Chapter 3

Sequent Calculus

In the previous chapter we developed linear logic in the form of natural deduction, which is appropriate for many applications of linear logic. It is also highly economical, in that we only needed one basic judgment (*A true*) and two judgment forms (linear and unrestricted hypothetical judgments) to explain the meaning of all connectives we have encountered so far. However, it is not well-suited directly proof search, because this involves mixing forward and backward reasoning even if we restrict ourselves to searching for normal deductions.

In this chapter we develop a sequent calculus as a calculus of proof search for normal natural deductions. We then extend it with a rule of Cut that allows us to model arbitrary natural deductions. The central theorem of this chapter is *cut elimination* which shows that the cut rule is admissible. We obtain the normalization theorem for natural deduction as a direct consequence of this theorem. It was this latter application which lead to the original discovery of the sequent calculus by Gentzen [Gen35]. There are many useful immediate corollaries of the cut elimination theorem, such as consistency of the logic, or the disjunction property.

3.1 Cut-Free Sequent Calculus

In this section we transcribe the process of searching for *normal* natural deductions into an inference system. In the context of sequent calculus, proof search is seen entirely as the bottom-up construction of a derivation. This means that elimination rules must be turned "upside-down" so they can also be applied bottom-up rather than top-down.

In terms of judgments we develop the sequent calculus via a splitting of the judgment "*A is true*" into two judgments: "*A is a resource*" (*A res*) and "*A is a goal*" (*A goal*). Ignoring unrestricted hypothesis for the moment, the main judgment

$$w_1{:}A_1\ res, \ldots, w_n{:}A_n\ res \Longrightarrow C\ goal$$

expresses

> Under the linear hypothesis that we have resources $A_1, \ldots, A_n$ we can achieve goal C.

In order to model validity, we add inexhaustible resources or *resource factories*, written A *fact*. We obtain

$$(v_1{:}B_1\,\mathit{fact}, \ldots, v_m{:}B_m\,\mathit{fact}); (w_1{:}A_1\ \mathit{res}, \ldots, w_n{:}A_n\ \mathit{res}) \Longrightarrow C\ \mathit{goal},$$

which expresses

> Under the unrestricted hypotheses that we have resource factories $B_1, \ldots, B_m$ and linear hyptheses that we have resources $A_1, \ldots, A_n$, we can achieve goal C.

As before, the order of the hypothesis (linear or unrestricted) is irrelevant, and we assume that all hypothesis labels v_j and w_i are distinct.

Resources and goals are related in that with the resource A we can achieve goal A. Recall that the linear hypothetical judgment requires us to use all linear hypotheses exactly once. We therefore have the following rule.

$$\frac{}{\Gamma; u{:}A\ \mathit{res} \Longrightarrow A\ \mathit{goal}}\ \mathsf{IN}_u$$

We call such as sequent *initial* and write IN. Note that, for the moment, we do not have the opposite: if we can achieve goal A we cannot assume A as a resource. The corresponding rule will be called Cut and is shown later to be admissible, that is, every instance of this rule can be eliminated from a proof. It is the desire to rule out Cut that necessitated splitting truth into two judgments.

We also need a rule that allows a factory to produce a resource. This rule is labelled DL for *dereliction*. We will alsw refer to it as a *copy rule*.

$$\frac{(\Gamma, v{:}A\,\mathit{fact}); (\Delta, w{:}A\ \mathit{res}) \Longrightarrow C\ \mathit{goal}}{(\Gamma, v{:}A\,\mathit{fact}); \Delta \Longrightarrow C\ \mathit{goal}}\ \mathsf{DL}_v$$

Note how this is different from the unrestricted hypothesis rule in natural deduction. Factories are directly related to resources and only indirectly to goals.

