

15-251

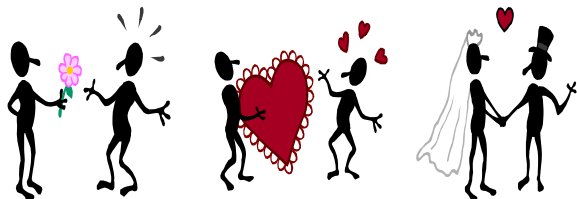
Great Theoretical Ideas in Computer Science

15-251

Dating for Computer Scientists

The Mathematics Of 1950's Dating: Who wins The Battle of The Sexes?

Lecture 12 (October 6, 2009)



WARNING: This lecture
contains mathematical
content that may be
shocking to some
students

Dating Scenario

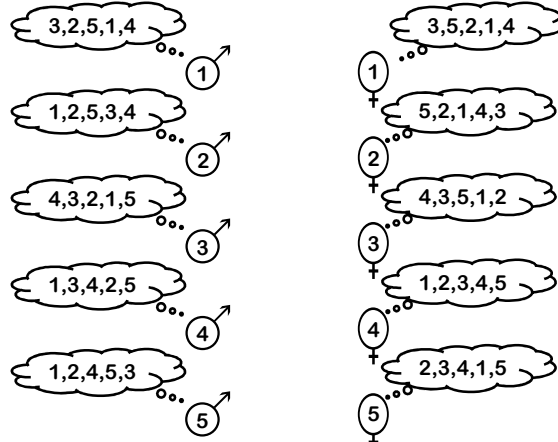
There are n boys and n girls

Each girl has her own ranked
preference list of all the boys

Each boy has his own ranked
preference list of the girls

The lists have no ties

Question: How do we pair them off?



More Than One Notion of What Constitutes A “Good” Pairing

Maximizing total satisfaction

Hong Kong and to an extent the USA

Maximizing the minimum satisfaction

Western Europe

Minimizing maximum difference in mate ranks

Sweden

Maximizing people who get their first choice

Barbie and Ken Land



We will ignore the issue of what is “best”!

Rogue Couples

Suppose we pair off all the boys and girls

Now suppose that some boy and some girl prefer each other to the people to whom they are paired

They will be called a rogue couple



Why be with them when we can be with each other?



What use is fairness, if it is not stable?

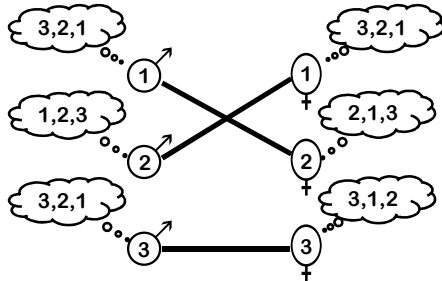
Any list of criteria for a good pairing must include stability. (A pairing is doomed if it contains a rogue couple.)

Stable Pairings

A pairing of boys and girls is called stable if it contains no rogue couples

Stable Pairings

A pairing of boys and girls is called stable if it contains no rogue couples



The study of stability will be the subject of the entire lecture

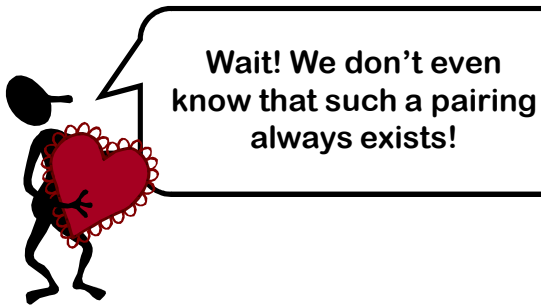
We will:

Analyze various mathematical properties of an algorithm that looks a lot like 1950's dating

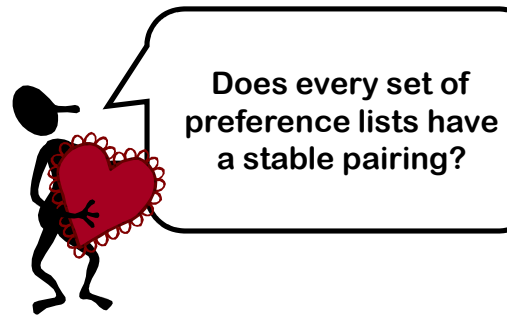
Discover the naked mathematical truth about which sex has the romantic edge

