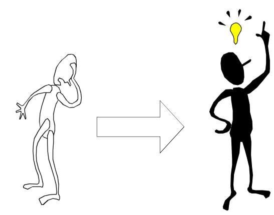
15-251

Great Theoretical Ideas in Computer Science

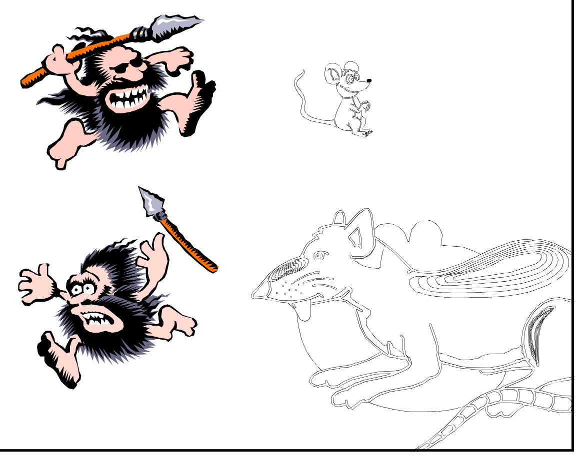
Bits of Wisdom on Solving Problems, Writing Proofs, and Enjoying the Pain: How to Succeed in This Class

Lecture 4 (September 3, 2009)



What did our brains evolve to do?

What were our brains designed to do?



Our brains probably did not evolve to do math!

Over the last 30,000 years, our brains have essentially stayed the same!

The human mind was designed by evolution to deal with foraging in small bands on the African Savannah . . . faulting our minds for succumbing to games of chance is like complaining that our wrists are poorly designed for getting out of handcuffs

Steven Pinker
"How the Mind Works"

Our brains can perform simple, concrete tasks very well

And that's how math is best approached!

Substitute concrete values for the variables: x=0, x=100, …

Draw simple pictures

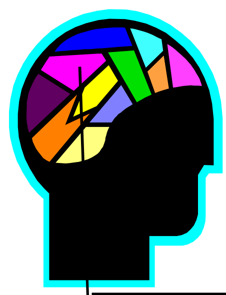
Try out small examples of the problem: What happens for n=1? n=2?

"I don't have any magical ability…I look at the problem, and it looks like one I've already done. When nothing's working out, then I think of a small trick that makes it a little better. I play with the problem, and after a while, I figure out what's going on."

Terry Tao (Fields Medalist, considered to be the best problem solver in the world)

Novice

Expert

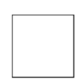


The better the problem solver, the less brain activity is evident. The real masters show almost no brain activity!

Simple and to the point

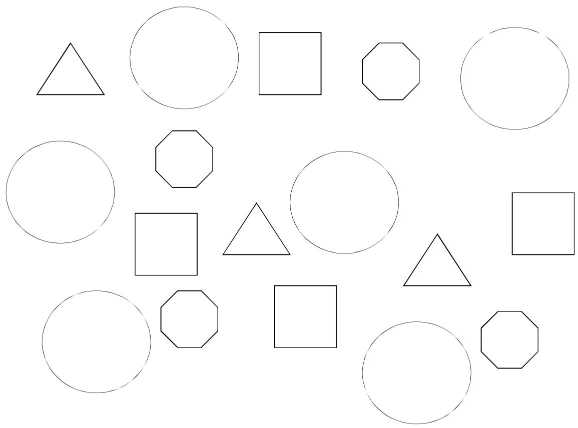
Use a lot of paper, or a board!!!

Quick Test...

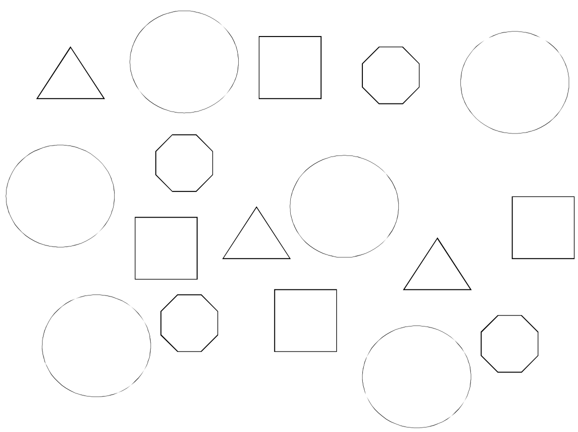


Count the green squares

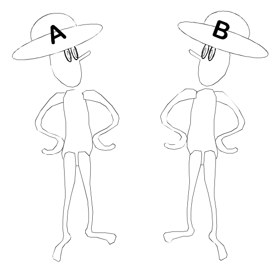
(you will have three seconds)



How many were there?