The remaining rules are divided into *right* and *left* rules, which correspond to the *introduction* and *elimination* rules of natural deduction, respectively. The right rules apply to the goal, while the left rules apply to resources. In the following, we adhere to common practice and omit labels on hypotheses and consequently also on the justifications of the inference rules. The reader should keep in mind, however, that this is just a short-hand, and that there are, for example, two *different* derivations of $(A, A); \cdot \Longrightarrow A$, one using the first copy of A and one using the second.

Hypotheses.

$$\frac{}{\Gamma; A \Longrightarrow A}\ \mathsf{IN} \qquad \frac{(\Gamma, A); (\Delta, A) \Longrightarrow C}{(\Gamma, A); \Delta \Longrightarrow C}\ \mathsf{DL}$$

Multiplicative Connectives.

$$\frac{\Gamma; \Delta, A \Longrightarrow B}{\Gamma; \Delta \Longrightarrow A \multimap B} \multimap R \qquad \frac{\Gamma; \Delta_1 \Longrightarrow A \qquad \Gamma; \Delta_2, B \Longrightarrow C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \Longrightarrow C} \multimap L$$

$$\frac{\Gamma; \Delta_1 \Longrightarrow A \qquad \Gamma; \Delta_2 \Longrightarrow B}{\Gamma; \Delta_1, \Delta_2 \Longrightarrow A \otimes B} \otimes R \qquad \frac{\Gamma; \Delta, A, B \Longrightarrow C}{\Gamma; \Delta, A \otimes B \Longrightarrow C} \otimes L$$

$$\frac{}{\Gamma; \cdot \Longrightarrow \mathbf{1}} \mathbf{1}R \qquad \frac{\Gamma; \Delta \Longrightarrow C}{\Gamma; \Delta, \mathbf{1} \Longrightarrow C} \mathbf{1}L$$

Additive Connectives.

$$\frac{\Gamma; \Delta \Longrightarrow A \qquad \Gamma; \Delta \Longrightarrow B}{\Gamma; \Delta \Longrightarrow A \& B} \& R \qquad \begin{array}{c} \dfrac{\Gamma; \Delta, A \Longrightarrow C}{\Gamma; \Delta, A \& B \Longrightarrow C} \& L_1 \\ \dfrac{\Gamma; \Delta, B \Longrightarrow C}{\Gamma; \Delta, A \& B \Longrightarrow C} \& L_2 \end{array}$$

$$\frac{}{\Gamma; \Delta \Longrightarrow \top} \top R \qquad \textit{No } \top \textit{ left rule}$$

$$\begin{array}{c} \dfrac{\Gamma; \Delta \Longrightarrow A}{\Gamma; \Delta \Longrightarrow A \oplus B} \oplus R_1 \\ \dfrac{\Gamma; \Delta \Longrightarrow B}{\Gamma; \Delta \Longrightarrow A \oplus B} \oplus R_2 \end{array} \qquad \frac{\Gamma; \Delta, A \Longrightarrow C \qquad \Gamma; \Delta, B \Longrightarrow C}{\Gamma; \Delta, A \oplus B \Longrightarrow C} \oplus L$$

$$\textit{No } \mathbf{0} \textit{ right rule} \qquad \frac{}{\Gamma; \Delta, \mathbf{0} \Longrightarrow C} \mathbf{0}L$$

Quantifiers.

$$\frac{\Gamma; \Delta \Longrightarrow [a/x]A}{\Gamma; \Delta \Longrightarrow \forall x.\, A} \forall R^a \qquad \frac{\Gamma; \Delta, [t/x]A \Longrightarrow C}{\Gamma; \Delta, \forall x.\, A \Longrightarrow C} \forall L$$

$$\frac{\Gamma; \Delta \Longrightarrow [t/x]A}{\Gamma; \Delta \Longrightarrow \exists x.\, A} \exists R \qquad \frac{\Gamma; \Delta, [a/x]A \Longrightarrow C}{\Gamma; \Delta, \exists x.\, A \Longrightarrow C} \exists L^a$$

Exponentials.

