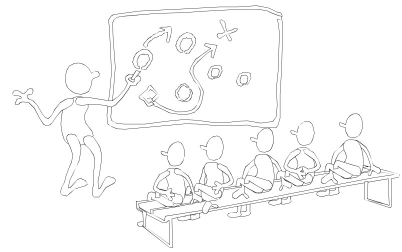
15-251

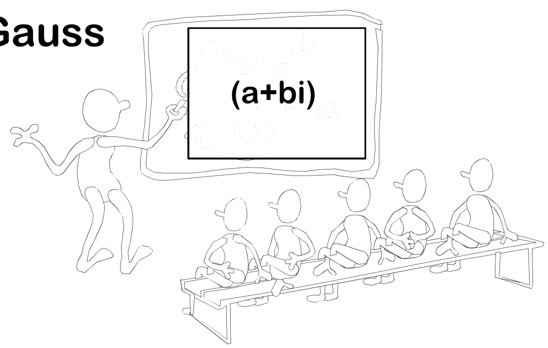
Great Theoretical Ideas in Computer Science

Grade School Revisited: How To Multiply Two Numbers

Lecture 22 (November 5, 2009)



Gauss



Gauss' Complex Puzzle

Remember how to multiply two complex numbers a + bi and c + di?

(a+bi)(c+di) = [ac –bd] + [ad + bc] i

Input: a,b,c,d
Output: ac-bd, ad+bc

If multiplying two real numbers costs $1 and adding them costs a penny, what is the cheapest way to obtain the output from the input?

Can you do better than $4.02?

Gauss' $3.05 Method

Input: a,b,c,d
Output: ac-bd, ad+bc

c	$X_1 = a + b$	
c	$X_2 = c + d$	
$	$X_3 = X_1 X_2$	$= ac + ad + bc + bd$
$	$X_4 = ac$	
$	$X_5 = bd$	
c	$X_6 = X_4 – X_5$	$= ac - bd$
cc	$X_7 = X_3 – X_4 – X_5$	$= bc + ad$

The Gauss optimization saves one multiplication out of four.

It requires 25% less work.

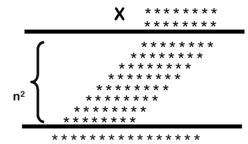
Time complexity of grade school addition

T(n) = amount of time grade school addition uses to add two n-bit numbers

We saw that T(n) was linear

$T(n) = \Theta(n)$

Time complexity of grade school multiplication

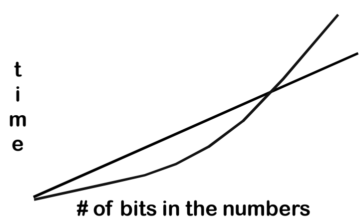


T(n) = The amount of time grade school multiplication uses to add two n-bit numbers

We saw that T(n) was quadratic

$T(n) = \Theta(n^2)$

Grade School Addition: Linear time
Grade School Multiplication: Quadratic time



No matter how dramatic the difference in the constants, the quadratic curve will eventually dominate the linear curve

Is there a sub-linear time method for addition?

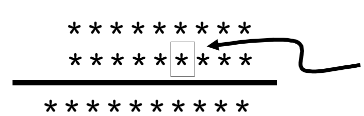
Any addition algorithm takes $\Omega(n)$ time

Claim: Any algorithm for addition must read all of the input bits

Proof: Suppose there is a mystery algorithm A that does not examine each bit

Give A a pair of numbers. There must be some unexamined bit position i in one of the numbers

Any addition algorithm takes $\Omega(n)$ time



A did not read this bit at position i

If A is not correct on the inputs, we found a bug

If A is correct, flip the bit at position i and give A the new pair of numbers. A gives the same answer as before, which is now wrong.

Grade school addition can't be improved upon by more than a constant factor

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Grade School Addition: $\Theta(n)$ time.
Furthermore, it is optimal

Grade School Multiplication: $\Theta(n^2)$ time

Is there a clever algorithm to multiply two numbers in linear time?

Despite years of research, no one knows! If you resolve this question, Carnegie Mellon will give you a PhD!

Can we even break the quadratic time barrier?

In other words, can we do something very different than grade school multiplication?