Hats with Consecutive Numbers

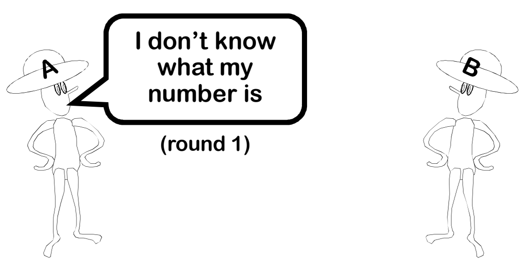


Alice Bob

$|A - B| = 1$

Alice starts: …

Hats with Consecutive Numbers

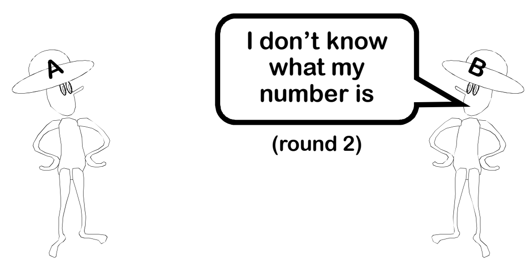


Alice Bob

$|A - B| = 1$ and $A, B > 0$

Alice starts: …

Hats with Consecutive Numbers



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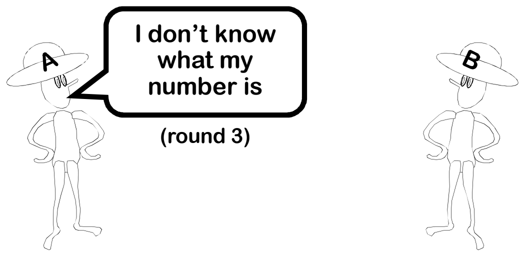
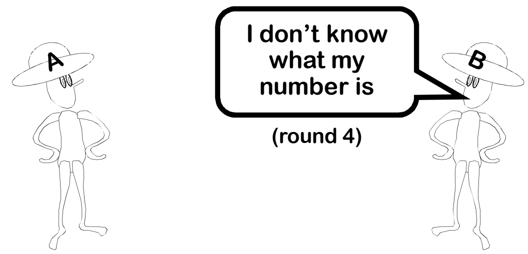


Alice Bob

| A - B | = 1 and A, B > 0

Alice starts: ...

Hats with Consecutive Numbers



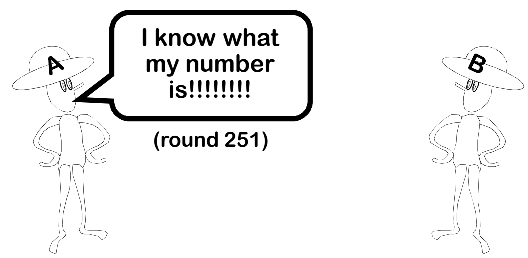
Alice Bob

| A - B | = 1 and A, B > 0

Alice starts: ...

• • •

Hats with Consecutive Numbers

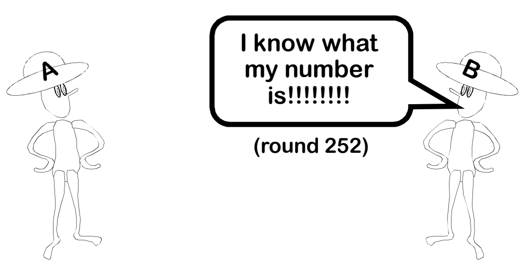


Alice Bob

| A - B | = 1 and A, B > 0

Alice starts: ...

Hats with Consecutive Numbers



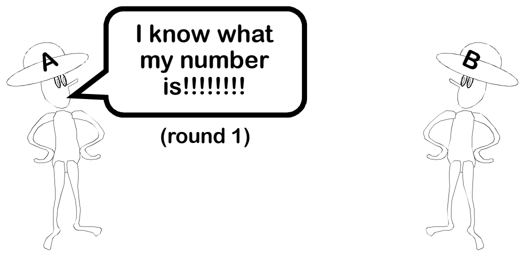
Alice Bob

| A - B | = 1 and A, B > 0

Alice starts: ...

Question: What are Alice and Bob's numbers?

Imagine Alice Knew Right Away

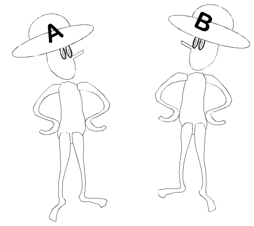


Alice Bob

| A - B | = 1 and A, B > 0

Then A = 2 and B = 1

1,2	N,Y
2,1	Y
2,3	N,Y
3,2	N,N,Y
3,4	N,N,N,Y
4,3	N,N,Y
4,5	N,N,N,Y



Inductive Claim

Claim: After 2k NOs, Alice knows that her number is at least 2k+1.

