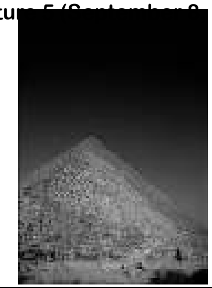
15-251

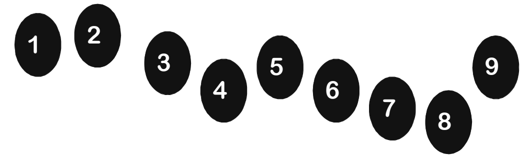
Great Theoretical Ideas in Computer Science

Ancient Wisdom: Unary and Binary

Lectu 2009)



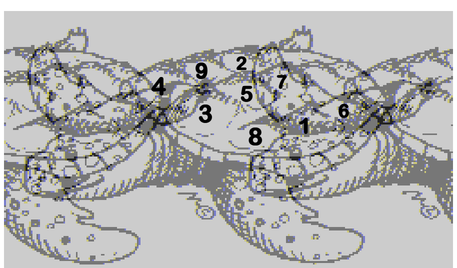
How to play the 9 stone game?



9 stones, numbered 1-9
Two players alternate moves.
Each move a player gets to take a new stone

Any subset of 3 stones adding to 15, wins.

Magic Square: Brought to humanity on the back of a tortoise from the river Lo in the days of Emperor Yu in ancient China



Magic Square: Any 3 in a vertical, horizontal, or diagonal line add up to 15.

4	9	2
3	5	7
8	1	6

Conversely, any 3 that add to 15 must be on a line.

4	9	2
3	5	7
8	1	6

TIC-TAC-TOE on a Magic Square Represents The Nine Stone Game

Alternate taking squares 1-9.
Get 3 in a row to win.

4	9	2
3	5	7
8	1	6

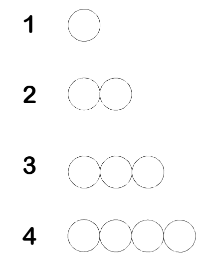
Basic Idea of this Lecture

Don't stick with the representation in which you encounter problems!

Always seek the more useful one!

This idea requires a lot of practice

Prehistoric Unary



Consider the problem of finding a formula for the sum of the first n numbers

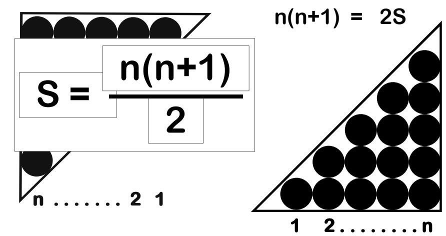
You already used induction to verify that the answer is ½n(n+1)

$$1 + 2 + 3 + \dots + n\text{-}1 + n = S$$
$$n + n\text{-}1 + n\text{-}2 + \dots + 2 + 1 = S$$
$$n\text{+}1 + n\text{+}1 + n\text{+}1 + \dots + n\text{+}1 + n\text{+}1 = 2S$$
$$n(n+1) = 2S$$

$$S = \frac{n(n+1)}{2}$$

$$1 + 2 + 3 + \dots + n\text{-}1 + n = S$$
$$n + n\text{-}1 + n\text{-}2 + \dots + 2 + 1 = S$$