$$\frac{(\Gamma, A); \Delta \Longrightarrow B}{\Gamma; \Delta \Longrightarrow A \supset B} \supset\mathrm{R} \qquad \frac{\Gamma; \cdot \Longrightarrow A \qquad \Gamma; \Delta, B \Longrightarrow C}{\Gamma; \Delta, A \supset B \Longrightarrow C} \supset\mathrm{L}$$

$$\frac{\Gamma; \cdot \Longrightarrow A}{\Gamma; \cdot \Longrightarrow !A} !\mathrm{R} \qquad \frac{(\Gamma, A); \Delta \Longrightarrow C}{\Gamma; (\Delta, !A) \Longrightarrow C} !\mathrm{L}$$

We have the following theorems relating normal natural deductions and sequent derivations.

Theorem 3.1 (Soundness of Sequent Derivations)
If $\Gamma; \Delta \Longrightarrow A$ *then* $\Gamma; \Delta \vdash A \uparrow$.

Proof: By induction on the structure of the derivation of $\Gamma; \Delta \Longrightarrow A$. Initial sequents are translated to the $\downarrow\uparrow$ coercion, and use of an unrestricted hypothesis follows by a substitution principle (Lemma 2.2). For right rules we apply the corresponding introduction rules. For left rules we either directly construct a derivation of the conclusion after an appeal to the induction hypothesis ($\otimes$L, **1**L, $\otimes$L, **0**L, $\exists$L, !L) or we appeal to a substitution principle of atomic natural deductions for hypotheses ($\multimap$ L, &L_1, &L_2, $\forall$L, $\supset$L). □

Theorem 3.2 (Completeness of Sequent Derivations)

1. *If* $\Gamma; \Delta \vdash A \uparrow$ *then there is a sequent derivation of* $\Gamma; \Delta \Longrightarrow A$*, and*
2. *if* $\Gamma; \Delta \vdash A \downarrow$ *then for any formula* C *and derivation of* $\Gamma; \Delta', A \Longrightarrow C$ *there is a derivation of* $\Gamma; (\Delta', \Delta) \Longrightarrow C$.

Proof: By simultaneous induction on the structure of the derivations of $\Gamma; \Delta \vdash A \uparrow$ and $\Gamma; \Delta \vdash A \downarrow$. □

3.2 Another Example: Petri Nets

In this section we show[1] how to represent Petri nets in linear logic. This example is due to Martì-Oliet and Meseguer [MOM91], but has been treated several times in the literature.

3.3 Deductions with Lemmas

One common way to find or formulate a proof is to introduce a lemma. In the sequent calculus, the introduction and use of a lemma during proof search is modelled by the rules of cut, *Cut* for lemmas used as linear hypotheses, and

[1] [*eventually*]

Cut! for lemmas used as factories or resources. The corresponding rule for intuitionistic logic is due to Gentzen [Gen35]. We write $\Gamma; \Delta \stackrel{+}{\Longrightarrow} A$ for the judgment that A can be derived with the rules from before, plus one of the two cut rules below.

$$\frac{\Gamma; \Delta \stackrel{+}{\Longrightarrow} A \quad \Gamma; (\Delta', A) \stackrel{+}{\Longrightarrow} C}{\Gamma; \Delta', \Delta \stackrel{+}{\Longrightarrow} C}\,\mathit{Cut} \qquad \frac{\Gamma; \cdot \stackrel{+}{\Longrightarrow} A \quad (\Gamma, A); \Delta' \stackrel{+}{\Longrightarrow} C}{\Gamma; \Delta' \stackrel{+}{\Longrightarrow} C}\,\mathit{Cut}!$$

Note that the linear context in the left premise of the *Cut*! rule must be empty, because the new hypothesis A in the right premise is unrestricted in its use.