Divide And Conquer

An approach to faster algorithms:

- DIVIDE a problem into smaller subproblems
- CONQUER them recursively
- GLUE the answers together so as to obtain the answer to the larger problem

---

Multiplication of 2 n-bit numbers

n bits

X = | X |

Y = | Y |

n/2 bits | n/2 bits

$X = a\,2^{n/2} + b \qquad Y = c\,2^{n/2} + d$

$X \times Y = ac\,2^n + (ad + bc)\,2^{n/2} + bd$

Multiplication of 2 n-bit numbers

X = | a | b |

Y = | c | d |

n/2 bits | n/2 bits

$X \times Y = ac\,2^n + (ad + bc)\,2^{n/2} + bd$

```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else break X into a;b and Y into c;d
    return MULT(a,c) 2^n + (MULT(a,d)
           + MULT(b,c)) 2^(n/2) + MULT(b,d)
```

Same thing for numbers in decimal!

n digits

X =	a	b
Y =	c	d

n/2 digits n/2 digits

$X = a\ 10^{n/2} + b \quad Y = c\ 10^{n/2} + d$

$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$

Multiplying (Divide & Conquer style)

$\boxed{1234}\boxed{5678} * \boxed{2139}\boxed{4276}$

1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39 34*21 34*39

1*2 1*1 2*2 2*1

2 1 4 2

Hence: $12*21 = 2*10^2 + (1 + 4)10^1 + 2 = 252$

a	b
c	d

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

$\boxed{252}*10^4 + \boxed{468}*10^2 + \boxed{714}*10^2 + \boxed{1326}*1 = 2639526$

a	b
c	d

Multiplying (Divide & Conquer style)

12345678 * 21394276

$\boxed{2639526}*10^8 + \boxed{5276584}*10^4 + \boxed{12145242}*10^4 + \boxed{24279128}*1$

= 264126842539128

a	b
c	d

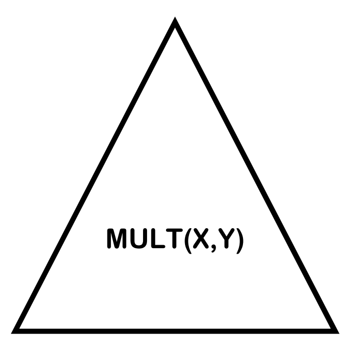
Multiplying (Divide & Conquer style)

12345678 * 21394276

= 264126842539128

a	b
c	d

Divide, Conquer, and Glue



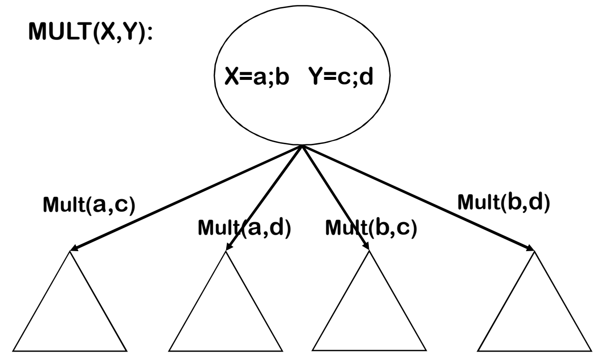
Divide, Conquer, and Glue

MULT(X,Y):

if |X| = |Y| = 1
then return XY,
else…

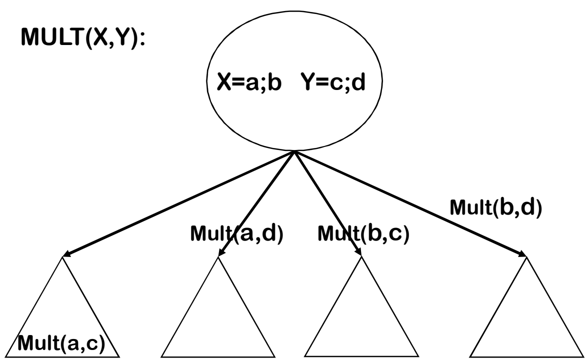
Divide, Conquer, and Glue

MULT(X,Y):



