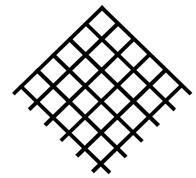
15-251

Great Theoretical Ideas in Computer Science

Counting II: Pigeons, Pirates and Pascal

Lecture 7 (September 15, 2009)



Addition Rule (2 Possibly Overlapping Sets)

Let A and B be two finite sets:

$$|A \cup B| =$$

$$|A| + |B| - |A \cap B|$$

Difference Method

To count the elements of a finite set S, find two sets A and B such that

$S = A \setminus B$

$S \cup B = A$

then $|S| = |A| - |B|$

Let $f : A \to B$ Be a Function From a Set A to a Set B

f is injective if and only if

$\forall x,y \in A,\ x \neq y \Rightarrow f(x) \neq f(y)$

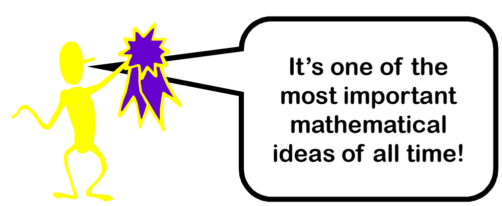
f is surjective if and only if

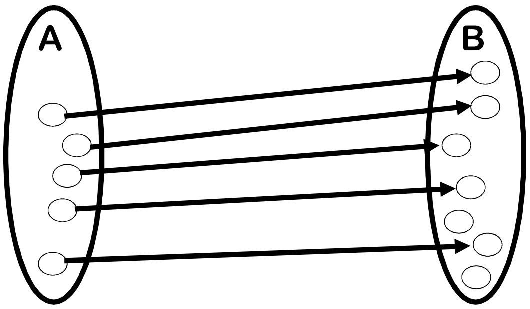
$\forall z \in B\ \exists x \in A\ \ f(x) = z$

f is bijective if f is both injective and surjective

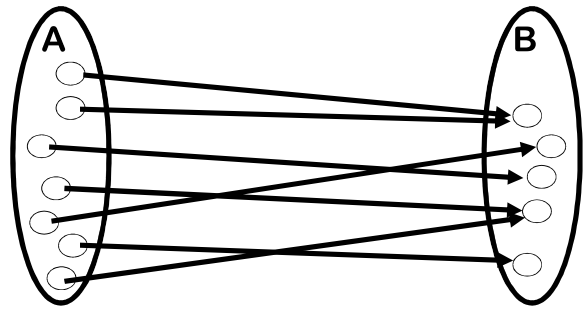
Correspondence Principle

If two finite sets can be placed into bijection, then they have the same size

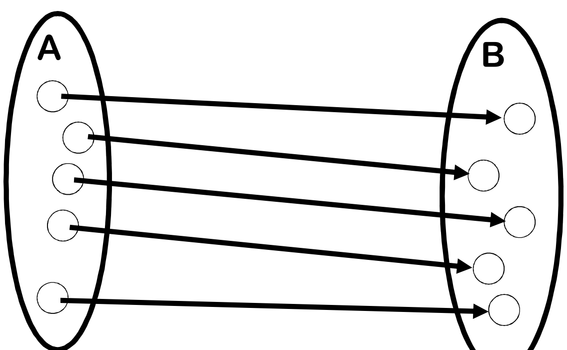




$\exists$ injective (1-1) f : A $\rightarrow$ B $\Rightarrow$ | A | $\leq$ | B |

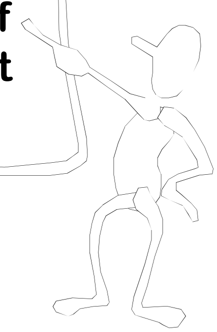


$\exists$ surjective (onto) f : A $\rightarrow$ B $\Rightarrow$ | A | $\geq$ | B |



$\exists$ bijective f : A $\rightarrow$ B $\Rightarrow$ | A | = | B |

The number of subsets of an n-element set is 2^n



Product Rule

Suppose every object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

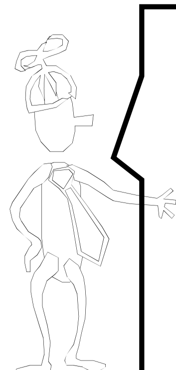
IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

