15-251

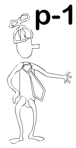
Great Theoretical Ideas in Computer Science

Fact:
GCD(x,y) × LCM(x,y) = x × y

You can use
MAX(a,b) + MIN(a,b) = a+b
to prove the above fact…

Number Theory and Modular Arithmetic

Lecture 13 (October 8, 2009)



$^{p-1} \equiv_p 1$

(a mod n) means the remainder when a is divided by n.

a mod n = r
⇔
a = dn + r for some integer d

Greatest Common Divisor:
GCD(x,y) =
greatest k ≥ 1 s.t. k|x and k|y.

Least Common Multiple:
LCM(x,y) =
smallest k ≥ 1 s.t. x|k and y|k.

Definition: Modular equivalence
a ≡ b [mod n]
⇔ (a mod n) = (b mod n)
⇔ n | (a-b)

31 ≡ 81 [mod 2]
$31 \equiv_2 81$

31 ≡ 80 [mod 7]
$31 \equiv_7 80$

Written as $a \equiv_n b$, and spoken
"a and b are equivalent or congruent modulo n"

$\equiv_n$ is an <u>equivalence relation</u>

In other words, it is

Reflexive: $a \equiv_n a$

Symmetric: $(a \equiv_n b) \Rightarrow (b \equiv_n a)$

Transitive: $(a \equiv_n b$ and $b \equiv_n c) \Rightarrow (a \equiv_n c)$

Why do we care about these residue classes?

Because we can replace any member of a residue class with another member when doing addition or multiplication mod n and the answer will not change

To calculate: 249 * 504 mod 251

just do -2 * 2 = -4 = 247

$\equiv_n$ induces a natural partition of the integers into n "residue" classes.

("residue" = what left over = "remainder")

Define residue class [k] = the set of all integers that are congruent to k modulo n.

Fundamental lemma of plus and times mod n:

If $(x \equiv_n y)$ and $(a \equiv_n b)$. Then

1) $x + a \equiv_n y + b$
2) $x * a \equiv_n y * b$

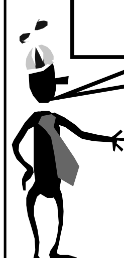
Residue Classes Mod 3:

[0] = { …, -6, -3, 0, 3, 6, ..}
[1] = { …, -5, -2, 1, 4, 7, ..}
[2] = { …, -4, -1, 2, 5, 8, ..}

[-6] = { …, -6, -3, 0, 3, 6, ..} = [0]
[7] = { …, -5, -2, 1, 4, 7, ..} = [1]
[-1] = { …, -4, -1, 2, 5, 8, ..} = [2]

Proof of 2: xa = yb (mod n)

(The other proof is similar…)



Another Simple Fact:
If ($x \equiv_n y$) and ($k|n$), then: $x \equiv_k y$

Example: $10 \equiv_6 16 \Rightarrow 10 \equiv_3 16$

Proof:

Unique representation system mod 4

Finite set S = {0, 1, 2, 3}

+ and * defined on S:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

A Unique Representation System Modulo n:

We pick one representative from each residue class and do all our calculations using these representatives.

Unsurprisingly, we use 0, 1, 2, …, n-1

Notation

$Z_n = \{0, 1, 2, \ldots, n-1\}$

Define operations $+_n$ and $*_n$:

$a +_n b = (a + b \bmod n)$
$a *_n b = (a * b \bmod n)$

Unique representation system mod 3

Finite set S = {0, 1, 2}

+ and * defined on S:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Some properties of the operation $+_n$

[“Closed”]
$x, y \in Z_n \Rightarrow x +_n y \in Z_n$

[“Associative”]
$x, y, z \in Z_n \Rightarrow (x +_n y) +_n z = x +_n (y +_n z)$

[“Commutative”]
$x, y \in Z_n \Rightarrow x +_n y = y +_n x$

Similar properties also hold for $*_n$

Unique representation system mod 3

Finite set S = {0, 1, 2}

+ and * defined on S:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Unique representation system mod 2

Finite set Z_2 = {0, 1}

two associative, commutative operators on Z_2

$+_2$ XOR	0	1
0	0	1
1	1	0

$*_2$ AND	0	1
0	0	0
1	0	1

Unique representation system mod 3

Finite set Z_3 = {0, 1, 2}

two associative, commutative operators on Z_3

Z_5 = {0,1,2,3,4}

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	
2	0				
3	0	3	1	4	
4	0	4	3	2	

Unique representation system mod 3

Finite set Z_3 = {0, 1, 2}

two associative, commutative operators on Z_3

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Z_6 = {0,1,2,3,4,5}

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

*	0	1	2	3	4	5
0	0	0	0	0	0	
1	0	1	2	3	4	
2	0	2	4	0	2	
3	0					
4	0	4	2	0	4	
5	0	5	4	3	2	

For addition tables, rows and columns <u>always</u> are a permutation of Z_n

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

For multiplication, if a row has a permutation you can solve, say,

5 * ___ = 4 (mod 6)

or, 5 * ___ = 1 (mod 6)

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

For multiplication, some rows and columns are permutation of Z_n, while others aren't…

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

what's happening here?