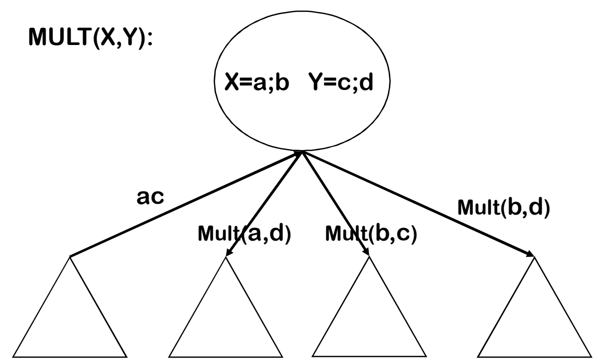
Divide, Conquer, and Glue

MULT(X,Y):



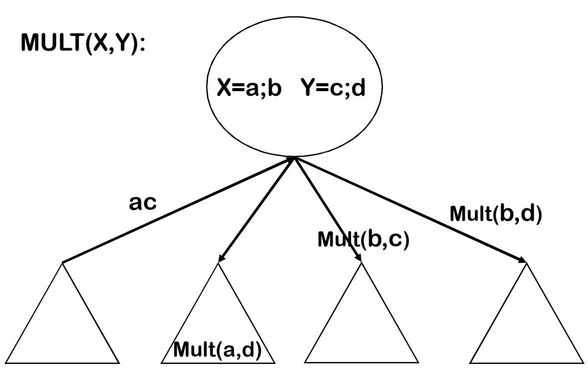
Divide, Conquer, and Glue

MULT(X,Y):



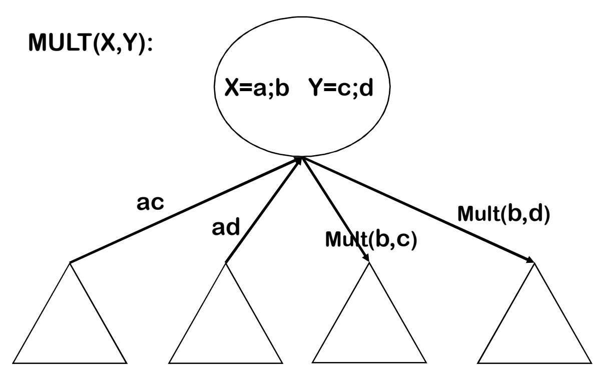
Divide, Conquer, and Glue

MULT(X,Y):



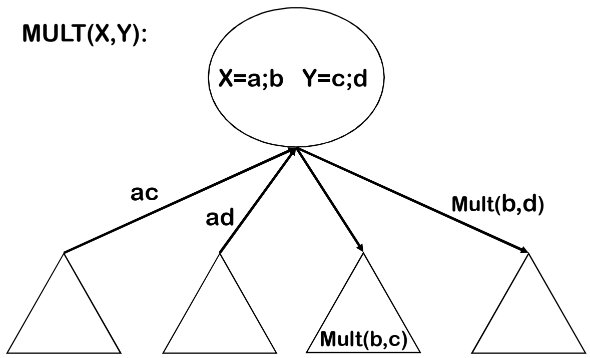
Divide, Conquer, and Glue

MULT(X,Y):



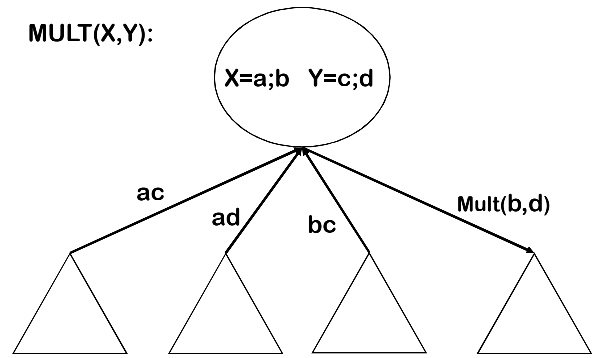
Divide, Conquer, and Glue

MULT(X,Y):



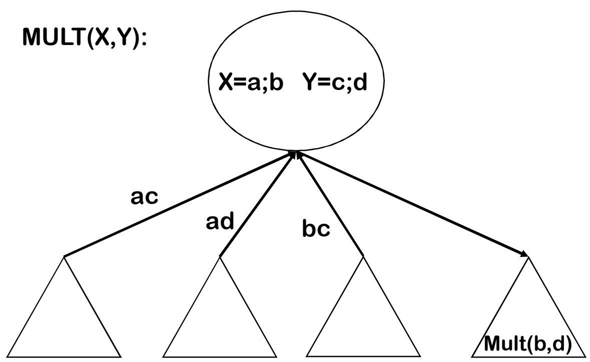
Divide, Conquer, and Glue

MULT(X,Y):