Learn how the world's largest, most successful dating service operates

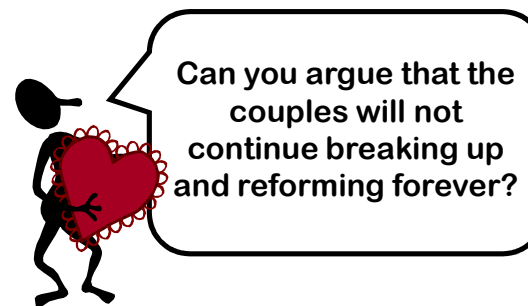
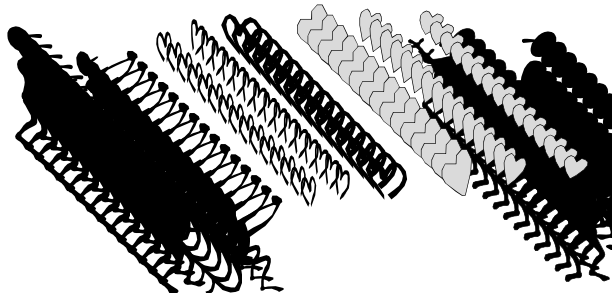
Given a set of preference lists, how do we find a stable pairing?

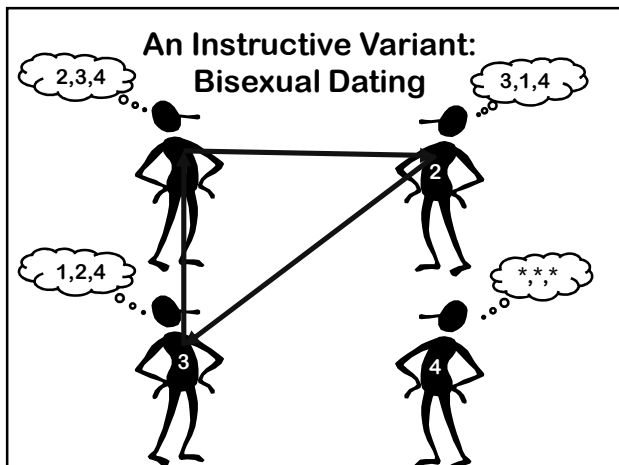


Better Question:



Idea: Allow the pairs to keep breaking up and reforming until they become stable

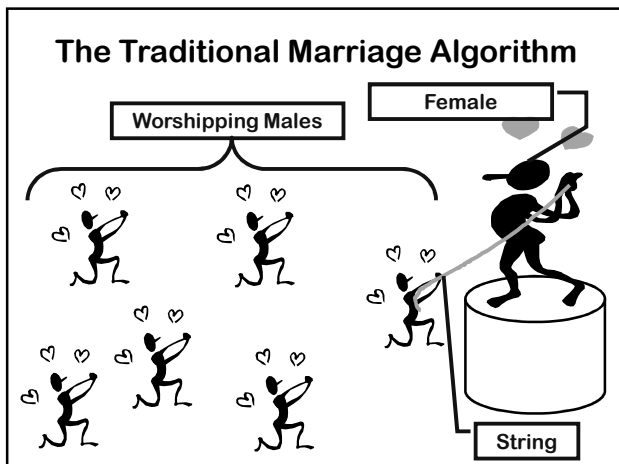




Insight

Any proof that heterosexual couples do not break up and re-form forever must contain a step that fails in the bisexual case

If you have a proof idea that works equally well in the hetero and bisexual versions, then your idea is not adequate to show the couples eventually stop



The Traditional Marriage Algorithm

For each day that some boy gets a "No" do:

Morning

- Each girl stands on her balcony
- Each boy proposes to the best girl whom he has not yet crossed off

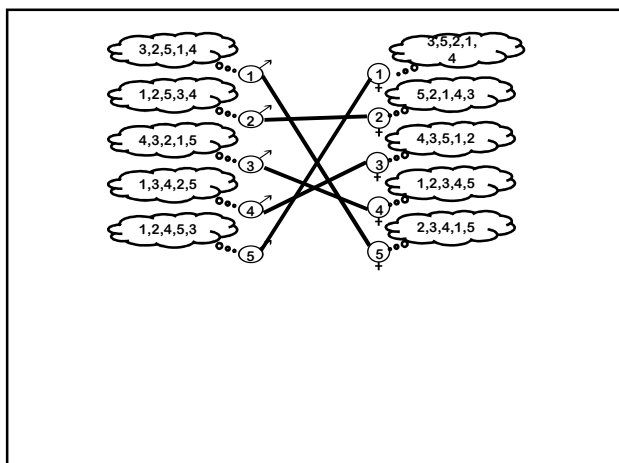
Afternoon (for girls with at least one suitor)