But if the row does not have the permutation property, how do you solve

no solutions! 3 * ___ = 4 (mod 6)

multiple solutions! 3 * ___ = 3 (mod 6)

3 * ___ = 1 (mod 6)

no multiplicative inverse!

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

For addition, the permutation property means you can solve, say,

4 + ___ = 1 (mod 6)

4 + ___ = x (mod 6) for any x in Z_6

Subtraction mod n is well-defined

Each row has a 0, hence –a is that element such that a + (-a) = 0

⇒ a – b = a + (-b)

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Division

If you define 1/a (mod n) = a^{-1} (mod n) as the element b in Z_n such that a * b = 1 (mod n)

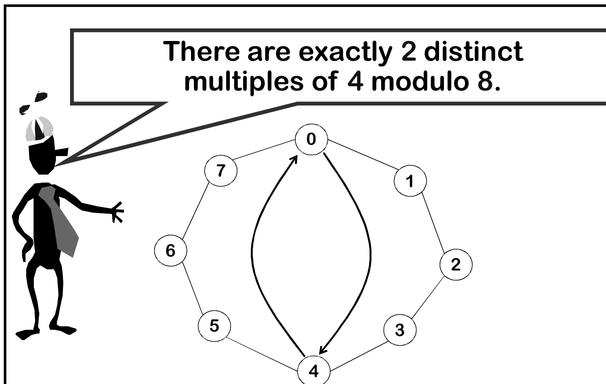
Then x/y (mod n)
=
x * 1/y (mod n)

Hence we can divide out by only the y's for which 1/y is defined!

And which rows do have the permutation property?

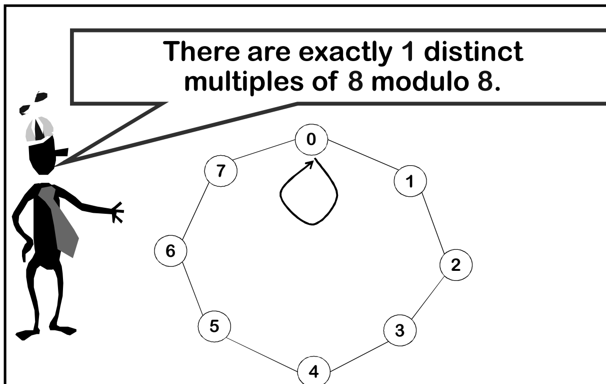
*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2						
3	0	3						
4	0	4						
5	0	5						
6	0	6						
7	0	7						

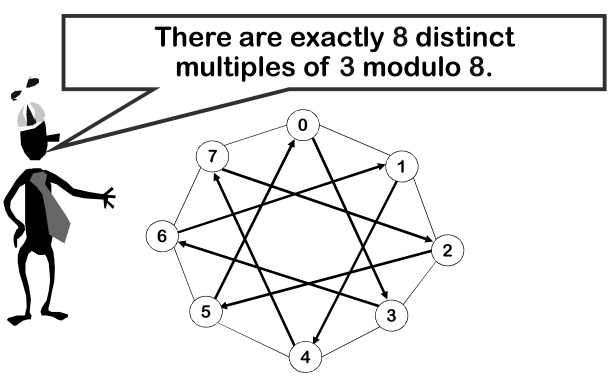
consider $*_8$ on Z_8



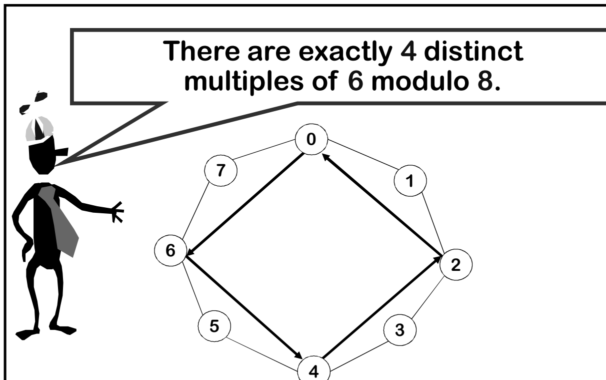
row 4 <u>does not</u> have "permutation property" for $*_8$ on Z_8

A visual way to understand multiplication and the "permutation property".





hit all numbers ⇔ row 3 has the "permutation property"



What's the pattern?