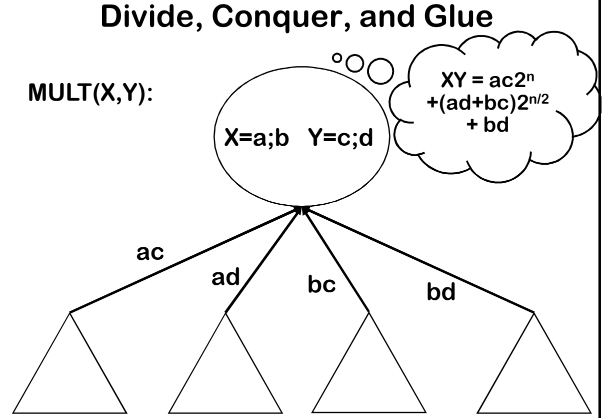
Divide, Conquer, and Glue

MULT(X,Y):



Divide, Conquer, and Glue

MULT(X,Y):



Time required by MULT

T(n) = time taken by MULT on two n-bit numbers

What is T(n)? What is its growth rate?

Big Question: Is it $\Theta(n^2)$?

$$T(n) = 4\,T(n/2) + (k'n + k'')$$

conquering time — $4\,T(n/2)$

divide and glue — $(k'n + k'')$

Recurrence Relation

$T(1) = k$ for some constant k

$T(n) = 4\,T(n/2) + k'n + k''$
for constants k' and k''

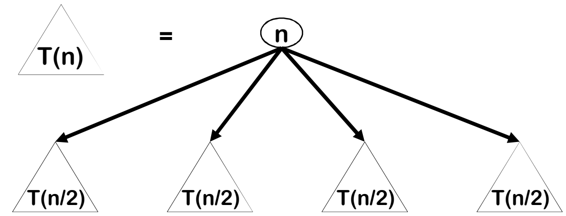
Simplified Recurrence Relation

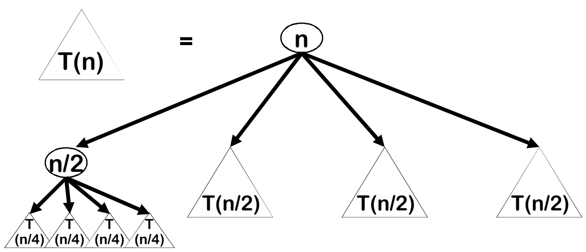
$T(1) = 1$

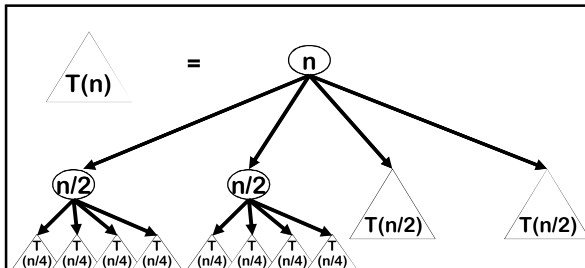
$T(n) = 4\,T(n/2) + n$

conquering time

divide and glue







0	n
1	n/2 + n/2 + n/2 + n/2
2	
i	Level i is the sum of 4^i copies of $n/2^i$
	. . . .
$\log_2(n)$	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

1n =	n
2n =	n/2 + n/2 + n/2 + n/2
4n =	
2^in =	Level i is the sum of 4^i copies of $n/2^i$
	. . . .
(n)n =	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

$n(1+2+4+8+ \ldots +n) = n(2n-1) = 2n^2-n$

Divide and Conquer MULT: $\Theta(n^2)$ time
Grade School Multiplication: $\Theta(n^2)$ time

Bummer!

MULT revisited

```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else  break X into a;b and Y into c;d
          return MULT(a,c) 2^n + (MULT(a,d)
          + MULT(b,c)) 2^(n/2) + MULT(b,d)
```

MULT calls itself 4 times. Can you see a way to reduce the number of calls?