THEN

There are $P_1P_2P_3 \ldots P_n$ objects of type S

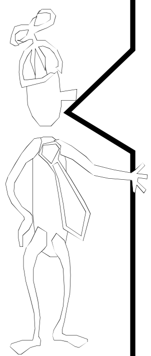


The three big mistakes people make in associating a choice tree with a set S are:

1. Creating objects not in S

2. Missing out some objects from the set S

3. Creating the same object two different ways



DEFENSIVE THINKING
ask yourself:

Am I creating objects of the right type?

Can I create every object of this type?

Can I reverse engineer my choice sequence from any given object?

Permutations vs. Combinations

$$\frac{n!}{(n-r)!} \qquad\qquad \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Ordered **Unordered**

Number of ways of ordering, permuting, or arranging r out of n objects

n choices for first place, n-1 choices for second place, . . .

$$n \times (n-1) \times (n-2) \times\ldots\times (n-(r-1))$$

$$= \frac{n!}{(n-r)!}$$

A combination or choice of r out of n objects is an (unordered) set of r of the n objects

The number of r combinations of n objects:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

n “choose” r

The Pigeonhole Principle

If there are n pigeons placed in n-1 holes then some pigeonhole contains at least two pigeons

also known the Dirichlet’s (box) principle

Example of how to use the pigeonhole principle…

At a party with n people, some handshaking took place.

Each pair shook hands at most once

Show that there exist two people who shook the same number of hands.

The number of shakes done by people lie in the set {0, 1, 2, …, n-1}

Claim: if someone shook n-1 hands, no one can have shaken 0 hands.

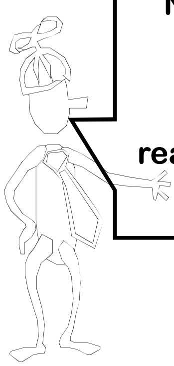
$\Rightarrow$ the number of shakes either all lie in
{0, 1, 2, …, n-2}
or {1, 2, …, n-1}

$\Rightarrow$ there are n people and n-1 possible values.

$\Rightarrow$ two people with the same number of shakes

The "Letterbox" Principle

If there are m letterboxes and n letters, there exists a letterbox with at least $\lceil n/m \rceil$ letters



Now, continuing on last week's theme…

How many ways to rearrange the letters in the word "SYSTEMS"?

SYSTEMS

7 places to put the Y,
6 places to put the T,
5 places to put the E,
4 places to put the M,
and the S's are forced

7 X 6 X 5 X 4 = 840

SYSTEMS

Let's pretend that the S's are distinct:

$S_1YS_2TEMS_3$

There are 7! permutations of $S_1YS_2TEMS_3$

But when we stop pretending we see that we have counted each arrangement of SYSTEMS 3! times, once for each of 3! rearrangements of $S_1S_2S_3$

$$\frac{7!}{3!} = 840$$

Arrange n symbols: r_1 of type 1, r_2 of type 2, …, r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n - r_1 - r_2 - \ldots - r_{k-1}}{r_k}$$

$$= \frac{n!}{(n-r_1)!r_1!}\;\frac{(n-r_1)!}{(n-r_1-r_2)!r_2!}\;\cdots$$

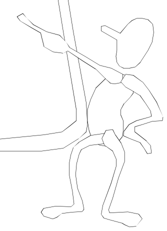
$$= \frac{n!}{r_1!r_2!\ldots r_k!}$$

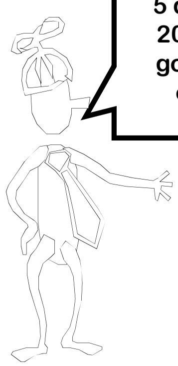
CARNEGIEMELLON

$$\frac{14!}{2!3!2!} = 3{,}632{,}428{,}800$$

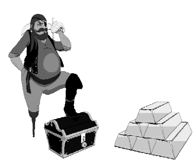
Remember:
The number of ways to arrange n symbols with r_1 of type 1, r_2 of type 2, …, r_k of type k is:

$$\frac{n!}{r_1!r_2! \dots r_k!}$$





5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?