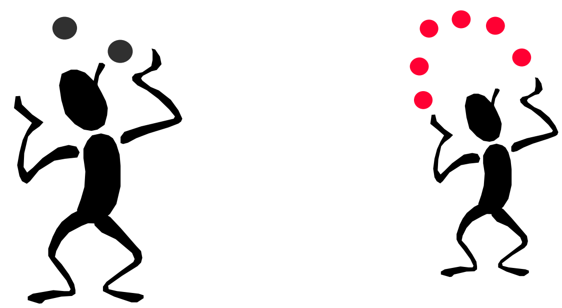
After 2k+1 NOs, Bob knows that his number is at least 2k+2.

Hence, after 250 NOs, Alice knows her number is at least 251. If she says YES, her number is at most 252.

If Bob's number is 250, her number must be 251. If his number is 251, her number must be 252.

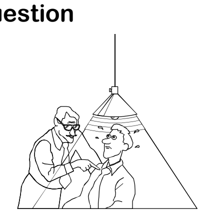
Exemplification:

Try out a problem or solution on small examples. Look for the patterns.

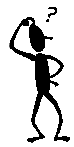


Relax

I am just going to ask you a Microsoft interview question



Four guys want to cross a bridge that can only hold two people at one time. It is pitch dark and they only have one flashlight, so people must cross either alone or in pairs (bringing the flashlight). Their walking speeds allow them to cross in 1, 2, 5, and 10 minutes, respectively. Is it possible for them to all cross in 17 minutes?

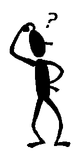


Get The Problem Right!

Given any context you should double check that you read/heard it correctly!

You should be able to repeat the problem back to the source and have them agree that you understand the issue

Four guys want to cross a bridge that can only hold two people at one time. It is pitch dark and they only have one flashlight, so people must cross either alone or in pairs (bringing the flashlight). Their walking speeds allow them to cross in 1, 2, 5, and 10 minutes, respectively. Is it possible for them to all cross in 17 minutes?



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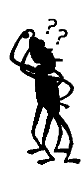


Intuitive, But False

“10 + 1 + 5 + 1+ 2 = 19, so the four guys just can’t cross in 17 minutes”

“Even if the fastest guy is the one to shuttle the others back and forth – you use at least 10 + 1 + 5 + 1 + 2 > 17 minutes”

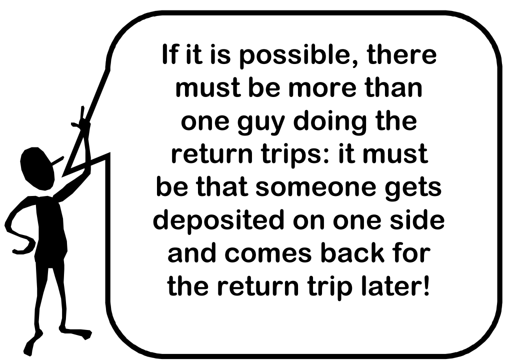
Vocabulary Self-Proofing

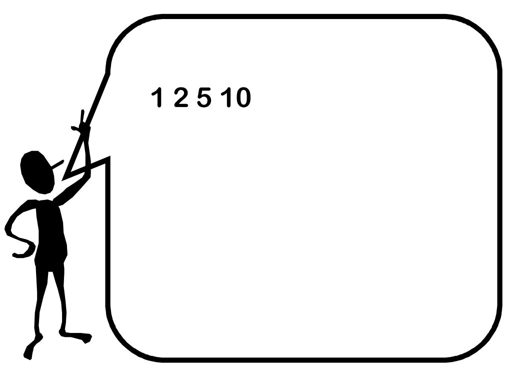
As you talk to yourself, make sure to tag assertions with phrases that denote degrees of conviction

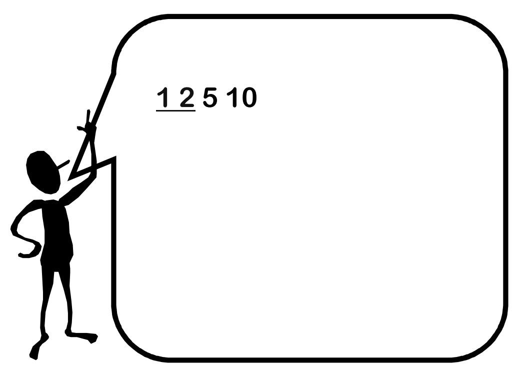
Keep track of what you actually know – remember what you merely suspect