Gauss' optimization

Input: a,b,c,d
Output: ac-bd, ad+bc

c	$X_1 = a + b$	
c	$X_2 = c + d$	
\$	$X_3 = X_1 X_2$	$= ac + ad + bc + bd$
\$	$X_4 = ac$	
\$	$X_5 = bd$	
c	$X_6 = X_4 - X_5$	$= ac - bd$
cc	$X_7 = X_3 - X_4 - X_5$	$= bc + ad$

Karatsuba, Anatolii Alexeevich (1937-)

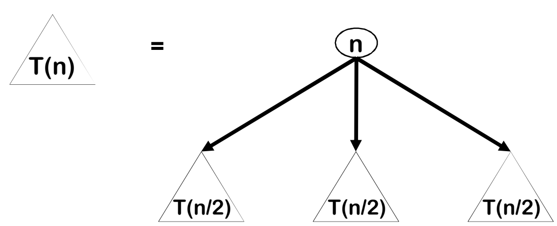
Sometime in the late 1950's Karatsuba had formulated the first algorithm to break the n^2 barrier!

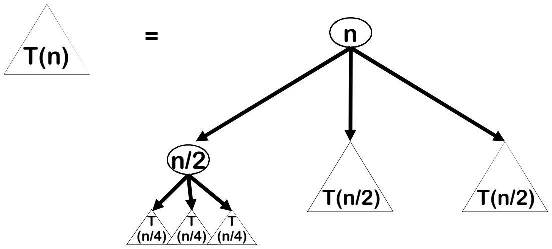
Gaussified MULT (Karatsuba 1962)

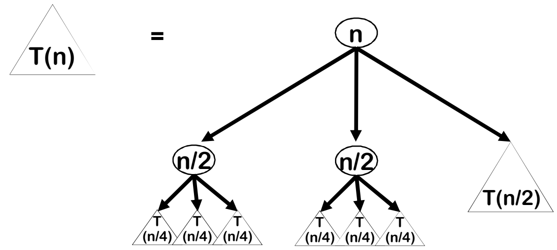
```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else  break X into a;b and Y into c;d
              e : = MULT(a,c)
              f  := MULT(b,d)
    return
    e 2^n + (MULT(a+b,c+d) – e –  f) 2^(n/2) + f
```

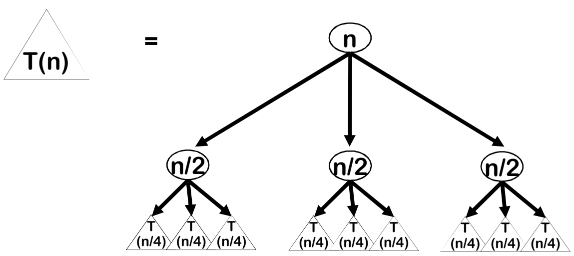
$T(n) = 3\,T(n/2) + n$

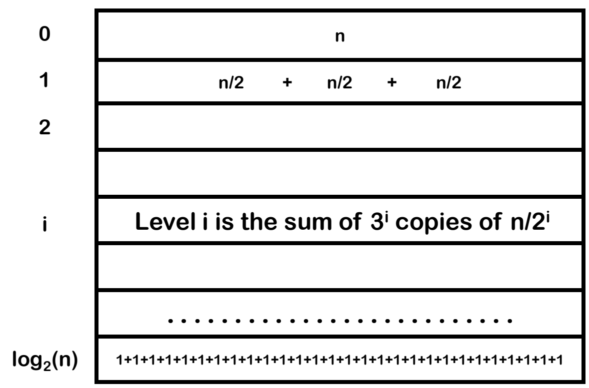
Actually: $T(n) = 2\,T(n/2) + T(n/2 + 1) + kn$

