

15-251

Great Theoretical Ideas in Computer Science

Graphs I

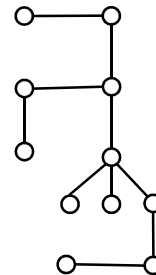
Lecture 18, October 22, 2009

What's a tree?

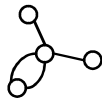
A tree is a connected
graph with no cycles



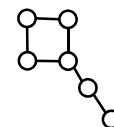
Tree

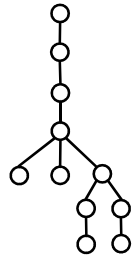


Not a tree



Not a tree







How Many n-Node Trees?

- 1: O

- $$2: \text{O}=\text{O}$$

- 3: ○—○—○

- 4: 

- 5: 

Notation

In this lecture:

n will denote the number of nodes in a graph

e will denote the number of edges in a graph

Theorem: Let G be a graph with n nodes and e edges

The following are equivalent:

1. G is a tree (connected, acyclic)
2. Every two nodes of G are joined by a unique path
3. G is connected and $n = e + 1$
4. G is acyclic and $n = e + 1$
5. G is acyclic and if any two non-adjacent nodes are joined by an edge, the resulting graph has exactly one cycle

To prove this, it suffices to show

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$$

1 \Rightarrow 2 1. G is a tree (connected, acyclic)

2. Every two nodes of G are joined by a unique path

Proof: (by contradiction)

Assume G is a tree that has two nodes connected by two different paths:



Then there exists a cycle!

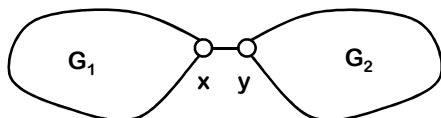
2 \Rightarrow 3 2. Every two nodes of G are joined by a unique path

3. G is connected and $n = e + 1$

Proof: (by induction)

Assume true for every graph with $< n$ nodes

Let G have n nodes and let x and y be adjacent



Let n_1, e_1 be number of nodes and edges in G_1

Then $n = n_1 + n_2 = e_1 + e_2 + 2 = e + 1$

3 \Rightarrow 4 3. G is connected and $n = e + 1$

4. G is acyclic and $n = e + 1$

Proof: (by contradiction)

Assume G is connected with $n = e + 1$, and G has a cycle containing k nodes



Note that the cycle has k nodes and k edges

Start adding nodes and edges until you cover the whole graph

Number of edges in the graph will be at least n

Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

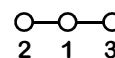
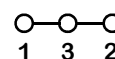
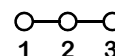
Proof (by contradiction):

Assume all but one of the points in the tree have degree at least 2

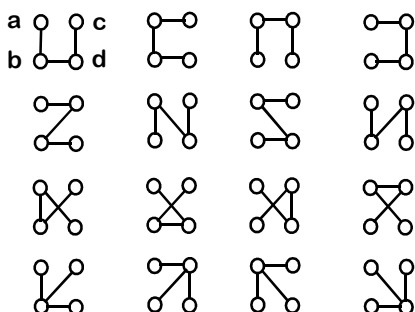
In any graph, sum of the degrees = $2e$

Then the total number of edges in the tree is at least $(2n-1)/2 = n - 1/2 > n - 1$

How many labeled trees are there with three nodes?



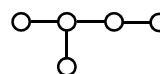
How many labeled trees are there with four nodes?



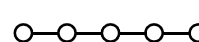
How many labeled trees are there with five nodes?



5
labelings



$5 \times 4 \times 3$
labelings



$5!/2$
labelings

125 labeled trees

How many labeled trees are there with n nodes?

3 labeled trees with 3 nodes

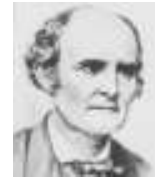
16 labeled trees with 4 nodes

125 labeled trees with 5 nodes

n^{n-2} labeled trees with n nodes

Cayley's Formula

The number of labeled trees on n nodes is n^{n-2}



The proof will use the correspondence principle

Each labeled tree on n nodes

corresponds to

A sequence in $\{1, 2, \dots, n\}^{n-2}$ (that is, $n-2$ numbers, each in the range $[1..n]$)

How to make a sequence from a tree?