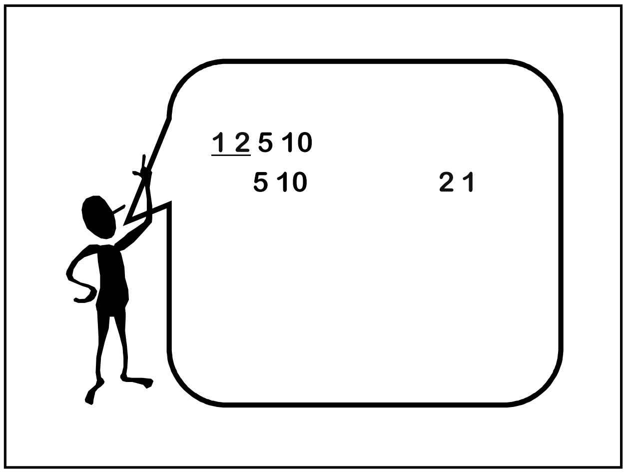
"10 + 1 + 5 + 1 + 2 = 19, so it would be weird if the four guys could cross in 17 minutes"

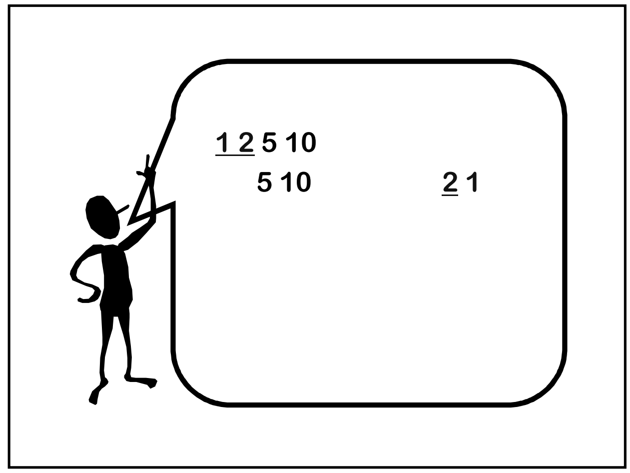
"~~even~~ if we use the fastest guy to shuttle the others, they take too long."