$$n(n+1) = 2S$$

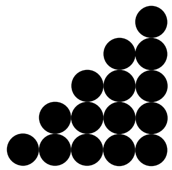


$$S = \frac{n(n+1)}{2}$$

n^{th} Triangular Number

$$\Delta_n = 1 + 2 + 3 + \ldots + n-1 + n$$

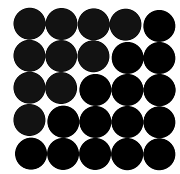
$$= n(n+1)/2$$

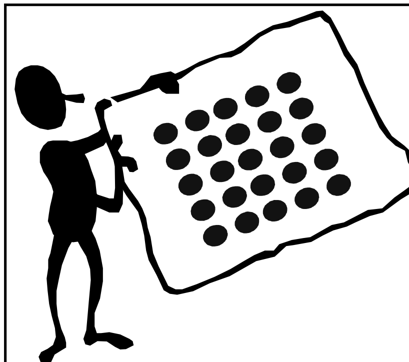


n^{th} Square Number

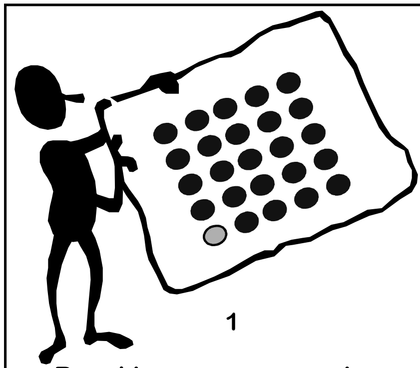
$$\square_n = n^2$$

$$= \Delta_n + \Delta_{n-1}$$





Breaking a square up in a new way



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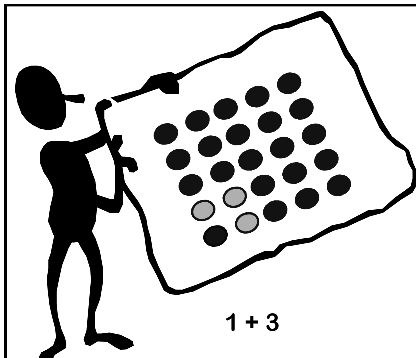
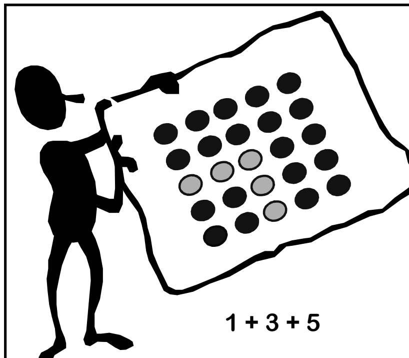
Breaking a square up in a new way



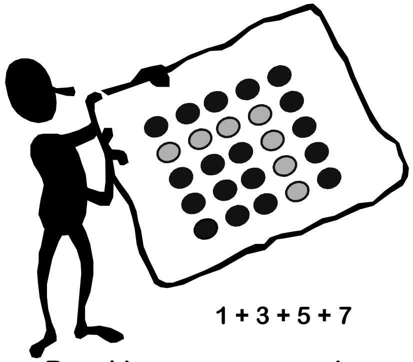
1 + 3

Breaking a square up in a new way



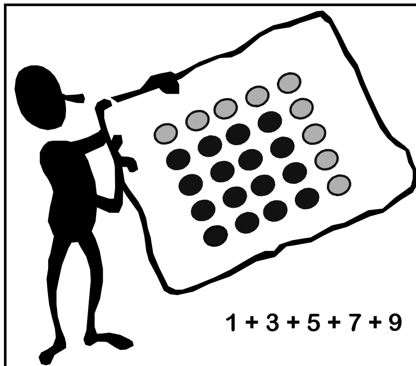
1 + 3 + 5

Breaking a square up in a new way



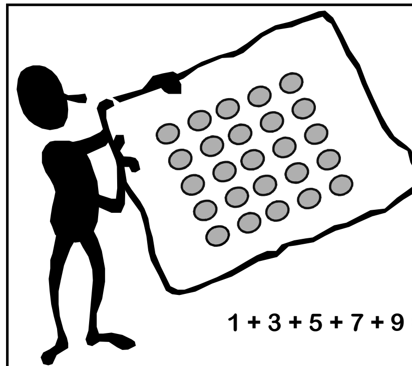
$1 + 3 + 5 + 7$

Breaking a square up in a new way



$1 + 3 + 5 + 7 + 9$

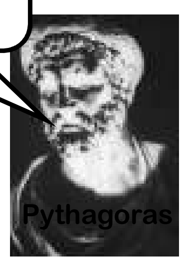
Breaking a square up in a new way



$1 + 3 + 5 + 7 + 9 = 5^2$

Breaking a square up in a new way

The sum of the first n odd numbers is n^2

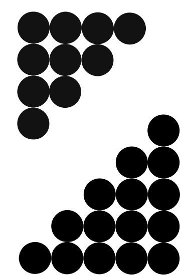


Here is an alternative dot proof of the same sum….

