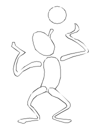
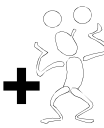
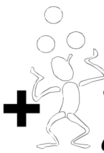
15-251

Great Theoretical Ideas in Computer Science

Counting III: Generating Functions

Lecture 8 (September 17, 2009)

 $X^1 +$  $X^2 +$  X^3

Recap

The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

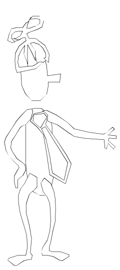
$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle: k^{th} row are coefficients of $(1+X)^k$

Inductive definition of k^{th} entry of n^{th} row:
Pascal(n,0) = Pascal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)

Pascal's Triangle



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

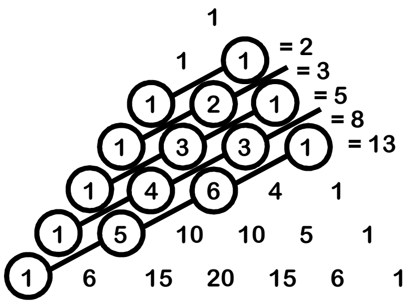
"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand"

Summing on k^{th} Avenue

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

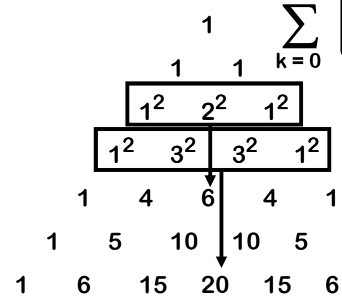
$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

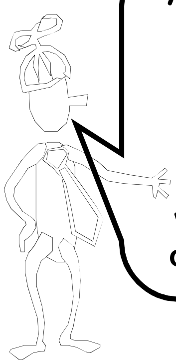
Fibonacci Numbers



Sums of Squares

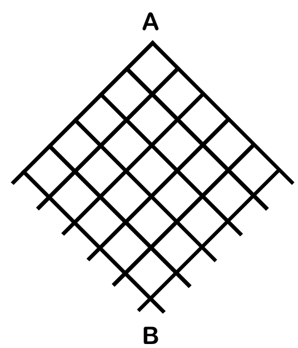
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$





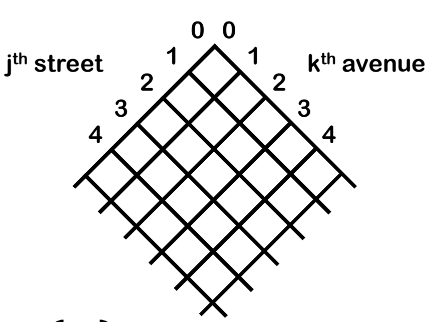
All these properties can be proved inductively and algebraically. We gave *combinatorial* proofs using the Manhattan block walking representation of binomial coefficients

How many shortest routes from A to B?



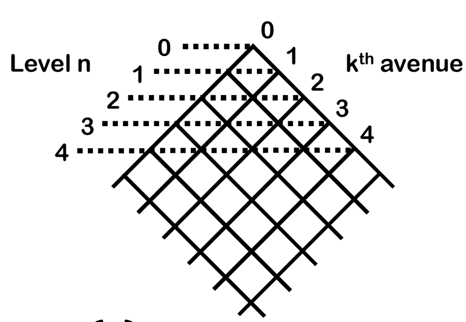
$$\binom{10}{5}$$

Manhattan



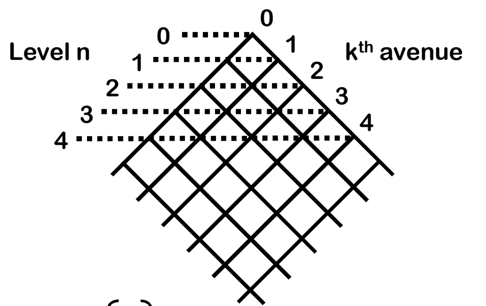
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k)

Manhattan

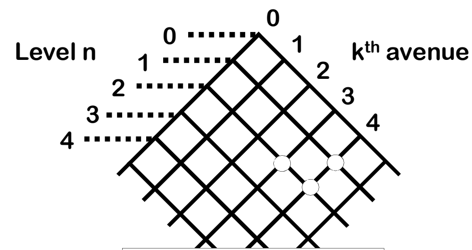


There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k)