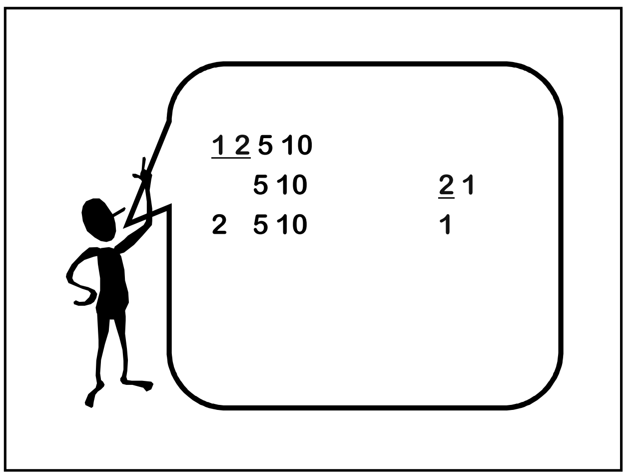


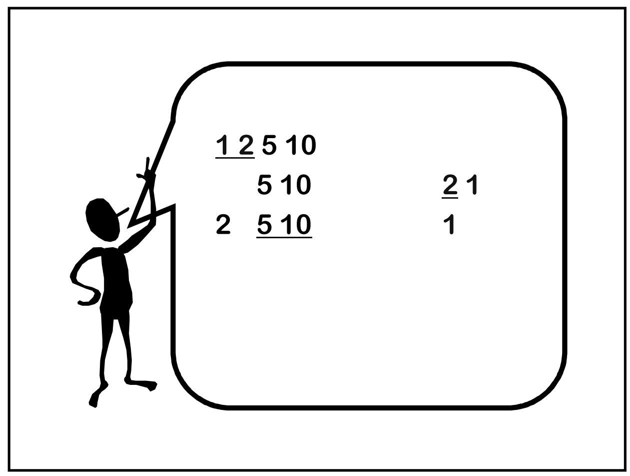


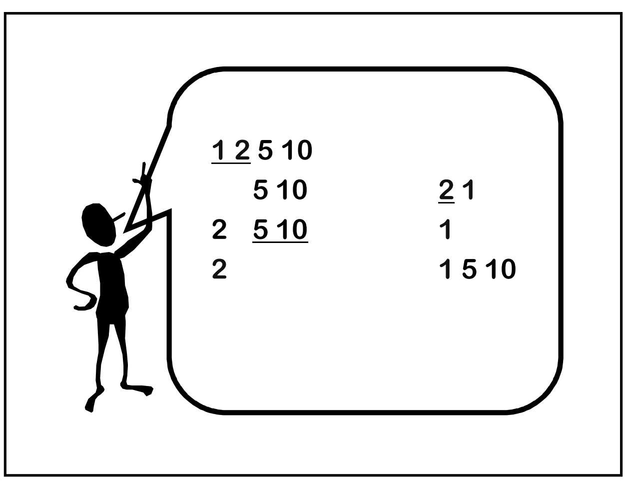


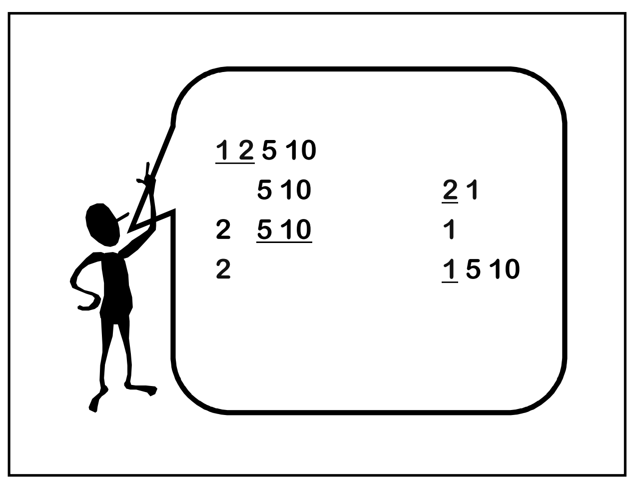

1 2 5 10
5 10 2 1

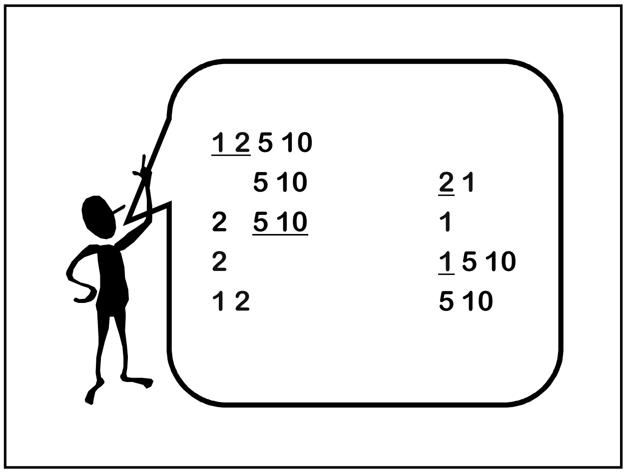

1 2 5 10
5 10 2 1

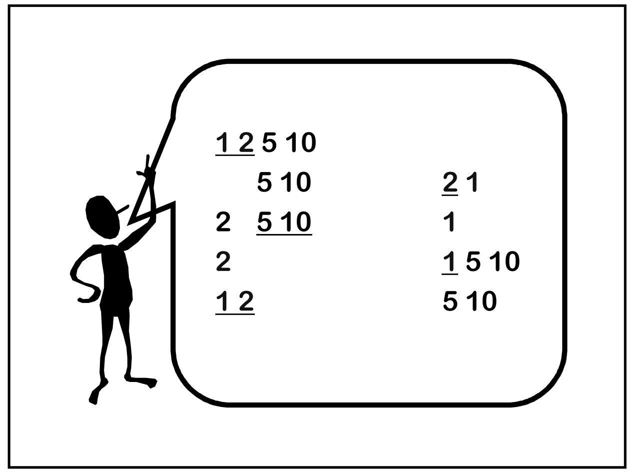

1 2 5 10
5 10 2 1
2 5 10 1

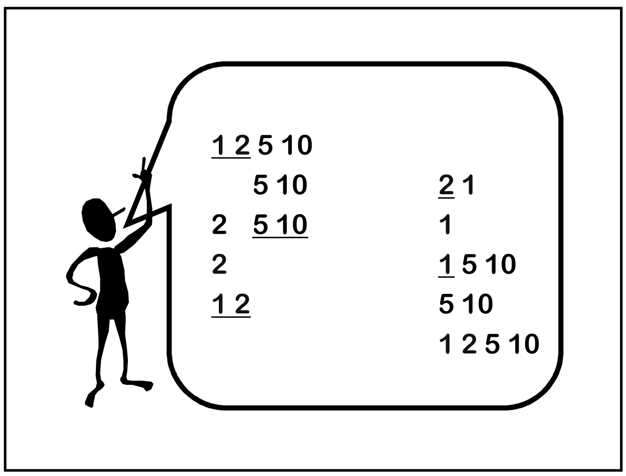

1 2 5 10
5 10 2 1
2 5 10 1

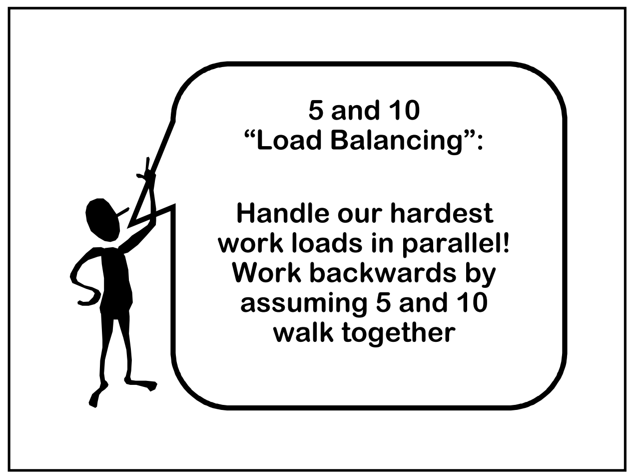

1 2 5 10
5 10 2 1
2 5 10 1
2 1 5 10

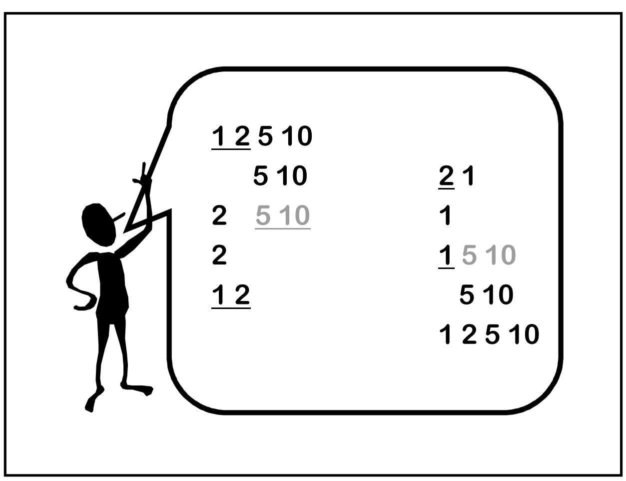

1 2 5 10
5 10 2 1
2 5 10 1
2 1 5 10


1 2 5 10
5 10
2 5 10
2
1 2
2 1
1
1 5 10
5 10


1 2 5 10
5 10
2 5 10
2
1 2
2 1
1
1 5 10
5 10

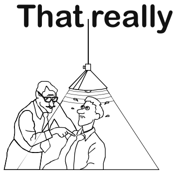

1 2 5 10
5 10
2 5 10
2
1 2
2 1
1
1 5 10
5 10
1 2 5 10


5 and 10
"Load Balancing":
Handle our hardest
