

Repetition: repeat each number k+1 times

8 8 8 5 5 5 12 12 12 12 12 12 15 15 15

At least one copy of each number will reach

8 8 8 5 * * 12 12 12 12 12 12 15 15 15

For arbitrary corruptions, repeat 2k+1 times and take majority

Very wasteful!

we'll get d+k+and

To send d+1 numbers with erasures, we sent (d+1)(k+1) numbers $\approx d+1\left(\frac{d}{10}+1\right)$ $= \theta\left(\frac{d^2}{100}\right)$

$k = \frac{d}{100}$

Can we do better?

Note that to send 1 number with k erasures we need to send k+1 numbers.

Think polynomials…

Encoding messages as polynomials:

HELLO 8 5 12 12 15

$8x^4 + 5x^3 + 12x^2 + 12x + 15 \in Z_{29}[x]$

I want to send you a polynomial P(x) of degree d.

Send it in the value representation!

Want to send a polynomial of degree-d subject to at most k erasures.

Evaluate P(x) at d+1+k points

Send P(0), P(1), P(2), …, P(d+k)

At least d+1 of these values will reach

Say P(0), * , P(2), *, …, *, P(d+k)

Can recover P(x) from these d+1 values

Example

$f(x) := 8x^4 + 5x^3 + 12x^2 + 12x + 15;$

$$x \rightarrow 8x^4 + 5x^3 + 12x^2 + 12x + 15 \quad (1)$$

Example

$f(x) := 8x^4 + 5x^3 + 12x^2 + 12x + 15;$

$$x \rightarrow 8x^4 + 5x^3 + 12x^2 + 12x + 15 \quad (1)$$

$f(0)$ **mod** 29

$$15 \quad (2)$$

$f(1)$ **mod** 29

$$23 \quad (3)$$

$f(2)$ **mod** 29

$$23 \quad (4)$$

$f(3)$ **mod** 29

$$14 \quad (5)$$

$f(4)$ **mod** 29

$$13 \quad (6)$$

$f(5)$ **mod** 29

$$26 \quad (7)$$

$f(6)$ **mod** 29

$$19 \quad (8)$$

CurveFitting[*PolynomialInterpolation*]([[0, 15], [1, 23], [4, 13], [5, 26], [6, 19]], *x*, *form* = *Lagrange*);

$$\frac{1}{8}(x-1)(x-4)(x-5)(x-6) - \frac{23}{60}x(x-4)(x-5)(x-6) + \frac{13}{24}x(x-1)(x-5)(x-6) - \frac{13}{10}x(x-1)(x-4)(x-6) + \frac{19}{60}x(x-1)(x-4)(x-5) \quad (9)$$

$g := eval(expand(\%) \ \mathbf{mod}\ 29)$

$$8x^4 + 5x^3 + 12x^2 + 12x + 15 \quad (10)$$

Much better!!!

Naïve Repetition:
To send d+1 numbers with k erasures, we sent (d+1)(k+1) numbers

Polynomial Coding:
To send d+1 numbers with k erasures, we sent (d+k+1) numbers

What about corruptions?

Want to send a polynomial of degree-d subject to at most k corruptions.

Suppose we try the same idea

Evaluate P(x) at d+1+k points

Send P(0), P(1), P(2), …, P(d+k)

At least d+1 of these values will be unchanged

Example

$P(x) = 2x^2 + 1$, and $k = 1$.

So I sent P(0)=1, P(1)=3, P(2)=9, P(3)=19

Corrupted email says (1, 4, 9, 19)

Choosing (1, 4, 9) will give us $Q(x) = x^2 + 2x + 1$

But we can at least detect errors!

Evaluate P(x) at d+1+k points

Send P(0), P(1), P(2), …, P(d+k)

At least d+1 of these values will be correct

Say P(0), P'(1) , P(2), P(3), P'(4), …, P(d+k)

Using these d+1 correct values will give P(x)

Using any of the incorrect values will give something else

Quick way of detecting errors

Interpolate first d+1 points to get Q(x)

Check that all other received values are consistent with this polynomial Q(x)

If all values consistent, no errors!

In that case, we know Q(x) = P(x)

else there were errors…

Number of numbers?

Naïve Repetition:
To send d+1 numbers with error detection, sent (d+1)(k+1) numbers

Polynomial Coding:
To send d+1 numbers with error detection, sent (d+k+1) numbers

How about error correction?

requires more work

To send d+1 numbers in such a way that we can correct up to k errors, need to send d+1+2k numbers.

Similar encoding scheme

Evaluate degree-d P(x) at d+1+2k points

Send P(0), P(1), P(2), …, P(d+2k)

At least d+1+k of these values will be correct

Say P(0), P(1) , P(2), P(3), P(4), …, P(d+2k)

How do we know which are correct?

how do we do this fast?

Theorem: A unique degree-d polynomial R(x) can agree with the received data on at least d+1+k points

Clearly, the original polynomial P(x) agrees with data on d+1+k points (since at most k errors, total d+1+2k points)

And if two different degree-d polynomials did so, they would have to agree with each other on d+1 points, and hence be the same.

So any such R(x) = P(x)

Theorem: A unique degree-d polynomial R(x) can agree with the received data on at least d+1+k points

Brute-force Algorithm:

Interpolate each subset of (d+1) points

Check if the resulting polynomial agrees with received data on d+1+k pts

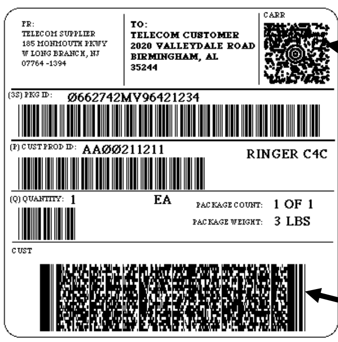
Takes too much time…

A fast algorithm to decode was given by Berlekamp and Welch which solves a system of linear equations

Recent research has given very fast encoding and decoding algorithms

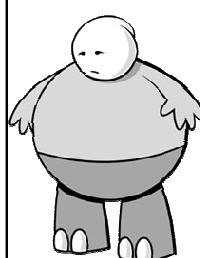
BTW, this coding scheme is called Reed-Solomon encoding

It's used in practice…



Maxicodes = "UPS codes" = another 2-d Reed-Solomon codes

PDF417 codes = 2-d Reed-Solomon codes



Here's What You Need to Know…

Polynomials
- Fundamental Theorem of polys:
  - Degree-d poly has at most d roots.
- Two deg-d polys agree on $\leq$ d points.

Lagrange Interpolation:
- Given d+1 pairs (a_k, b_k), can find unique poly P with $P(a_k) = b_k$ for all these k.
- Gives us value represent'n for polys.

Error Correction
- Erasure codes
- Detection and correction