Sequences with 20 G’s and 4 /’s

GG/G//GGGGGGGGGGGGGGGGG/

represents the following division among the pirates: 2, 1, 0, 17, 0

In general, the ith pirate gets the number of G’s after the i-1st / and before the ith /

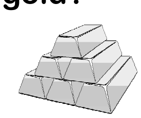
This gives a correspondence between divisions of the gold and sequences with 20 G’s and 4 /’s

How many different ways to divide up the loot?

Sequences with 20 G’s and 4 /’s

$$\binom{24}{4}$$

How many different ways can n distinct pirates divide k identical, indivisible bars of gold?



$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Think of x_k are being the number of gold bars that are allotted to pirate k

$$\binom{24}{4}$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + \ldots + x_n = k$$
$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + \ldots + x_n = k$$
$$x_1, x_2, x_3, \ldots, x_n \geq 1$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + \ldots + x_n = k$$
$$x_1, x_2, x_3, \ldots, x_n \geq 1$$

in bijection with solutions to

$$x_1 + x_2 + x_3 + \ldots + x_n = k\text{-}n$$
$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

Multisets

A multiset is a set of elements, each of which has a multiplicity

The size of the multiset is the sum of the multiplicities of all the elements

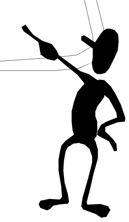
Example:
{X, Y, Z} with m(X)=0 m(Y)=3, m(Z)=2

Unary visualization: {Y, Y, Y, Z, Z}

Counting Multisets

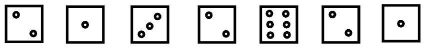
The number of ways to choose a multiset of size k from n types of elements is:

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$



Identical/Distinct Dice

Suppose that we roll seven dice



How many different outcomes are there, if order matters? 6^7

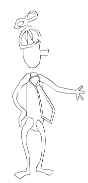
What if order doesn't matter? (E.g., Yahtzee) $\binom{12}{7}$

Remember to distinguish between Identical / Distinct Objects

If we are putting k objects into n distinct bins.

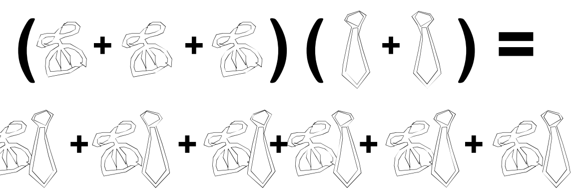
Objects are distinguishable	n^k
Objects are indistinguishable	$\binom{k+n-1}{k}$

On to Pascal…



Polynomials Express Choices and Outcomes

Products of Sum = Sums of Products



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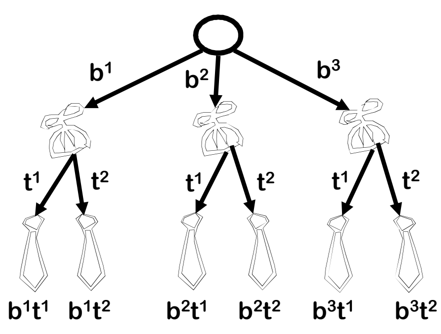
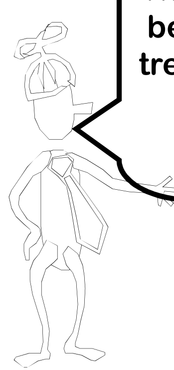
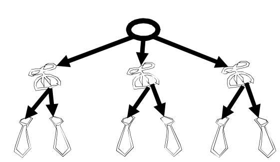


$(b^1+b^2+b^3)(t^1+t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2 + b^3t^1 + b^3t^2$

4

There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!