work loads in parallel!
Work backwards by
assuming 5 and 10
walk together


1 2 5 10
5 10
2 5 10
2
1 2
2 1
1
1 5 10
5 10
1 2 5 10


Words To The Wise
Keep It Simple
Don't Fool Yourself

That really was a Microsoft question

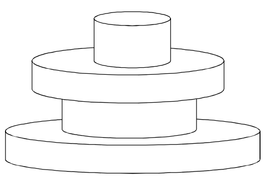


Why do you think that they ask such questions, as opposed to asking for a piece of code to do binary search?

The future belongs to the computer scientist who has

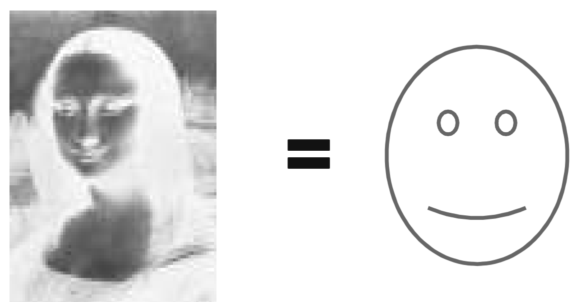
- Content: An up to date grasp of fundamental problems and solutions
- Method: Principles and techniques to solve the vast array of unfamiliar problems that arise in a rapidly changing field

Representation:
Understand the relationship between different representations of the same information or idea