n^{th} Square Number

$$\square_n = \Delta_n + \Delta_{n-1}$$

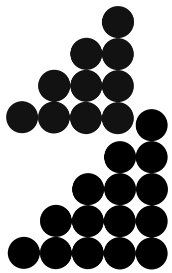
$$= n^2$$



n^{th} Square Number

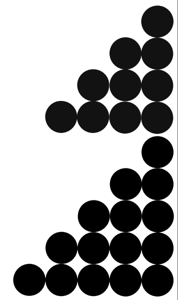
$\square_n = \Delta_n + \Delta_{n-1}$

$= n^2$



n^{th} Square Number

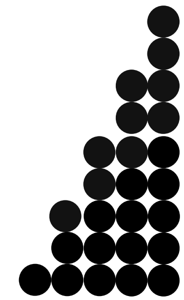
$\square_n = \Delta_n + \Delta_{n-1}$



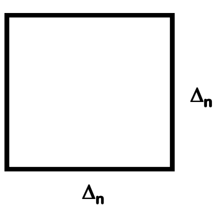
n^{th} Square Number

$\square_n = \Delta_n + \Delta_{n-1}$

= Sum of first n odd numbers



Area of square $= (\Delta_n)^2$



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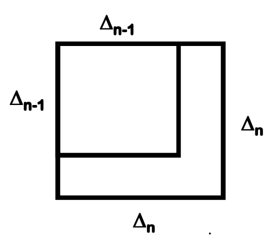

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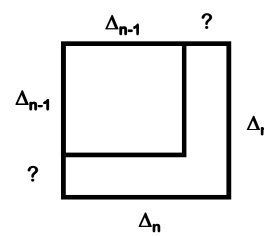
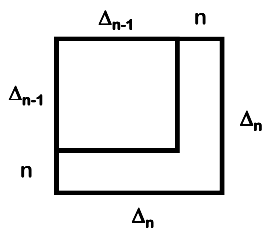
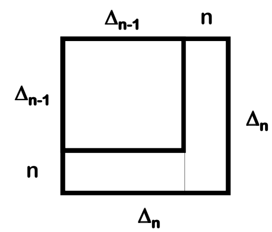


Area of square $= (\Delta_n)^2$



Area of square $= (\Delta_n)^2$



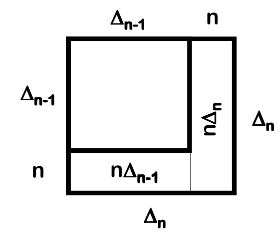
Area of square $= (\Delta_n)^2$

$= (\Delta_{n-1})^2 + n\Delta_{n-1} + n\Delta_n$

$= (\Delta_{n-1})^2 + n(\Delta_{n-1} + \Delta_n)$

$= (\Delta_{n-1})^2 + n(\square_n)$

$= (\Delta_{n-1})^2 + n^3$



$(\Delta_n)^2 = n^3 + (\Delta_{n-1})^2$

$= n^3 + (n-1)^3 + (\Delta_{n-2})^2$

$= n^3 + (n-1)^3 + (n-2)^3 + (\Delta_{n-3})^2$

$= n^3 + (n-1)^3 + (n-2)^3 + \ldots + 1^3$

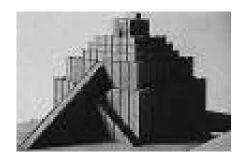
$(\Delta_n)^2 \quad = 1^3 + 2^3 + 3^3 + \ldots + n^3$

$= [\, n(n+1)/2 \,]^2$

$\Delta_n = 1^1 + 2^1 + \cdots + n^1$

Can you find a formula for the sum of the first n squares?

$\frac{n(n+1)(2n+1)}{6}$



Babylonians needed this sum to compute the number of blocks in their pyramids

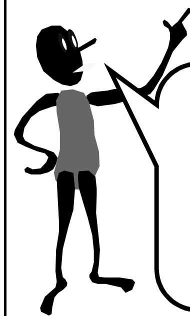
Rhind Papyrus

Scribe Ahmes was Martin Gardener of his day!

A man has 7 houses,
Each house contains 7 cats,
Each cat has killed 7 mice,
Each mouse had eaten 7 ears of spelt,
Each ear had 7 grains on it.
What is the total of all of these?