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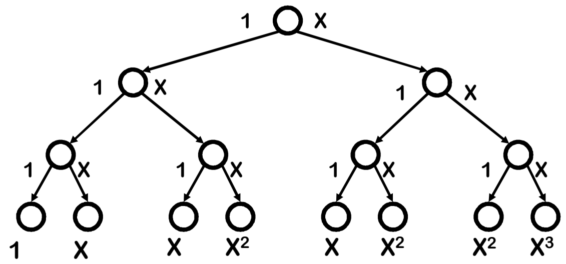
Choice Tree for Terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

4

The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

What is a Closed Form Expression For c_k?

$$(1+X)^n = c_0 + c_1X + c_2X^2 + \ldots + c_nX^n$$

$(1+X)(1+X)(1+X)(1+X)\ldots(1+X)$

After multiplying things out, but before combining like terms, we get 2^n cross terms, each corresponding to a path in the choice tree

c_k, the coefficient of X^k, is the number of paths with exactly k X's

$$c_k = \binom{n}{k}$$

The Binomial Formula

$$(1+X)^n = \binom{n}{0}X^0 + \binom{n}{1}X^1 + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

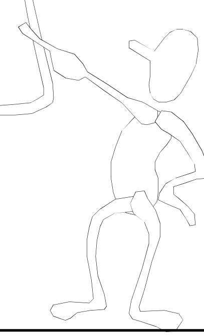
binomial expression

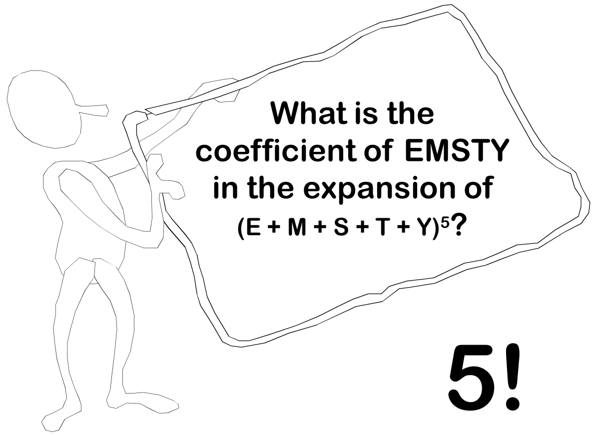
The Binomial Formula

$$(X+Y)^n = \binom{n}{0}X^nY^0 + \binom{n}{1}X^{n-1}Y^1 + \ldots + \binom{n}{k}X^{n-k}Y^k + \ldots + \binom{n}{n}X^0Y^n$$

The Binomial Formula

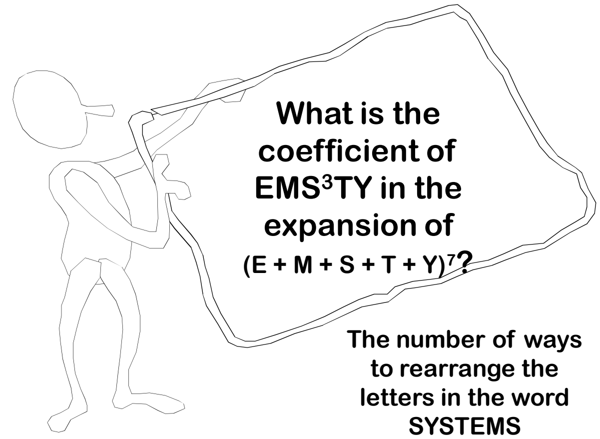
$$(X+Y)^n = \sum_{k=0}^{n}\binom{n}{k}X^{n-k}Y^k$$





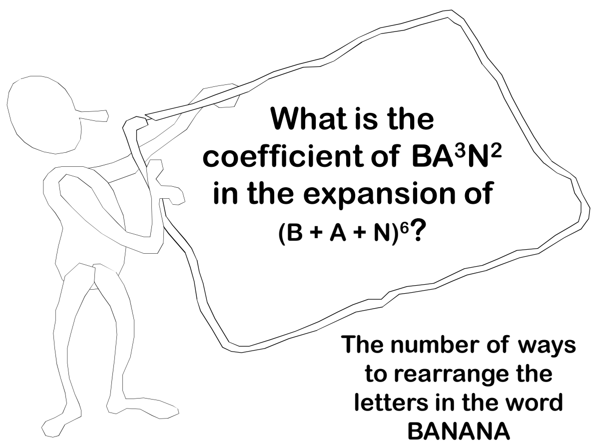
What is the coefficient of EMSTY in the expansion of $(E + M + S + T + Y)^5$?

