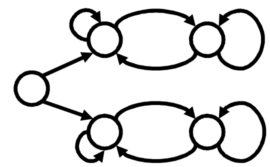
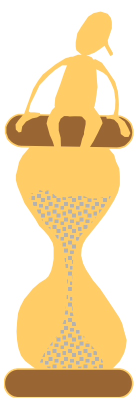
15-251

Great Theoretical Ideas in Computer Science

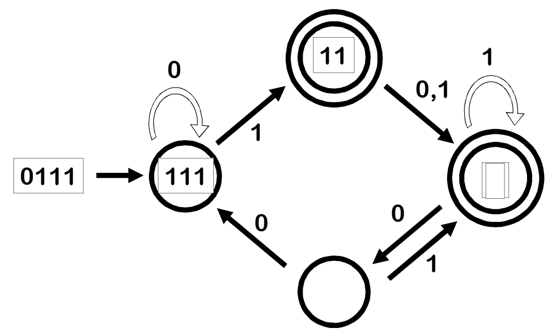
Deterministic Finite Automata

Lecture 20 (October 29, 2009)



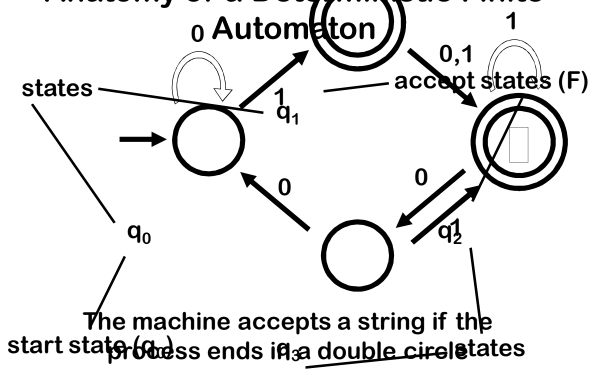


A machine so simple that you can understand it in less than one minute



The machine accepts a string if the process ends in a double circle

Anatomy of a Deterministic Finite Automaton



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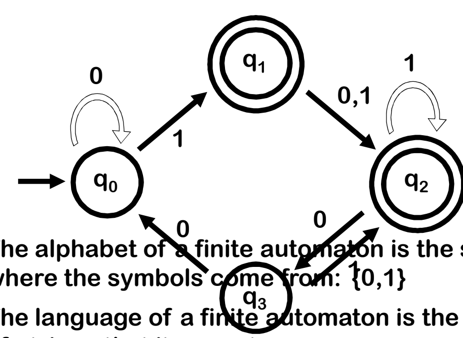
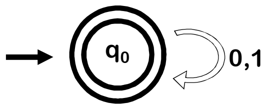


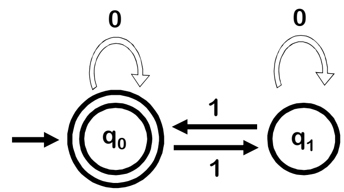
The alphabet of a finite automaton is the set where the symbols come from: {0,1}

The language of a finite automaton is the set of strings that it accepts



L(M) = ∅ All strings of 0s and 1s

The Language of Machine M



L(M) = { w | w has an even number of 1s}

Notation

An alphabet Σ is a finite set (e.g., Σ = {0,1})

A string over Σ is a finite-length sequence of elements of Σ

For x a string, |x| is the length of x

The unique string of length 0 will be denoted by ε and will be called the empty or null string

A language over Σ is a set of strings over Σ

A finite automaton is a 5-tuple M = (Q, Σ, δ, q_0, F)

Q is the set of states

Σ is the alphabet

δ : Q × Σ → Q is the transition function

$q_0 \in Q$ is the start state

F ⊆ Q is the set of accept states

L(M) = the language of machine M
= set of all strings machine M accepts

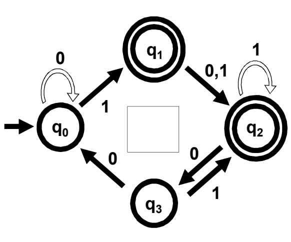
M = (Q, Σ, δ, q_0, F) where Q = {q_0, q_1, q_2, q_3}