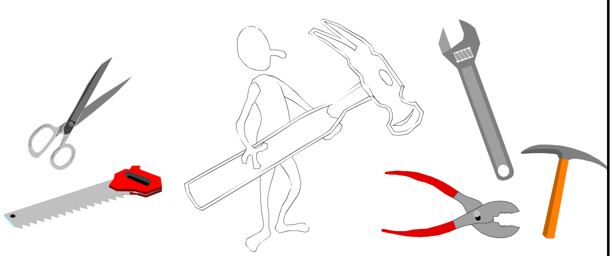


1
3
2
4

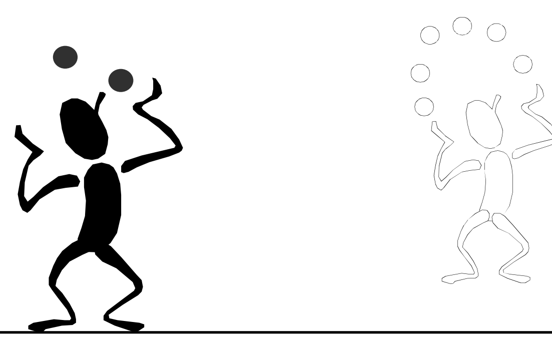
Abstraction:
Abstract away the inessential features of a problem

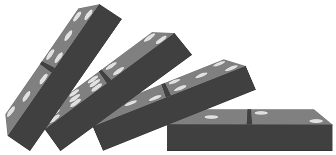


Toolkit:
Name abstract objects and ideas, and put them in your toolkit. Know their advantages and limitations.



Exemplification:
Try out a problem or solution on small examples. Look for the patterns.



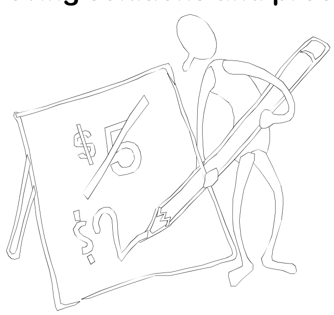


<u>Induction</u> has many guises.
Master their interrelationship.

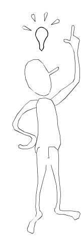
- Formal Arguments
- Invariants
- Recursion
- Recurrences

Modularity:
Decompose a complex problem into simpler sub-problems

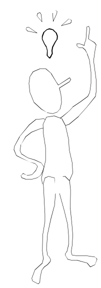
Improvement:
The best solution comes from a process of repeatedly refining and improving solutions and proofs.



Bracketing:
What are the best lower and upper bounds that I can prove?



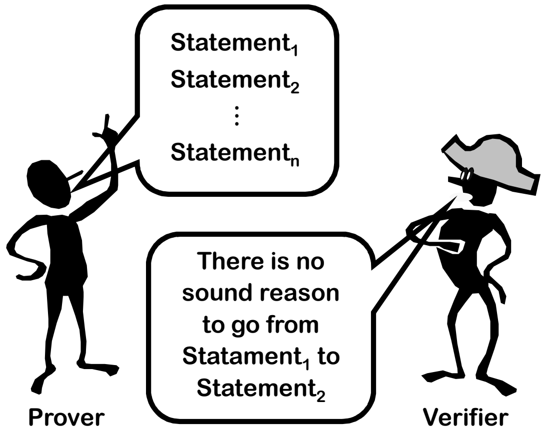
$$\leq f(x) \leq$$



In this course you will have to write a lot of proofs!

Think of Yourself as a (Logical) Lawyer

Your arguments should have no holes, because the opposing lawyer will expose them



The verifier is very thorough, (he can catch all your mistakes), but he will not supply missing details of a proof

A valid complaint on his part is: I don't understand

The verifier is similar to a computer running a program that you wrote!

Writing Proofs Is A Lot Like Writing Programs

You have to write the correct sequence of statements to satisfy the verifier

Errors than can occur with a program and with a proof!

- Syntax error
- Undefined term
- Infinite Loop
- Output is not quite what was needed

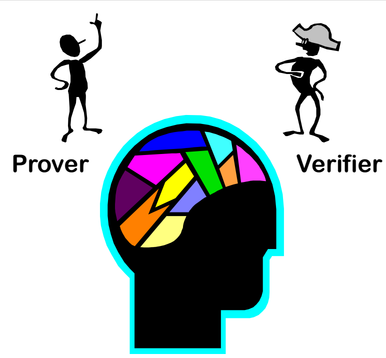
Good code is well-commented and written in a way that is easy for other humans (and yourself) to understand

Similarly, good proofs should be easy to understand. Although the formal proof does not require certain explanatory sentences (e.g., "the idea of this proof is basically X"), good proofs usually do

Writing Proofs is Even Harder than Writing Programs

The proof verifier will not accept a proof unless every step is justified!

It's as if a compiler required your programs to have every line commented (using a special syntax) as to why you wrote that line



A successful mathematician plays both roles in their head when writing a proof

Gratuitous Induction Proof

S_n = “sum of first n integers = n(n+1)/2”
Want to prove: S_n is true for all n > 0

Base case: S_1 = “1 = 1(1+1)/2”

I.H. Suppose S_k is true for some k > 0

Induction step:

1 + 2 + …. + k + (k+1) = k(k+1)/2 + (k+1) (by I.H.)