5!



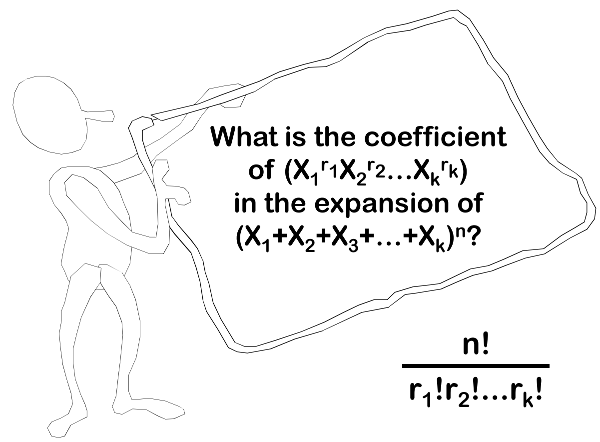
What is the coefficient of EMS^3TY in the expansion of $(E + M + S + T + Y)^7$?

The number of ways to rearrange the letters in the word SYSTEMS



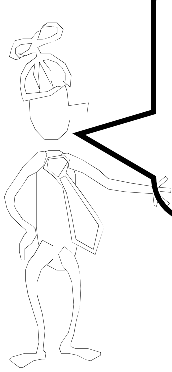
What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?

The number of ways to rearrange the letters in the word BANANA



What is the coefficient of $(X_1^{r_1}X_2^{r_2}\ldots X_k^{r_k})$ in the expansion of $(X_1+X_2+X_3+\ldots+X_k)^n$?

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$



Polynomials can encode counting questions in very non-trivial ways…

Power Series Representation

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

"Product form" or "Generating form"

$$= \sum_{k=0}^{\infty} \binom{n}{k} X^k$$

For k>n, $\binom{n}{k} = 0$

"Power Series" or "Taylor Series" Expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

Let x = 1, $$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

The number of subsets of an n-element set

By varying x, we can discover new identities:

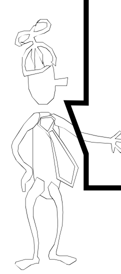
$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$

Let x = -1, $$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Equivalently, $$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$

The number of subsets with even size is the same as the number of subsets with odd size

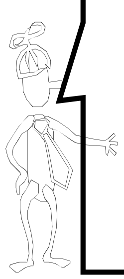
$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$



Proofs that work by manipulating algebraic forms are called “algebraic” arguments.

Proofs that build a bijection are called “combinatorial” arguments

$$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$



Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We gave an algebraic proof that

$|O_n| = |E_n|$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones

Let E_n be the set of binary strings of length n with an even number of ones

A combinatorial proof must construct a bijection between O_n and E_n

An Attempt at a Bijection

Let f_n be the function that takes an n-bit string and flips all its bits

f_n is clearly a one-to-one and onto function

for odd n. E.g. in f_7 we have:

0010011 → 1101100
1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100
101010 → 010101

Uh oh. Complementing maps evens to evens!

A Correspondence That Works for all n

Let f_n be the function that takes an n-bit string and flips only the first bit. For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

Another combinatorial proof

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

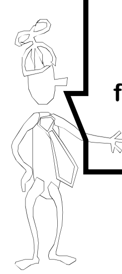
A = all possible subsets of size k out of n elements

B = all possible subsets of size k out of n elements that contain element 1

C = all possible subsets of size k out of n elements that do not contain element 1

|A| = |B| + |C|

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

The Binomial Formula

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle: k^{th} row are coefficients of $(1+X)^k$

Inductive definition of k^{th} entry of n^{th} row:
Pascal(n,0) = Pascal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)

"Pascal's Triangle"

$\binom{0}{0} = 1$

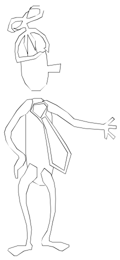
$\binom{1}{0} = 1$ $\binom{1}{1} = 1$

$\binom{2}{0} = 1$ $\binom{2}{1} = 2$ $\binom{2}{2} = 1$

$\binom{3}{0} = 1$ $\binom{3}{1} = 3$ $\binom{3}{2} = 3$ $\binom{3}{3} = 1$

Al-Karaji, Baghdad 953-1029
Chu Shin-Chieh 1303
Blaise Pascal 1654

Pascal's Triangle



"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand"