Σ = {0,1}

δ : Q × Σ → Q transition function*

$q_0 \in Q$ is start state

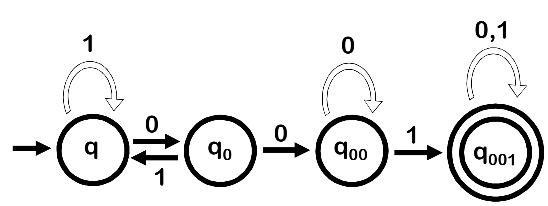
F = {q_1, q_2} ⊆ Q accept states



*

δ	0	1
q_0	q_0	q_1
q_1	q_2	q_2
q_2	q_3	q_2
q_3	q_0	q_2

Build an automaton that accepts all and only those strings that contain 001



Build an automaton that accepts all strings whose length is divisible by 2 but not 3

Build an automaton that accepts exactly the strings that contain 01011 as a substring?

How about an automaton that accepts exactly the strings that contain an even number of 01 pairs?

A language is regular if it is recognized by a deterministic finite automaton

L = { w | w contains 001} is regular

L = { w | w has an even number of 1s} is regular

Union Theorem

Given two languages, L_1 and L_2, define the union of L_1 and L_2 as

$$L_1 \cup L_2 = \{ w \mid w \in L_1 \text{ or } w \in L_2 \}$$

Theorem: The union of two regular languages is also a regular language

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Proof Sketch: Let

$M_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$ be finite automaton for L_1

and

$M_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$ be finite automaton for L_2

We want to construct a finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $L = L_1 \cup L_2$

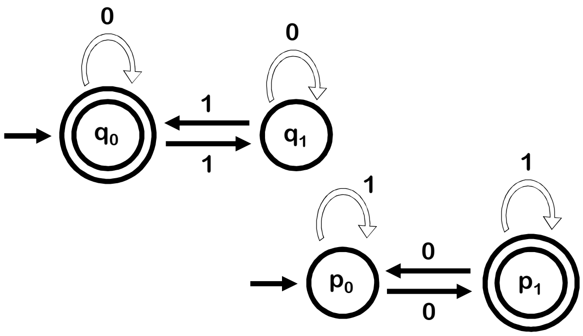
Idea: Run both M_1 and M_2 at the same time!

Q = pairs of states, one from M_1 and one from M_2

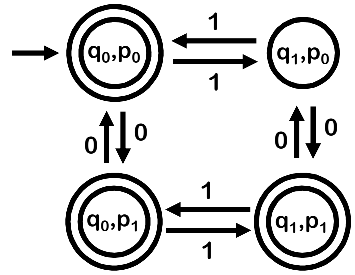
= $\{ (q_1, q_2) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \}$

= $Q_1 \times Q_2$

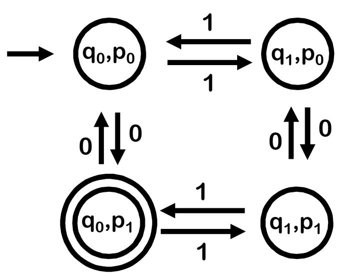
Theorem: The union of two regular languages is also a regular language



Automaton for Union



Automaton for Intersection



Theorem: The union of two regular languages is also a regular language

Corollary: Any finite language is regular

The Regular Operations

Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$

Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$

Negation: $\neg A = \{ w \mid w \notin A \}$

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

Star: $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

Regular Languages Are Closed Under The Regular Operations

We have seen part of the proof for Union. The proof for intersection is very similar. The proof for negation is easy.

The "Grep" Problem

Input: Text T of length t, string S of length n

Problem: Does string S appear inside text T?

Naïve method:

$\boxed{a_1, a_2, a_3,} a_4, a_5, \ldots, a_t$

Cost: Roughly nt comparisons

Automata Solution

Build a machine M that accepts any string with S as a consecutive substring

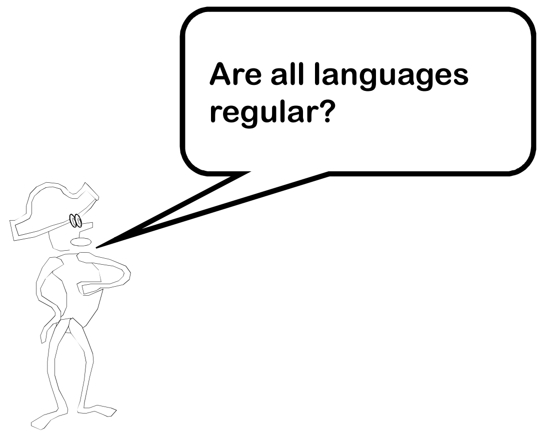
Feed the text to M

Cost: t comparisons + time to build M

As luck would have it, the Knuth, Morris, Pratt algorithm builds M quickly

Real-life Uses of DFAs

- Grep
- Coke Machines
- Thermostats (fridge)
- Elevators
- Train Track Switches
- Lexical Analyzers for Parsers