Sum of powers of 7

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$



We'll use this fundamental sum again and again:

The Geometric Series

A Frequently Arising Calculation

$$(X-1)\,(1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1})$$

$$= X + X^2 + X^3 \cdots + X^{n-1} + X^n$$
$$-1 - X - X^2 - X^3 \cdots - X^{n-1}$$

$$-1 \qquad + X^n \qquad = X^n - 1$$

dividing by $X-1$ (if $X \neq 1$)

$$1 + X + X^2 \cdots + X^{n-1} = \frac{X^n - 1}{X - 1}$$

A Frequently Arising Calculation

$$(X-1)\,(1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1})$$

$$= \quad X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n$$
$$- \; 1 - X^1 - X^2 - X^3 - \ldots - X^{n-2} - X^{n-1}$$

$$= X^n - 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$
(when $x \neq 1$)

Geometric Series for X=2

$$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$
(when $x \neq 1$)

Geometric Series for X=½

$$1 + ½ + ½^2 + ½^3 + \ldots + ½^{n-1} = \frac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1}$$

$$= 2\left(1 - (\tfrac{1}{2})^n\right)$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$
(when $x \neq 1$)

A Similar Sum

$a^n + a^{n-1}b^1 + a^{n-2}b^2 + + \ldots + a^1b^{n-1} + b^n$

$$\sout{(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n}$$

$$a^n\left(1 + \frac{b}{a} + \left(\frac{b}{a}\right)^2 + \cdots + \left(\frac{b}{a}\right)^n\right)$$

$$= a^n\left(\frac{\left(\frac{b}{a}\right)^{n+1} - 1}{b/a - 1}\right) = \frac{b^{n+1} - a^{n+1}}{b - a}$$

A slightly different one

$S = 0.2^0 + 1.2^1 + 2.2^2 + 3.2^3 + \ldots + n2^n = ?$

$$-(S = 0.2^0 + 1.2^1 + 2.2^2 + \ldots + n2^n)$$

$$2S = 0.2^1 + 1.2^2 + \cdots + (n-1)2^n + n2^{n+1}$$

$$S = -0.2^0 - 1.2^1 - 1.2^2 \cdots - 1.2^n + n.2^{n+1}$$

$$= -(2^1 + 2^2 + \cdots + 2^n) + n.2^{n+1}$$

$$= -(2^{n+1} - 1 - 1) + n.2^{n+1}$$

$$= n.2^{n+1} - 2^{n+1} + 2 = (n-1)2^{n+1} + 2.$$

Two Case Studies

$$-(S = 0^2.2^0 + 1^2.2^1 + 2^2.2^2 \quad \cdots + n^2.2^n)$$

$$2S = 0^2.2^1 + 1^2.2^2 \quad \cdots + (n-1)^2.2^n + n^2.2^{n+1}$$

$$S = -(1^2.2^1 + (1^2 - 2^2).2^2 + \cdots + ((n-1)^2 - n^2)2^n) + n^2.2^{n+1}$$

Bases and Representation

$$= n^2.2^{n+1} - \sum_{j=1}^{n}(2j-1)2^j$$

BASE X Representation

$S = a_{n-1}\, a_{n-2} \ldots a_1\, a_0$ represents the number:

$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0$

$9321 = 9.10^3 + 3.10^2 + 2.10^1 + 7.10^0$

Base 2 [Binary Notation]

101 represents: $1\,(2)^2 + 0\,(2^1) + 1\,(2^0)$

= ○○○○○

Base 7

015 represents: $0\,(7)^2 + 1\,(7^1) + 5\,(7^0)$

= ○○○○○○○○○○○○

Bases In Different Cultures

- Sumerian-Babylonian: 10, 60, 360
- Egyptians: 3, 7, 10, 60
- Maya: 20
- Africans: 5, 10
- French: 10, 20
- English: 10, 12, 20

BASE X Representation

$S = (a_{n-1}\, a_{n-2} \ldots a_1\, a_0)_X$ represents the number:

$a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0$

Largest number representable in base-X with n "digits"

$= (X\text{-}1\ X\text{-}1\ X\text{-}1\ X\text{-}1\ X\text{-}1 \ldots X\text{-}1)_X$