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Summing the Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1	= 1
1 + 1	= 2
1 + 2 + 1	= 4
1 + 3 + 3 + 1	= 8
1 + 4 + 6 + 4 + 1	= 16
1 + 5 + 10 + 10 + 5 + 1	= 32
1 + 6 + 15 + 20 + 15 + 6 + 1	= 64

Odds and Evens

$$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1 + 15 + 15 + 1 = 6 + 20 + 6

Summing on 1st Avenue

$$\sum_{i=1}^{n} i = \sum_{i=1}^{n} \binom{i}{1} = \binom{n+1}{2}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Summing on k^{th} Avenue

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Fibonacci Numbers

1
1 1 = 2
1 2 1 = 3
1 3 3 1 = 5
1 4 6 4 1 = 8
1 5 10 10 5 1 = 13
1 6 15 20 15 6 1

Sums of Squares

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

1

1 1

$\boxed{1^2 \quad 2^2 \quad 1^2}$

$\boxed{1^2 \quad 3^2 \quad 3^2 \quad 1^2}$

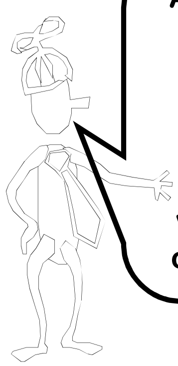
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

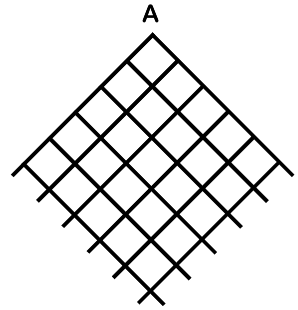
Al-Karaji Squares

1	
1 1	= 1
1 2 +2·1	= 4
1 3 +2·3 1	= 9
1 4 +2·6 4 1	= 16
1 5 +2·10 10 5 1	= 25
1 6 +2·15 20 15 6 1	= 36



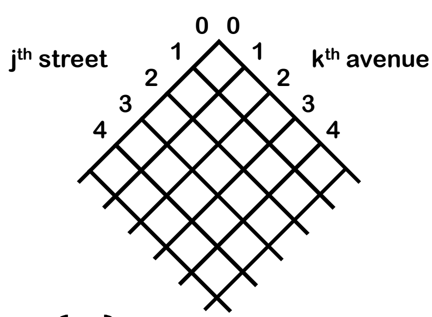
All these properties can be proved inductively and algebraically. We will give *combinatorial* proofs using the Manhattan block walking representation of binomial coefficients

How many shortest routes from A to B?



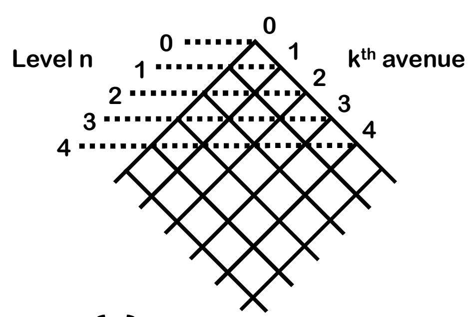
$$\binom{10}{5}$$

Manhattan



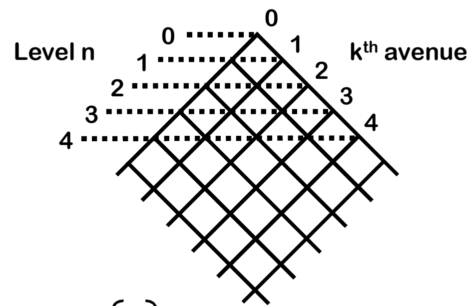
There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k)

Manhattan

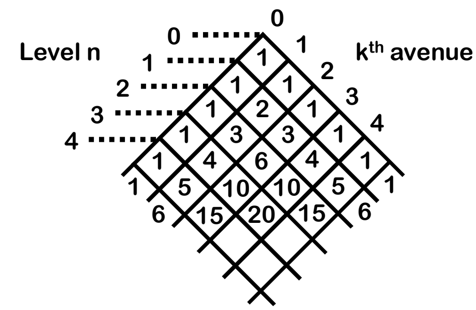


There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k)

Manhattan



There are $\binom{n}{k}$ shortest routes from (0,0) to level n and k^{th} avenue



Level n $\qquad$ k^{th} avenue

0 ....