exactly 8 distinct multiples of 3 modulo 8.
exactly 2 distinct multiples of 4 modulo 8
exactly 1 distinct multiple of 8 modulo 8
exactly 4 distinct multiples of 6 modulo 8

exactly ________________ distinct

multiples of x modulo y

Theorem: There are exactly
LCM(n,c)/c = n/GCD(c,n)
distinct multiples of c modulo n

Hence,
only those values of c with GCD(c,n) = 1
have n distinct multiples
(i.e., the permutation property for $*_n$ on Z_n)

And remember, permutation property means
you can divide out by c (working mod n)

Theorem: There are exactly
LCM(n,c)/c = n/GCD(c,n)
distinct multiples of c modulo n

Fundamental lemma of division modulo n:
if GCD(c,n)=1, then ca $\equiv_n$ cb $\Rightarrow$ a $\equiv_n$ b

Proof:

Theorem: There are exactly k = n/GCD(c,n)
distinct multiples of c modulo n, and these
multiples are { c*i mod n | 0 ≤ i < k }

Proof:
Clearly, c/GCD(c,n) ≥ 1 is a whole number

ck = cn/GCD(c,n) = n(c/GCD(c,n)) $\equiv_n$ 0
$\Rightarrow$There are ≤ k distinct multiples of c mod n:
c*0, c*1, c*2, …, c*(k-1)

Also, k = factors of n missing from c
$\Rightarrow$ cx $\equiv_n$ cy $\Leftrightarrow$ n|c(x-y) $\Rightarrow$ k|(x-y) $\Rightarrow$ x-y ≥ k
$\Rightarrow$ There are ≥ k multiples of c.

Hence exactly k.

If you want to extend to
general c and n

ca $\equiv_n$ cb $\Rightarrow$ a $\equiv_{n/gcd(c,n)}$ b

Fundamental lemmas mod n:

If $(x \equiv_n y)$ and $(a \equiv_n b)$. Then

1) $x + a \equiv_n y + b$
2) $x * a \equiv_n y * b$
3) $x - a \equiv_n y - b$
4) $cx \equiv_n cy \Rightarrow a \equiv_n b$ if gcd(c,n)=1

We've got closure

Recall we proved that Z_n was "closed" under addition and multiplication?

What about Z_n^* under multiplication?

Fact: if $a,b \in Z_n^*$, then ab (mod n) in Z_n^*

Proof: if gcd(a,n) = gcd(b,n) = 1,
then gcd(ab, n) = 1
then gcd(ab mod n, n) = 1

New definition:

$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$

Multiplication over this set Z_n^* has the cancellation property.

$Z_{12}^* = \{0 \le x < 12 \mid gcd(x,12) = 1\}$
$= \{1,5,7,11\}$

$*_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

$Z_6 = \{0, 1,2,3,4,5\}$
$Z_6^* = \{1,5\}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Z_{15}^*

*	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

$Z_5^* = \{1,2,3,4\}$ $= Z_5 \setminus \{0\}$

$*_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$Z_{12}^* = \{0 \le x < 12 \mid \gcd(x,12) = 1\}$
$= \{1,5,7,11\}$

$\phi(12) = 4$

$*_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Fact:
For prime p, the set $Z_p^* = Z_p \setminus \{0\}$

Proof:
It just follows from the definition!

For prim p, all $0 < x < p$ satisfy $\gcd(x,p) = 1$

Theorem: if p,q distinct primes then $\phi(pq) = (p-1)(q-1)$

How about p = 3, q = 5?

Euler Phi Function $\phi(n)$

$\phi(n)$ = size of Z_n^*
= number of $1 \le k < n$ that are relatively prime to n.

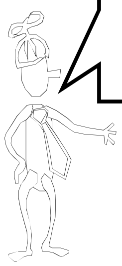
p prime
$\Rightarrow Z_p^* = \{1,2,3,\ldots,p-1\}$
$\Rightarrow \phi(p) = p-1$

Theorem: if p,q distinct primes then $\phi(pq) = (p-1)(q-1)$

pq = # of numbers from 1 to pq
p = # of multiples of q up to pq
q = # of multiples of p up to pq
1 = # of multiple of <u>both</u> p and q up to pq

$\phi(pq) = pq - p - q + 1 = (p-1)(q-1)$

Additive and Multiplicative Inverses



Multiplicative inverse of a mod n
= number b such that a*b=1 (mod n)

What is the multiplicative inverse
of a = 342952340 in
$Z_{4230493243} = Z_n$?

Answer: a^{-1} = 583739113

Additive inverse of a mod n
= number b such that a+b=0 (mod n)

What is the additive inverse
of a = 342952340 in
$Z_{4230493243} = Z_n$?