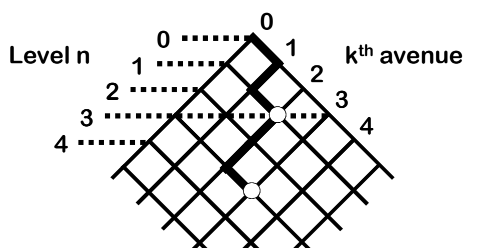
Manhattan



There are $\binom{n}{k}$ shortest routes from (0,0) to level n and k^{th} avenue

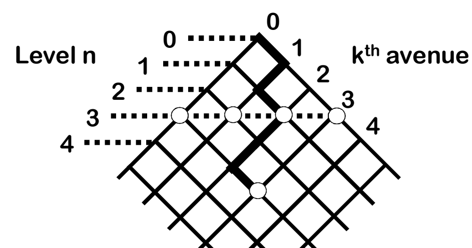


$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

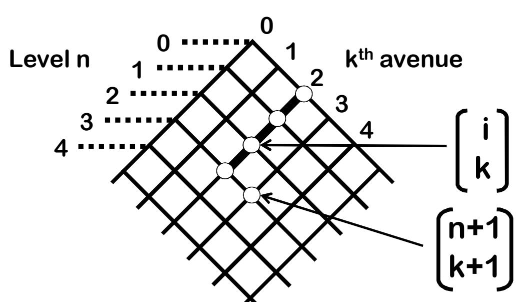


How many ways to get to ○ via ○ ?

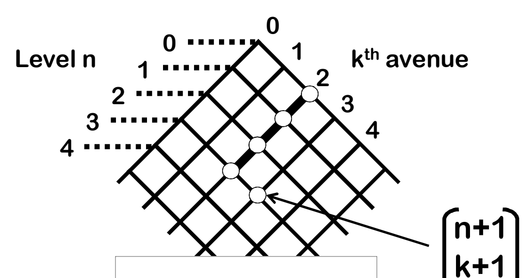
$$\binom{n}{k}^2$$



$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

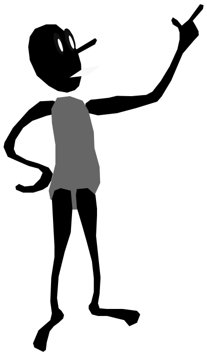


How many ways to get to ○ via ○ ?



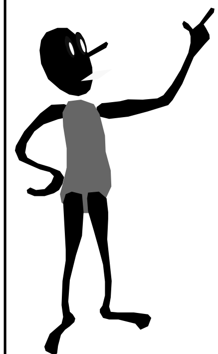
$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1} = \frac{1 - X^n}{1 - X}$$



Recall the Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots = \frac{1}{1 - X}$$



the <u>Infinite</u> Geometric Series

Holds when $|X| < 1$

Also makes sense if we view the infinite sum on the left as a formal power series

$$
\begin{aligned}
P(X) &= 1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots \\
-X * P(X) &= \quad - X^1 - X^2 - X^3 - \ldots - X^n - X^{n+1} - \ldots \\
\hline
(1 - X)\, P(X) &= 1
\end{aligned}
$$

$$\Rightarrow P(X) = \frac{1}{1 - X}$$

Formal Power Series

$$P(X) = \sum_{i=0}^{\infty} a_i X^i$$

There are no worries about convergence issues.

This is a purely syntactic object.

Formal Power Series

$$P(X) = \sum_{i=0}^{\infty} a_i X^i$$

If you want, think of as the infinite vector $V = < a_0, a_1, a_2, \ldots, a_n, \ldots >$

But, as you will see, thinking of as a "polynomial" is very natural.