- To today's best: "Maybe, return tomorrow"
- To any others: "No, I will never marry you"

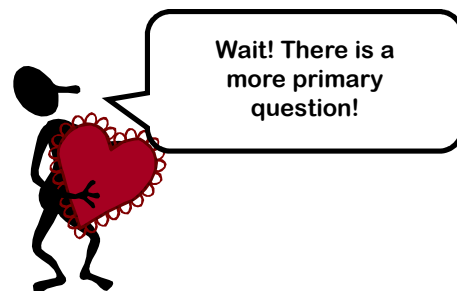
Evening

- Any rejected boy crosses the girl off his list

If no boys get a "No", each girl marries boy to whom she just said "maybe"



Does Traditional Marriage Algorithm always produce a stable pairing?



Does TMA Always Terminate?

It might encounter a situation where algorithm does not specify what to do next (e.g. "core dump error")

It might keep on going for an infinite number of days

The Traditional Marriage Algorithm

For each day that some boy gets a "No" do:

Morning

- Each girl stands on her balcony
- Each boy proposes to the best girl whom he has not yet crossed off

Afternoon (for girls with at least one suitor)

- To today's best: "Maybe, return tomorrow"
- To any others: "No, I will never marry you"

Evening

- Any rejected boy crosses the girl off his list

If no boys get a "No", each girl marries boy to whom she just said "maybe"

Does TMA Always Terminate?

It might encounter a situation where algorithm does not specify what to do next

this could happen if some boy
crosses all names off his list!
we'll show it doesn't...

It might keep on going for an infinite number of days

cannot happen, every day
some name is crossed off

Improvement Lemma:

If a girl has a boy on a string, then she will always have someone at least as good on a string (or for a husband)

She would only let go of him in order to "maybe" someone better

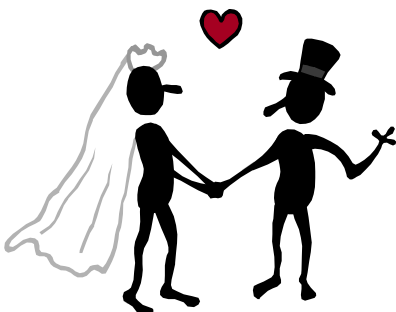
She would only let go of that guy for someone even better

She would only let go of that guy for someone even better

AND SO ON...



Corollary: Each girl will marry her absolute favorite of the boys who visit her during the TMA



Lemma: No boy can be rejected by all the girls

Proof (by contradiction):

Suppose boy b is rejected by all the girls

At that point:

Each girl must have a suitor other than b

(By Improvement Lemma, once a girl has a suitor she will always have at least one)

The n girls have n suitors, and b is not among them. Thus, there are at least $n+1$ boys



Theorem: The TMA always terminates in at most n^2 days

A “master list” of all n of the boys lists starts with a total of $n \times n = n^2$ girls on it

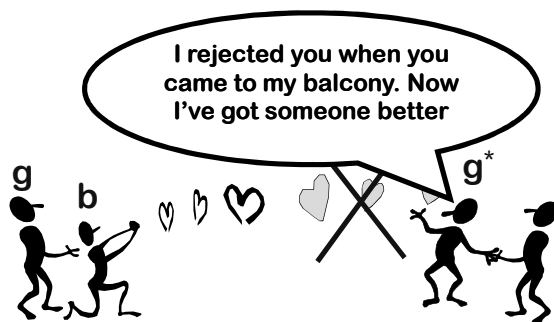
Each day that at least one boy gets a “No”, so at least one girl gets crossed off the master list

Therefore, the number of days is bounded by the original size of the master list

Great! We know that TMA will terminate and produce a pairing

But is it stable?

Theorem: The pairing T produced by TMA is stable



Let b and g be any couple in T , and b prefers g^* to g . We claim that g^* prefers her husband to b .

During TMA, b proposed to g^* before he proposed to g .

Hence, at some point g^* rejected b for someone she preferred to b .

By the Improvement lemma, the person she married was also preferable to b . □

Thus, every boy will be rejected by any girl he prefers to his wife.
 $\Rightarrow T$ is stable.

Opinion Poll



Forget TMA for a Moment...

How should we define “the optimal girl for boy b ”?