= (k + 1)(k+2)/2

Thus S_{k+1}

Gratuitous Induction Proof

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Induction step:

1 + 2 + …. + n + (n+1) = n(n+1)/2 + (n+1) (by I.H.)

= (n + 1)(n+2)/2

Thus S_{k+1} wrong variable

10 Proof by Throwing in the Kitchen Sink

The author writes down every theorem or result known to mankind and then adds a few more just for good measure

When questioned later, the author correctly observes that the proof contains all the key facts needed to actually prove the result

Very popular strategy on 251 exams

Believed to result in partial credit with sufficient whining

10 Proof by Throwing in the Kitchen Sink

The author writes down every theorem

Like writing a program with functions that do most everything you’d ever want to do (e.g. sorting integers, calculating derivatives), which in the end simply prints “hello world”

sufficient whining

9 Proof by Example

The author gives only the case n = 2 and suggests that it contains most of the ideas of the general proof.

Like writing a program that only works for a few inputs

8

Proof by Cumbersome Notation

Best done with access to at least four alphabets and special symbols.

Helps to speak several foreign languages.

Like writing a program that's really hard to read because the variable names are screwy

7

Proof by Lengthiness

An issue or two of a journal devoted to your proof is useful. Works well in combination with Proof strategy #10 (throwing in the kitchen sink) and Proof strategy #8 (cumbersome notation).

Like writing 10,000 lines of code to simply print "hello world"

6

Proof by Switcharoo

Concluding that p is true when both $p \Rightarrow q$ and q are true

Makes as much sense as:

```
If (PRINT "X is prime") {
    PRIME(X);
}
```

Switcharoo Example

S_n = "sum of first n integers = n(n+1)/2"
Want to prove: S_n is true for all n > 0

Base case: S_1 = "1 = 1(1+1)/2"

I.H. Suppose S_k is true for some k > 0

Induction step: by S_{k+1}

$1 + 2 + ... + k + (k+1) = (k + 1)(k+2)/2$

Hence blah blah, S_k is true

(Partial) Switcharoo Example

S_n = "sum of first n integers = n(n+1)/2"
Want to prove: S_n is true for all n > 0

Base case: S_1 = "1 = 1(1+1)/2"

I.H. Suppose S_k is true for some k > 0

Induction step:

$1 + 2 + ... + k + (k+1) = (k + 1)(k+2)/2$???

By I.H., $1 + 2 + ... + k = k(k+1)/2$

Subtracting, we get k+1 = k+1

Hence S_{k+1} is true

5

Proof by "It is Clear That..."

"It is clear that that the worst case is this:"

Like a program that calls a function that you never wrote

4 Proof by Assuming The Result

Assume X is true

⋮

Therefore, X is true!

Like a program with this code:

```
RECURSIVE(X) {
        :       :
        return RECURSIVE(X);
}
```

"Assuming the Result" Example

S_n = "sum of first n integers = n(n+1)/2"
Want to prove: S_n is true for all n > 0

Base case: S_1 = "1 = 1(1+1)/2"

I.H. Suppose S_k is true for all k > 0

Induction step:

1 + 2 + …. + k + (k+1) = k(k+1)/2 + (k+1) (by I.H.)

= (k + 1)(k+2)/2

Thus S_{k+1}

3 Not Covering All Cases

Usual mistake in inductive proofs: A proof is given for N = 1 (base case), and another proof is given that, for any N > 2, if it is true for N, then it is true for N+1

Like a program with this function:

```
RECURSIVE(X) {
        if (X > 2) { return 2*RECURSIVE(X-1); }
        if (X = 1) { return 1; }
}
```

"Not Covering All Cases" Example

S_n = "sum of first n integers = n(n+1)/2"
Want to prove: S_n is true for all n > 0

Base case: S_0 = "0 = 0(0+1)/2"

I.H. Suppose S_k is true for some k > 0

Induction step:

1 + 2 + …. + k + (k+1) = k(k+1)/2 + (k+1) (by I.H.)

= (k + 1)(k+2)/2

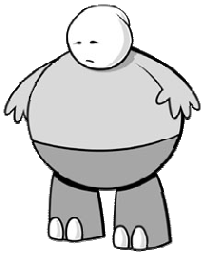
Thus S_{k+1}

2 Incorrectly Using "By Definition"

"By definition, { $a^n b^n$ | n > 0 } is not a regular language"

Like a program that assumes a procedure does something other than what it actually does

1 Proof by OMGWTFBBQ



Here's What You Need to Know…

Solving Problems
- Always try small examples!
- Use enough paper

Writing Proofs
- Writing proofs is sort of like writing programs, except every step in a proof has to be justified
- Be careful; search for your own errors