Operations

$$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$$
$$B(X) = b_0 + b_1 X + b_2 X^2 + \ldots$$

adding them together

$$(A+B)(X) = (a_0+b_0) + (a_1+b_1) X + (a_2+b_2) X^2 + \ldots$$

like adding the vectors position-wise

<4,2,3,…> + <5,1,1,….> = <9,3,4,…>

Operations

$A(X) = a_0 X^0 + a_1 X^1 + a_2 X^2 + \ldots$

multiplying by X

$X * A(X) = 0 X^0 + a_0 X^1 + a_1 X^2 + a_2 X^3 + \ldots$

like shifting the vector entries

SHIFT<4,2,3,…> = <0,4,2,3,…>

Example

Example:	Store:
V := <1,0,0,…>;	V = <1,0,0,0,…>
	V = <1,1,0,0,…>
Loop n times V := V + SHIFT(V);	V = <1,2,1,0,…>
	V = <1,3,3,1,…>

V = n^{th} row of Pascal's triangle

Example

Example:

V := <1,0,0,…>; $P_V := 1;$

Loop n times
V := V + SHIFT(V); $P_V := P_V(1+X);$

V = n^{th} row of Pascal's triangle

Example

Example:

V := <1,0,0,…>;

Loop n times
V := V + SHIFT(V);

} $P_V = (1+X)^n$

V = n^{th} row of Pascal's triangle

As expected, the coefficients of P_V give the n^{th} row of Pascal's triangle

To summarize…

Given a sequence V = < $a_0, a_1, a_2, \ldots, a_n, \ldots$ >

associate a formal power series with it

$$P(X) = \sum_{i=0}^{\infty} a_i X^i$$

This is the "generating function" for V

Fibonaccis

$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$

i.e., the sequence <0,1,1,2,3,5,8,13…>

is represented by the power series

$0 + 1X^1 + 1X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \ldots$

Two Representations

$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$

$A(X) = 0 + 1X^1 + 1X^2 + 2\,X^3 + 3\,X^4 + 5\,X^5 + 8\,X^6 + \ldots$

Can we write A(X) more succinctly?

$$A(X) = F_0 \;+ F_1 X^1 + F2\, X^2 + F_3 X^3 + \ldots + F_n X^n + \ldots$$

$$X\,A(X) = \;0 + F_0 X^1 + F_1 X^2 + F2\, X^3 + \ldots + F_{n-1} X^n + \ldots$$

$$X^2 A(X) = \;0 + 0\,X^1 + F_0 X^2 + F1\, X^3 + \ldots + F_{n-2} X^n + \ldots$$

$(1 - X - X^2)\,A(X)$

$= \;(F_0 - 0 - 0) \;\; + (F_1 - F_0 - 0)X^1 \;\; + 0$

$= \;X$

$$A(X) = \frac{X}{(1 - X - X^2)}$$

Fibonaccis

$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$

has the generating function

$$A(X) = \frac{X}{(1 - X - X^2)}$$

i.e., the coefficient of X^n in A(X) is F_n

$$
\begin{array}{l}
\qquad\qquad\; X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 \\
1 - X - X^2 \;\big|\; X \\
\qquad\qquad -(X - X^2 - X^3) \\
\qquad\qquad\quad X^2 + X^3 \\
\qquad\qquad\quad -(X^2 - X^3 - X^4) \\
\qquad\qquad\qquad 2X^3 + X^4 \\
\qquad\qquad\qquad -(2X^3 - 2X^4 - 2X^5) \\
\qquad\qquad\qquad\quad 3X^4 + 2X^5 \\
\qquad\qquad\qquad\quad -(3X^4 - 3X^5 - 3X^6) \\
\qquad\qquad\qquad\qquad 5X^5 + 3X^6 \\
\qquad\qquad\qquad\qquad -(5X^5 - 5X^6 - 5X^7) \\
\qquad\qquad\qquad\qquad\quad 8X^6 + 5X^7 \\
\qquad\qquad\qquad\qquad\quad -(8X^6 - 8X^7 - 8X^8)
\end{array}
$$

Two representations of the same thing…

$F_0 = 0, F_1 = 1,$
$F_n = F_{n-1} + F_{n-2}$

$$A(X) = \frac{X}{(1 - X - X^2)}$$

Closed form?