Flawed Attempt:
 “The girl at the top of b ’s list”

The Optimal Girl

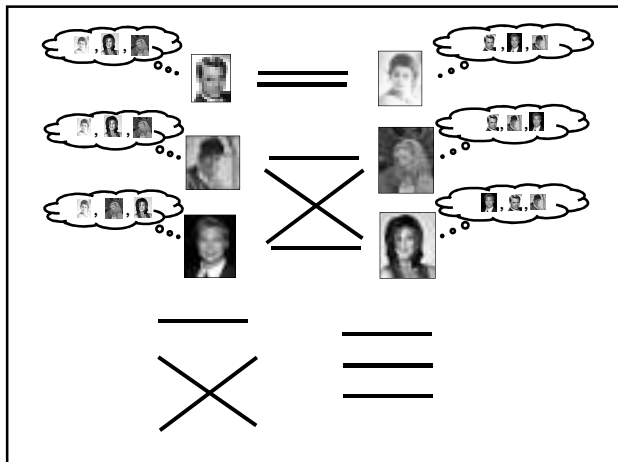
A boy's optimal girl is the highest ranked girl for whom there is some stable pairing in which the boy gets her

She is the best girl he can conceivably get in a stable world. Presumably, she might be better than the girl he gets in the stable pairing output by TMA

The Pessimal Girl

A boy's pessimal girl is the lowest ranked girl for whom there is some stable pairing in which the boy gets her

She is the worst girl he can conceivably get in a stable world



Dating Heaven and Hell

A pairing is male-optimal if every boy gets his optimal mate. This is the best of all possible worlds for every boy simultaneously

A pairing is male-pessimal if every boy gets his pessimal mate. This is the worst of all possible worlds for every boy simultaneously

BTW, is a male-optimal pairing stable?
How about a male-pessimal one?

Dating Heaven and Hell

A pairing is female-optimal if every girl gets her optimal mate. This is the best of all possible stable worlds for every girl simultaneously

A pairing is female-pessimal if every girl gets her pessimal mate. This is the worst of all possible stable worlds for every girl simultaneously

not clear if either exists!
we'll show that both exist...

The Naked Mathematical Truth!

The Traditional Marriage Algorithm always produces a male-optimal, female-pessimal pairing

Suppose TMA is not male-optimal

likes b^* more than b

consider the first moment in TMA when some boy is rejected by his optimal girl

likes g^* at least as much as his optimal (not been rejected by optimal yet)

since g is b 's optimal, there is a stable matching S where g & b are matched

likes b^* more than b

contradicts stability of S !!!

Formal proof on next two slides

Theorem: TMA produces a male-optimal pairing

Suppose, for a contradiction, that some boy gets rejected by his optimal girl during TMA

Let t be the earliest time at which this happened

At time t , boy b got rejected by his optimal girl g because she said "maybe" to a preferred b^*

By the definition of t , b^* had not yet been rejected by his optimal girl

Therefore, b^* likes g at least as much as his optimal

Some boy b got rejected by his optimal girl g because she said "maybe" to a preferred b^* . b^* likes g at least as much as his optimal girl

Also, there must exist a stable pairing S in which b and g are married (by def. of "optimal")

b^* wants g more than his wife in S :
 g is at least as good as his optimal and he is not matched to g in stable pairing S

g wants b^* more than her husband in S :
 b is her husband in S and she rejects him for b^* in TMA

Contradiction

Theorem: The TMA pairing, T , is female-pessimal

suppose not

T (boy-optimal)

there exists a girl g ...

...and a stable matching S that matches g - b^* , where b^* is worse for g

g is his optimal girl

prefers b to b^*

contradicts stability of S !!!

Formal proof on next slide

Theorem: The TMA pairing, T , is female-pessimal

We know it is male-optimal. Suppose there is a stable pairing S where some girl g does worse than in T

Let b be her mate in T

Let b^* be her mate in S

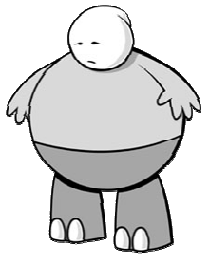
By assumption, g likes b better than her mate b^* in S

b likes g better than his mate in S (we already know that g is his optimal girl)

Therefore, S is a stable pairing

Contradiction

The largest, most successful dating service in the world uses a computer to run TMA!



**Here's What
You Need to
Know...**

Definitions of:

- **Stable Pairing**
- **Traditional Marriage Algorithm**
- **optimal and pessimal mates**

Proof that:

- **TMA Produces a Stable Pairing**
- **TMA Produces a Male-Optimal, Female-Pessimal Pairing**