$= (X-1)(X^{n-1} + X^{n-2} + \ldots + X^0)$

$= (X^n - 1)$

Fundamental Theorem For Binary

Each of the numbers from 0 to 2^n-1is uniquely represented by an n-bit number in binary

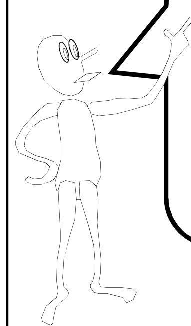
k uses $\lfloor \log_2 k \rfloor$ + 1 digits in base 2

$= \lceil \log_2(k+1) \rceil$

Fundamental Theorem For Base-X

Each of the numbers from 0 to X^n-1 is uniquely represented by an n-“digit” number in base X

k uses $\lfloor \log_X k \rfloor$ + 1 digits in base X



n has length n in unary, but has length $\lfloor \log_2 n \rfloor$ + 1 in binary

Unary is exponentially longer than binary

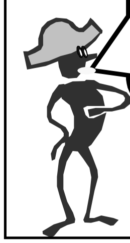
Other Representations: Egyptian Base 3

Conventional Base 3:
Each digit can be 0, 1, or 2

Here is a strange new one:
Egyptian Base 3 uses -1, 0, 1

Example: $(1\ \text{-}1\ \text{-}1)_{EB3}$ = 9 - 3 - 1 = 5

We can prove a unique representation theorem



How could this be Egyptian? Historically, negative numbers first appear in the writings of the Hindu mathematician Brahmagupta (628 AD)

One weight for each power of 3
Left = “negative”. Right = “positive”

Two Case Studies

Bases and Representation

Solving Recurrences
using a good representation

Example

$T(1) = 1$
$T(n) = 4T(n/2) + n$

Notice that T(n) is inductively defined only for positive powers of 2, and undefined on other values

$T(1) = 1$ $T(2) = 6$ $T(4) = 28$ $T(8) = 120$

Give a closed-form formula for T(n)

Technique 1
Guess Answer, Verify by Induction

$T(1) = 1,\ T(n) = 4\,T(n/2) + n$

Base Case: $G(1) = 1$ and $T(1) = 1$
Induction Hypothesis: $T(x) = G(x)$ for $x < n$
Hence: $T(n/2) = G(n/2) = 2(n/2)^2 - n/2$

$T(n) = 4\,T(n/2) + n$
$= 4\,G(n/2) + n$
$= 4\,[2(n/2)^2 - n/2] + n$
$= 2n^2 - 2n + n$
$= 2n^2 - n = G(n)$

Guess:
$G(n) = 2n^2 - n$

Technique 2
Guess Form, Calculate Coefficients

$T(1) = 1,\ T(n) = 4\,T(n/2) + n$

Guess: $T(n) = an^2 + bn + c$
for some a,b,c

Calculate: $T(1) = 1$, so $a + b + c = 1$

$T(n) = 4\,T(n/2) + n$

$an^2 + bn + c = 4\,[a(n/2)^2 + b(n/2) + c] + n$
$= an^2 + 2bn + 4c + n$

$(b+1)n + 3c = 0$

Therefore: $b = -1$ $c = 0$ $a = 2$

Technique 3
The Recursion Tree Approach

$T(1) = 1,\ T(n) = 4\,T(n/2) + n$

A slight variation

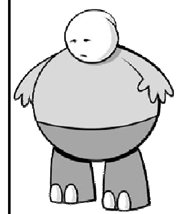
$T(1) = 1,\ T(n) = 4\,T(n/2) + n^2$

How about this one?

$T(1) = 1, T(n) = 3\,T(n/2) + n$

… and this one?

$T(1) = 1, T(n) = T(n/4) + T(n/2) + n$



Here's What You Need to Know…

Unary and Binary
- Triangular Numbers
- Dot proofs

$(1+x+x^2 + \ldots + x^{n-1}) = (x^n - 1)/(x-1)$

Base-X representations

k uses $\lfloor \log_2 k \rfloor + 1 = \lceil \log_2 (k+1) \rceil$ digits in base 2

Solving Simple Recurrences

Bhaskara's "proof" of Pythagoras' theorem