$F_0 = 0, F_1 = 1,$
$F_n = F_{n-1} + F_{n-2}$

$$A(X) = \frac{X}{(1 - X - X^2)}$$

let's factor $(1 - X - X^2)$

$(1 - X - X^2) = (1 - \varphi X)(1 - (-1/\varphi)X)$

where $\varphi = \dfrac{1 + \sqrt{5}}{2}$

Simplify, simplify…

$F_0 = 0, F_1 = 1,$
$F_n = F_{n-1} + F_{n-2}$

$$A(X) = \frac{X}{(1 - \varphi X)(1 - (-1/\varphi)X)}$$

some elementary algebra omitted…*

$$A(X) = \frac{1}{\sqrt{5}} \frac{1}{(1 - \varphi X)} + \frac{-1}{\sqrt{5}} \frac{1}{(1 - (-1/\varphi)X)}$$

*you are not allowed to say this in <u>your</u> answers…

$$A(X) = \frac{1}{\sqrt{5}} \frac{1}{(1 - \varphi X)} + \frac{-1}{\sqrt{5}} \frac{1}{(1 - (-1/\varphi)X)}$$

$$\frac{1}{(1 - \varphi X)} = 1 + \varphi X + \varphi^2 X^2 + \dots + \varphi^n X^n + \dots$$

$$\frac{1}{1 - Y} = 1 + Y^1 + Y^2 + Y^3 + \dots + Y^n + \dots$$

the <u>Infinite</u> Geometric Series

$$A(X) = \boxed{\frac{1}{\sqrt{5}}} \frac{1}{(1 - \varphi X)} \boxed{+} \boxed{\frac{-1}{\sqrt{5}}} \frac{1}{(1 - (-1/\varphi)X)}$$

$$\frac{1}{(1 - \varphi X)} = 1 + \varphi X + \varphi^2 X^2 + \dots + \boxed{\varphi^n} X^n + \dots$$

$$\frac{1}{(1 - (-1/\varphi)X)} = 1 + (-1/\varphi) X + \dots + \boxed{(-1/\varphi)^n} X^n + \dots$$

⇒ the coefficient of X^n in A(X) is…

$$\frac{1}{\sqrt{5}} \varphi^n + \frac{-1}{\sqrt{5}} (-1/\varphi)^n$$

Closed form for Fibonaccis

$$\boxed{F_n = \frac{1}{\sqrt{5}} \varphi^n + \frac{-1}{\sqrt{5}} (-1/\varphi)^n}$$

where $\varphi = \frac{1 + \sqrt{5}}{2}$

"golden ratio"

Abraham de Moivre (1730)
Leonhard Euler (1765)
J.P.M. Binet (1843)

Closed form for Fibonaccis

$$\boxed{F_n = \frac{1}{\sqrt{5}} \varphi^n + \frac{-1}{\sqrt{5}} (-1/\varphi)^n}$$

$F_n =$ closest integer to $\frac{1}{\sqrt{5}} \varphi^n$

To recap…

Given a sequence V = < a_0, a_1, a_2, …, a_n, … >

associate a formal power series with it

$$P(X) = \sum_{i=0}^{\infty} a_i X^i$$

This is the "generating function" for V

We just used this for the Fibonaccis…

Multiplication

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$

$B(X) = b_0 + b_1 X + b_2 X^2 + \ldots$

multiply them together

$$(A*B)(X) = (a_0 * b_0) + (a_0 b_1 + a_1 b_0) X + (a_0 b_2 + a_1 b_1 + a_2 b_0) X^2 + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) X^3 + \ldots$$

seems a bit less natural in the vector representation (it's called a "convolution" there)

Mult.: special case

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$

Special case: $B(X) = 1 + X + X^2 + \ldots = \frac{1}{1-X}$

multiply them together

$$(A*B)(X) = a_0 + (a_0 + a_1) X + (a_0 + a_1 + a_2) X^2 + (a_0 + a_1 + a_2 + a_3) X^3 + \ldots$$

it gives us partial sums!

For example…

Suppose $A(X) = 1 + X + X^2 + \ldots = \frac{1}{1-X}$

and $B(X) = 1 + X + X^2 + \ldots = \frac{1}{1-X}$

then $(A*B)(X) = 1 + 2X + 3X^2 + 4X^3 + \ldots$

$$= \frac{1}{1-X} * \frac{1}{1-X} = \frac{1}{(1-X)^2}$$

Generating function for the sequence <0,1,2,3,4…>

What happens if we again take prefix sums?