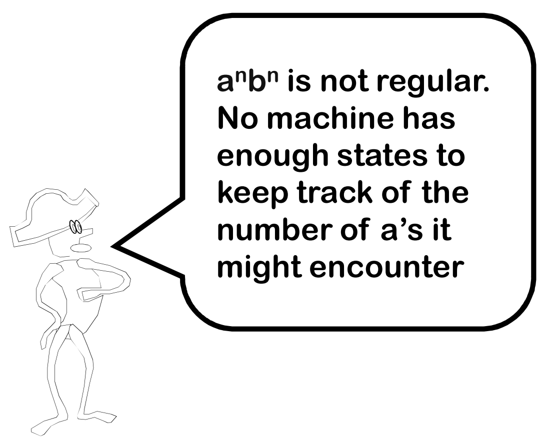


Consider the language $L = \{ a^n b^n \mid n > 0 \}$

i.e., a bunch of a's followed by an equal number of b's

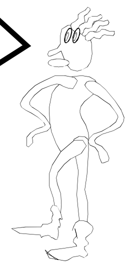
No finite automaton accepts this language

Can you prove this?



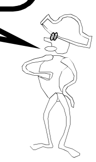
That is a fairly weak argument

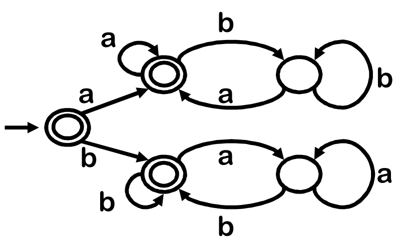
Consider the following example…



L = strings where the # of occurrences of the pattern ab is equal to the number of occurrences of the pattern ba

Can't be regular. No machine has enough states to keep track of the number of occurrences of ab

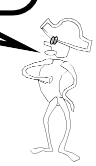




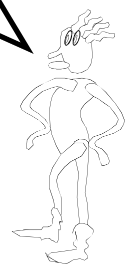
M accepts only the strings with an equal number of ab's and ba's!

L = strings where the # of occurrences of the pattern ab is equal to the number of occurrences of the pattern ba

~~Can't be regular. No machine has enough states to keep track of the number of occurrences of ab~~



Let me show you a professional strength proof that $a^n b^n$ is not regular…



This is the kind of proof we expect from you…

Pigeonhole principle:

Given n boxes and m > n objects, at least one box must contain more than one object

Letterbox principle:

If the average number of letters per box is x, then some box will have at least x letters (similarly, some box has at most x)

Theorem: L= $\{a^n b^n \mid n > 0\}$ is not regular

Proof (by contradiction):

Assume that L is regular

Then there exists a machine M with k states that accepts L

For each $0 \leq i \leq k$, let S_i be the state M is in after reading a^i

$\exists i,j \leq k$ such that $S_i = S_j$, but $i \neq j$

M will do the same thing on $a^i b^i$ and $a^j b^i$

But a valid M must reject $a^j b^i$ and accept $a^i b^i$

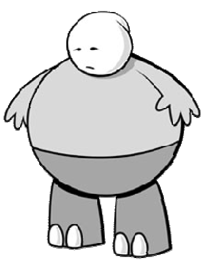
How to prove a language is not regular… (most of the time)

Assume it is regular, hence is accepted by a DFA M with n states.

Show that there are two strings s and s' which both reach some state in M (usually by pigeonhole principle)

Then show there is some string t such that string st is in the language, but s't is not. However, M accepts either both or neither.

What are s, s', t? That's where the work is…



Here's What You Need to Know…

Deterministic Finite Automata

- Definition
- Testing if they accept a string
- Building automata

Regular Languages

- Definition
- Closed Under Union, Intersection, Negation
- Using Pigeonhole Principle to show language ain't regular