Answer: n – a
= 4230493243-342952340
=3887540903

How do you find multiplicative inverses <u>fast</u>?

Multiplicative inverse of a mod n
= number b such that a*b=1 (mod n)

Remember,
only defined for numbers a in Z_n^*

Theorem: given positive integers X, Y, there exist integers r, s such that
r X + s Y = gcd(X, Y)

and we can find these integers fast!

Now take n, and $a \in Z_n^*$

gcd(a, n) ?   a in $Z_n^* \Rightarrow$ gcd(a, n) = 1

suppose ra + sn = 1

then ra $\equiv_n$ 1

so, r = a^{-1} mod n

Theorem: given positive integers X, Y, there exist integers r, s such that

r X + s Y = gcd(X, Y)

and we can find these integers fast!

How?

Extended Euclid Algorithm

Finally, a puzzle…

You have a 5 gallon bottle,
a 3 gallon bottle,
and lots of water.

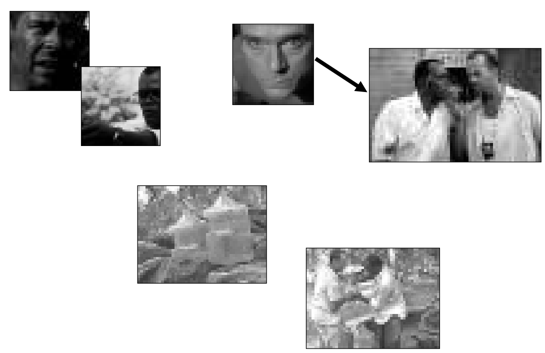
How can you measure out exactly 4 gallons?

Euclid’s Algorithm for GCD

```
Euclid(A,B)
If B=0 then return A
        else return Euclid(B, A mod B)
```

Euclid(67,29)	67 – 2*29 = 67 mod 29 = 9
Euclid(29,9)	29 – 3*9 = 29 mod 9 = 2
Euclid(9,2)	9 – 4*2 = 9 mod 2 = 1
Euclid(2,1)	2 – 2*1 = 2 mod 1 = 0
Euclid(1,0) outputs 1	

why?



Extended Euclid Algorithm

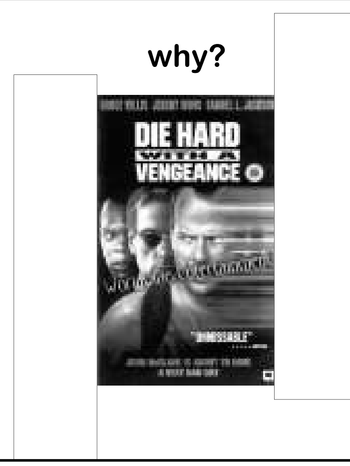
Let <r,s> denote the number r*67 + s*29. Calculate all intermediate values in this representation.

67=<1,0>   29=<0,1>

Euclid(67,29)	9=<1,0> – 2*<0,1>	9 =<1,-2>
Euclid(29,9)	2=<0,1> – 3*<1,-2>	2=<-3,7>
Euclid(9,2)	1=<1,-2> – 4*<-3,7>	1=<13,-30>
Euclid(2,1)	0=<-3,7> – 2*<13,-30>	0=<-29,67>

Euclid(1,0) outputs   1 = 13*67 – 30*29

why?



Invariant

Suppose stage of system is given by (L,S)
(L gallons in larger one, S in smaller)

Set of valid moves
1. empty out either bottle
2. fill up bottle (completely) from water source
3. pour bottle into other until first one empty
4. pour bottle into other until second one full

Invariant: L,S are both multiples of 3.

Diophantine equations

Does the equality
$3x + 5y = 4$
have a solution where x,y are integers?

Generalized bottles of water

You have a P gallon bottle,
a Q gallon bottle,
and lots of water.

When can you measure out
exactly 1 gallon?

New bottles of water puzzle

You have a 6 gallon bottle,
a 3 gallon bottle,
and lots of water.

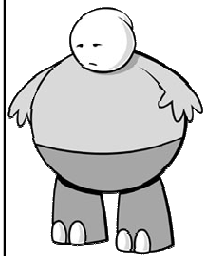
How can you measure out
exactly 4 gallons?

Recall that

if P and Q have gcd(P, Q) = 1
then you can find integers a and b so that
$a*P + b*Q = 1$

Suppose a is positive, then fill out P a times
and empty out Q b times

(and move water from P to Q as needed…)



Here's What You Need to Know…

Working modulo integer n

Definitions of Z_n, Z_n^*
and their properties

Fundamental lemmas of +,-,*,/
When can you divide out

Extended Euclid Algorithm
How to calculate c^{-1} mod n.

Euler phi function $\phi(n) = |Z_n^*|$