Take $1 + 2X + 3X^2 + 4X^3 + \ldots = \frac{1}{(1-X)^2}$

multiplying through by 1/(1-X)

$$\Delta_1 + \Delta_2 X^1 + \Delta_3 X^2 + \Delta_4 X^3 + \ldots = \frac{1}{(1-X)^3}$$

Generating function for the triangular numbers!

What's the pattern?

<1,1,1,1,…> $= \frac{1}{1-X}$

<1,2,3,4,…> $= \frac{1}{(1-X)^2}$

< $\Delta_1, \Delta_2, \Delta_3, \Delta_4$,…> $= \frac{1}{(1-X)^3}$

??? $= \frac{1}{(1-X)^n}$

What's the pattern?

$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \ldots \quad = \frac{1}{1-X}$

$<1,2,3,4,\ldots> \quad = \frac{1}{(1-X)^2}$

$<\Delta_1, \Delta_2, \Delta_3, \Delta_4, \ldots> \quad = \frac{1}{(1-X)^3}$

??? $\quad = \frac{1}{(1-X)^n}$

What's the pattern?

$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \ldots \quad = \frac{1}{1-X}$

$\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \binom{4}{1}, \ldots \quad = \frac{1}{(1-X)^2}$

$<\Delta_1, \Delta_2, \Delta_3, \Delta_4, \ldots> \quad = \frac{1}{(1-X)^3}$

??? $\quad = \frac{1}{(1-X)^n}$

What's the pattern?

$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \ldots \quad = \frac{1}{1-X}$

$\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \binom{4}{1}, \ldots \quad = \frac{1}{(1-X)^2}$

$\binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \binom{5}{2}, \ldots \quad = \frac{1}{(1-X)^3}$

??? $\quad = \frac{1}{(1-X)^n}$

What's the pattern?

$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \ldots \quad = \frac{1}{1-X}$

$\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \binom{4}{1}, \ldots \quad = \frac{1}{(1-X)^2}$

$\binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \binom{5}{2}, \ldots \quad = \frac{1}{(1-X)^3}$

$= \frac{1}{(1-X)^n}$

What's the pattern?

$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \ldots \quad = \frac{1}{1-X}$

$\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \binom{4}{1}, \ldots \quad = \frac{1}{(1-X)^2}$

$\binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \binom{5}{2}, \ldots \quad = \frac{1}{(1-X)^3}$

$$\sum_{k=0}^{\infty} \binom{k+n-1}{n-1} X^k = \frac{1}{(1-X)^n}$$

From Lecture #7

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

$$\sum_{i=0}^{k} \binom{i}{n} = \binom{k+1}{n+1}$$

Can be used to derive the coefficient of X^k

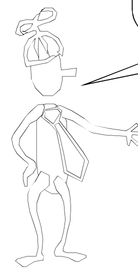
What's the pattern?

$$\sum_{k=0}^{\infty} \binom{k+n-1}{n-1} X^k = \frac{1}{(1-X)^n}$$

Another way to see it…

What is the coefficient of X^k in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + \dots)^n \ ?$$



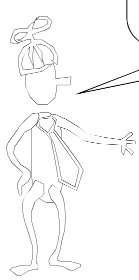
Each path in the choice tree for the cross terms has n choices of exponent $e_1, e_2, \dots, e_n \geq 0$. Each exponent can be any natural number.

Coefficient of X^k is the number of non-negative solutions to:
$e_1 + e_2 + \dots + e_n = k$

Another way to see it…

What is the coefficient of X^k in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + \dots)^n \ ?$$



$$\binom{n+k-1}{n-1}$$

Recap: getting partial sums

$$A(X) = a_0 + a_1 X + a_2 X^2 + \dots$$

$$\frac{A(X)}{1-X} = a_0 + (a_0 + a_1) X + (a_0 + a_1 + a_2) X^2 + (a_0 + a_1 + a_2 + a_3) X^3 + \dots$$

dividing by (1-X) gives us partial sums!