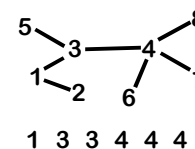
Loop through i from 1 to $n-2$

Let L be the degree-1 node with the lowest label

Define the i^{th} element of the sequence as the label of the node adjacent to L

Delete the node L from the tree

Example:



How to reconstruct the unique tree from a sequence S :

Let $I = \{1, 2, 3, \dots, n\}$

Loop until S is empty

Let i = smallest # in I but not in S

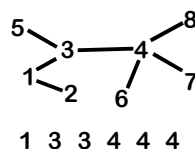
Let s = first label in sequence S

Add edge $\{i, s\}$ to the tree

Delete i from I

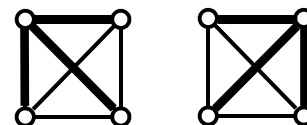
Delete s from S

Add edge $\{a, b\}$, where $I = \{a, b\}$



Spanning Trees

A spanning tree of a graph G is a tree that touches every node of G and uses only edges from G

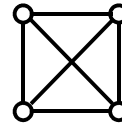
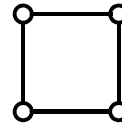


Every connected graph has a spanning tree

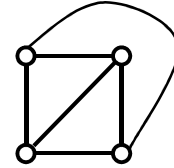
A graph is planar if it can be drawn in the plane without crossing edges



Examples of Planar Graphs

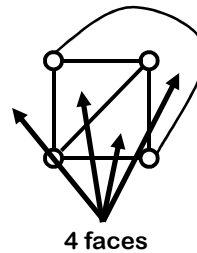


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<http://www.planarity.net>

Faces



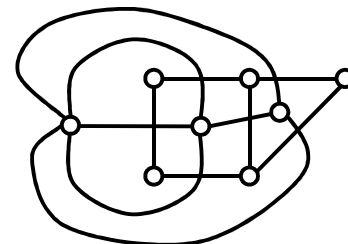
A planar graph splits the plane into disjoint faces

Euler's Formula

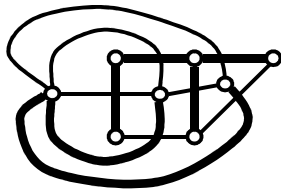
If G is a connected planar graph with n vertices, e edges and f faces, then $n - e + f = 2$



Rather than using induction, we'll use the important notion of the dual graph



Dual = put a node in every face, and an edge for each edge joining two adjacent faces



Let G^* be the dual graph of G

Let T be a spanning tree of G

Let T^* be the graph where there is an edge in dual graph for each edge in $G - T$

Then T^* is a spanning tree for G^*

$$\begin{aligned} n &= e_T + 1 & n + f &= e_T + e_{T^*} + 2 \\ f &= e_{T^*} + 1 & &= e + 2 \end{aligned}$$

Corollary: Let G be a simple planar graph with $n > 2$ vertices. Then:

1. G has a vertex of degree at most 5
2. G has at most $3n - 6$ edges

Proof of 1:

In any graph, (sum of degrees) = $2e$

Assume all vertices have degree ≥ 6

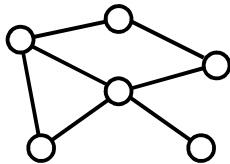
Then $3n \leq e$

Furthermore, since G is simple, $3f \leq 2e$

So $3n + 3f \leq 3e \Rightarrow 3(n - e + f) \leq 0$, contradict.

Graph Coloring

A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color



Graph Coloring

Arises surprisingly often in CS

Register allocation: assign temporary variables to registers for scheduling instructions. Variables that interfere, or are simultaneously active, cannot be assigned to the same register

Theorem: Every planar graph can be 6-colored

Proof Sketch (by induction):

Assume every planar graph with less than n vertices can be 6-colored

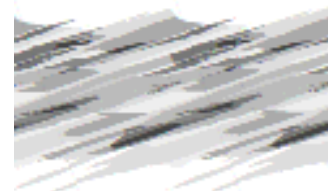
Assume G has n vertices

Since G is planar, it has some node v with degree at most 5

Remove v and color by Induction Hypothesis

Not too difficult to give an inductive proof of 5-colorability, using same fact that some vertex has degree ≤ 5

4-color theorem remains challenging!



Implementing Graphs

Adjacency Matrix

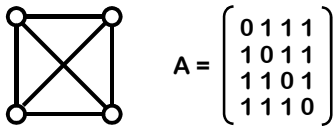
Suppose we have a graph G with n vertices. The adjacency matrix is the $n \times n$ matrix $A=[a_{ij}]$ with:

$a_{ij} = 1$ if (i,j) is an edge

$a_{ij} = 0$ if (i,j) is not an edge

Good for dense graphs!

Example



Counting Paths

The number of paths of length k from node i to node j is the entry in position (i,j) in the matrix A^k

$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

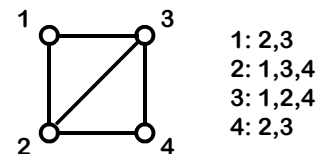
$$= \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

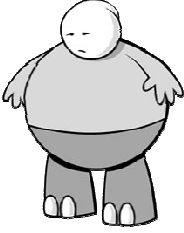
Adjacency List

Suppose we have a graph G with n vertices. The adjacency list is the list that contains all the nodes that each node is adjacent to

Good for sparse graphs!

Example





Trees

- Counting Trees
- Different Characterizations

Planar Graphs

- Definition
- Euler's Theorem
- Coloring Planar Graphs

Adjacency Matrix and List

- Definition
- Useful for counting

Here's What
You Need to
Know...