1 1

2 .... 1 1

3 .... 1 2 1

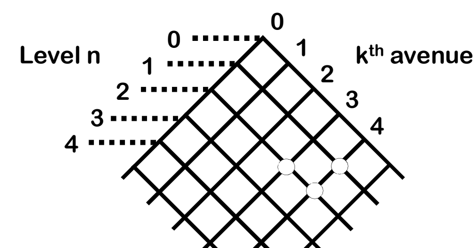
4 .... 1 3 3 1

1 4 6 4 1

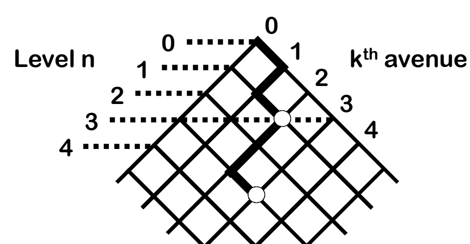
1 5 10 + 10 5 1

6 15 20 15 6

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

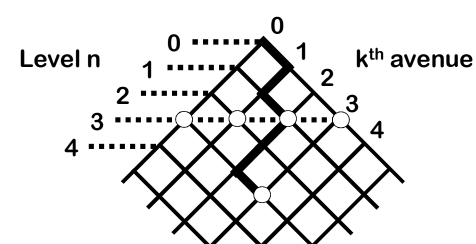


$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

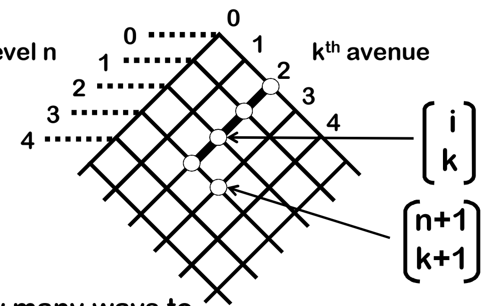


How many ways to get to ○ via ○ ?

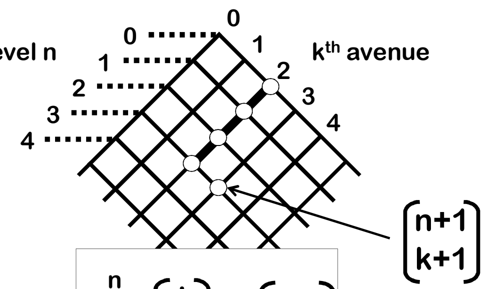
$$\binom{n}{k}^2$$



$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

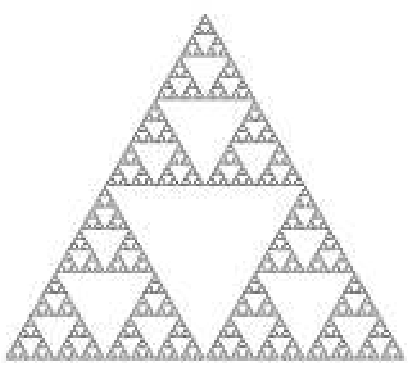


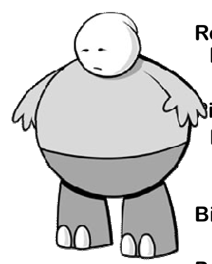
How many ways to get to ○ via ○ ?



$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Pascal Mod 2





Here's What You Need to Know…

Pigeonhole Principle

Rearranging words
binomial and multinomial coefficients

irates and Gold
How many integer solutions to
$x_1 + x_2 + \ldots + x_k = n$

Binomial Formula

Pascal's triangle
and some of its properties
Combinatorial and Algebraic proofs
manhattan representation of Pascal's
path-counting proofs