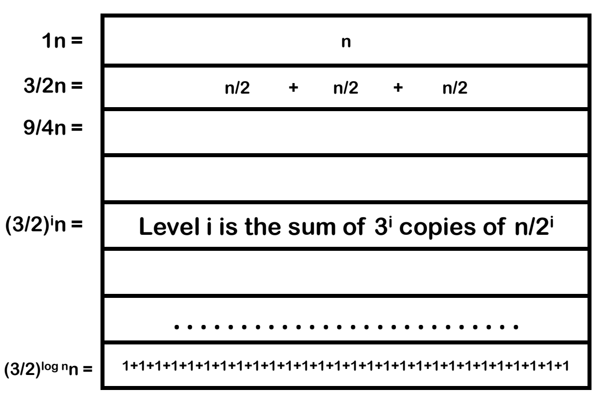










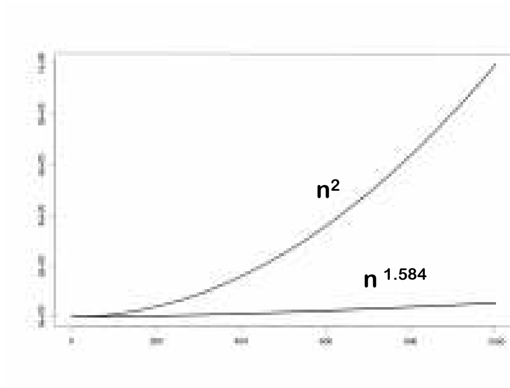


$n(1+3/2+(3/2)^2+ \ldots + (3/2)^{\log_2 n}) \quad = \quad 3n^{1.58\ldots} - 2n$

Dramatic Improvement for Large n

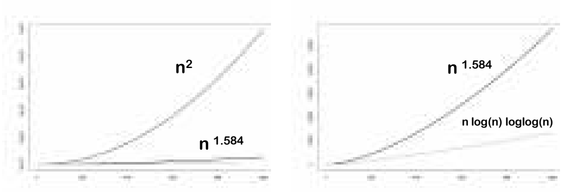
$$
\begin{aligned}
T(n) &= 3n^{\log_2 3} - 2n \\
&= \Theta(n^{\log_2 3}) \\
&= \Theta(n^{1.58\ldots})
\end{aligned}
$$

A huge savings over $\Theta(n^2)$ when n gets large.



Multiplication Algorithms

Kindergarten	$n2^n$
Grade School	n^2
Karatsuba	$n^{1.58...}$
Fastest Known	n logn loglogn



A case study

Anagram Programming Task.

You are given a 70,000 word dictionary. Write an anagram utility that given a word as input returns <u>all</u> anagrams of that word appearing in the dictionary.

Examples

Input: CAT
Output: ACT, CAT, TAC

Input: SUBESSENTIAL
Output: SUITABLENESS

(Novice Level Solution)

Loop through all possible ways of rearranging the input word

Use binary search to look up resulting word in dictionary.

If found, output it

Performance Analysis
Counting without executing

On the word “microphotographic”,
we loop 17! ≈ 3 * 10^14 times.

Even at 1 microsecond per iteration,
this will take 3 *10^8 seconds.

Almost a decade!

(There are about π seconds in a nanocentury.)

“Expert” Level Solution

Module ANAGRAM(X,Y) returns TRUE
exactly when X and Y are anagrams.
(Works by sorting the letters in X and Y)

Input: X
Loop through <u>all dictionary words</u> Y
If ANAGRAM(X,Y) output Y

The hacker is satisfied and reflects no further

Comparing an input word with
each of 70,000 dictionary entries
takes about 15 seconds

The master keeps trying to <u>refine</u> the solution

The master’s program runs in less
than 1/1000 seconds.

Master Solution

Don’t just keep the dictionary
in sorted order!

Rearranging the
dictionary into
“anagram classes”
makes the original
problem simpler.

Suppose the dictionary was the list below.

ASP
DOG
LURE
GOD
NICE
RULE
SPA

After each word, write its "signature" (sort its letters)

ASP	APS
DOG	DGO
LURE	ELRU
GOD	DGO
NICE	CEIN
RULE	ELRU
SPA	APS

Sort by the signatures

ASP	APS
SPA	APS
NICE	CEIN
DOG	DGO
GOD	DGO
LURE	ELRU
RULE	ELRU

The Master's Program

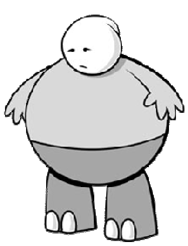
Input word W

X := signature of W (sort the letters)

Use binary search to find the anagram class of W and output it.

A useful tool: preprocessing…

Of course, it takes about 30 seconds to create the dictionary, but it is perfectly fair to think of this as programming time. The building of the dictionary is a one-time cost that is part of writing the program.



Here's What You Need to Know…

- Gauss's Multiplication Trick
- Proof of Lower bound for addition
- Divide and Conquer
- Solving Recurrences
- Karatsuba Multiplication
- Preprocessing