Here's an interesting use…

$$\sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} X^k = \frac{1}{(1-X)^3}$$

$\binom{k+2}{2}$

subtract off $\frac{3/2}{(1-X)^2}$ and add $\frac{1}{2}\frac{1}{1-X}$

$$\sum_{k=0}^{\infty} \frac{k^2}{2} X^k = \frac{X(1+X)}{2(1-X)^3}$$

From the previous page…

$$\sum_{k=0}^{\infty} k^2 X^k = \frac{X(1+X)}{(1-X)^3}$$

hence if we take partial sums…

$$\sum_{k=0}^{\infty} \binom{\text{sum of first}}{\text{k squares}} X^k = \frac{X(1+X)}{(1-X)^3} * \frac{1}{1-X}$$

Coefficient of X^k in $(X^2+X)/(1-X)^4$ is the sum of the first k squares:

$$\frac{X^2+X}{(1-X)^4} = \sum_{k=0}^{\infty}\left(\binom{k+2}{3}+\binom{k+1}{3}\right)X^k$$

$$\frac{1}{(1-X)^4} = \sum_{k=0}^{\infty}\binom{k+3}{3}X^k$$

So finally…

$$\sum_{i=0}^{i=n} i^2 = \binom{n+2}{3}+\binom{n+1}{3}$$

Finally, a different counting problem…

Let c_n = number of ways to pick exactly n fruits.

E.g., $c_5 = 6$

What is a closed form for c_n?

Recall Multiplication

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$

$B(X) = b_0 + b_1 X + b_2 X^2 + \ldots$

multiply them together

$(A*B)(X) = (a_0 b_0) + (a_0 b_1 + a_1 b_0) X$
$+ (a_0 b_2 + a_1 b_1 + a_2 b_0) X^2$
$+ (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) X^3$
$+ \ldots$

So if A(x), B(x), O(x) and P(x) are the generating functions for the number of ways to fill baskets using only one kind of fruit

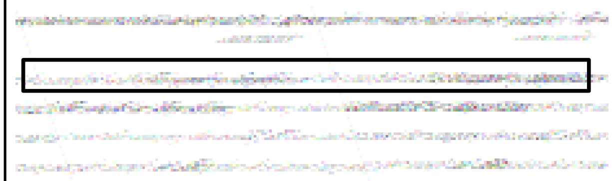
the generating function for number of ways to fill basket using any of these fruit is given by
C(x) = A(x)B(x)O(x)P(x)

Suppose we only pick bananas

b_n = number of ways to pick n fruits, only bananas.

<1,0,1,0,1,0,...>

$B(x) = 1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1-X^2}$



Suppose we only pick apples

a_n = number of ways to pick n fruits, only apples.

<1,0,0,0,0,1,...>

$A(x) = 1 + x^5 + x^{10} + x^{15} + \ldots = \frac{1}{1-X^5}$

Suppose we only pick oranges

o_n = number of ways to pick n fruits, only oranges.

<1,1,1,1,1,0,0,0,...>

$O(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1-X^5}{1-X}$

Suppose we only pick pears

p_n = number of ways to pick n fruits, only pears.

<1,1,0,0,0,0,0,...>

$P(x) = 1 + x = \frac{1-X^2}{1-X}$

Let c_n = number of ways to pick exactly n fruits of any type

$\sum c_n x^n = A(x)\, B(x)\, O(x)\, P(x)$

$= \frac{1}{1-X^5}\ \frac{1}{1-X^2}\ \frac{1-X^5}{1-X}\ \frac{1-X^2}{1-X} = \frac{1}{(1-X)^2}$

Let c_n = number of ways to pick exactly n fruits of any type

c_n is coefficient of X^n in $\frac{1}{(1-X)^2}$

<1,2,3,4,...>

$c_n = n+1.$

More operations: Differentiation

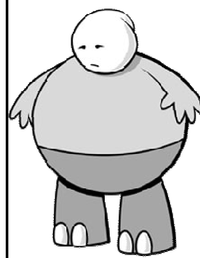
$$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$$

differentiate it…

$$A'(X) = a_1 + 2a_2 X + 3a_3 X^2 \ldots$$

$$A'(X) = \sum_{i=0}^{\infty} (i+1) a_{i+1} X^i$$

$$X A'(X) = \sum_{i=0}^{\infty} i a_i X^i$$



Here's What You Need to Know…

- Formal Power Series
- Basic operations on Formal Power Series
- Writing the generating function for a recurrence
  - binomial and multinomial coefficients
- Solving G.F. to get closed form
- G.F.s for common sequences
- Prefix sums using G.F.s
- Using G.F.s to solve counting problems