On the side of natural deduction, these rules correspond to substitution principles. They can be related to normal and atomic derivations only if we allow an additional coercion from normal to atomic derivations. This is because the left premise corresponds to a derivation of $\Gamma; \Delta \vdash A \uparrow$ which can be substituted into a derivation of $\Gamma; \Delta', A \vdash C \uparrow$ only have the additional coercion has been applied. Of course, the resulting deductions are no longer normal in the sense we defined before, so we write $\Gamma; \Delta \vdash^{+} A \downarrow$ and $\Gamma; \Delta \vdash^{+} A \uparrow$. These judgments are defined with the same rules as $\Gamma; \Delta \vdash A \uparrow$ and $\Gamma; \Delta \vdash A \downarrow$, plus the following coercion.

$$\frac{\Gamma; \Delta \vdash^{+} A \uparrow}{\Gamma; \Delta \vdash^{+} A \downarrow}\,\uparrow\downarrow$$

It is now easy to prove that arbitrary natural deductions can be annotated with $\uparrow$ and $\downarrow$, since we can arbitrarily coerce back and forth between the two judgments.

Theorem 3.3 *If* $\Gamma; \Delta \vdash A$ *then* $\Gamma; \Delta \vdash^{+} A \uparrow$ *and* $\Gamma; \Delta \vdash^{+} A \downarrow$

Proof: By induction on the structure of $\mathcal{D} :: (\Gamma; \Delta \vdash A)$. □

Theorem 3.4

1. *If* $\Gamma; \Delta \vdash^{+} A \uparrow$ *then* $\Gamma; \Delta \vdash A$.
2. *If* $\Gamma; \Delta \vdash^{+} A \downarrow$ *then* $\Gamma; \Delta \vdash A$.

Proof: My mutual induction on $\mathcal{N} :: (\Gamma; \Delta \vdash^{+} A \uparrow)$ and $\mathcal{A} :: (\Gamma; \Delta \vdash^{+} A \downarrow)$. □

It is also easy to relate the Cut rules to the new coercions (and thereby to natural deductions), plus four substitution principles.

Property 3.5

1. *If* $\Gamma; (\Delta', w{:}A) \vdash^{+} C \uparrow$ *and* $\Gamma; \Delta \vdash^{+} A \downarrow$ *then* $\Gamma; (\Delta', \Delta) \vdash^{+} C \uparrow$.
2. *If* $\Gamma; (\Delta', w{:}A) \vdash^{+} C \downarrow$ *and* $\Gamma; \Delta \vdash^{+} A \downarrow$ *then* $\Gamma; (\Delta', \Delta) \vdash^{+} C \downarrow$.

3. *If* $(\Gamma, u{:}A); \Delta' \vdash^{+} C \uparrow$ *and* $\Gamma; \cdot \vdash^{+} A \downarrow$ *then* $\Gamma; \Delta' \vdash^{+} C \uparrow$.

4. *If* $(\Gamma, u{:}A); \Delta' \vdash^{+} C \downarrow$ *and* $\Gamma; \cdot \vdash^{+} A \downarrow$ *then* $\Gamma; \Delta' \vdash^{+} C \downarrow$.

Proof: By mutual induction on the structure of the given derivations. □

We can now extend Theorems 3.1 and 3.2 to relate sequent derivations with Cut to natural deductions with explicit lemmas.

Theorem 3.6 (Soundness of Sequent Derivations with Cut)
If $\Gamma; \Delta \stackrel{+}{\Longrightarrow} A$ *then* $\Gamma; \Delta \vdash^{+} A \uparrow$.

Proof: As in Theorem 3.1 by induction on the structure of the derivation of $\Gamma; \Delta \stackrel{+}{\Longrightarrow} A$. An inference with one of the new rules *Cut* or *Cut*! is translated into an application of the $\uparrow\downarrow$ coercion followed by an appeal to one of the substitution principles in Property 3.5. □

Theorem 3.7 (Completeness of Sequent Derivations with Cut)

1. *If* $\Gamma; \Delta \vdash^{+} A \uparrow$ *then there is a sequent derivation of* $\Gamma; \Delta \stackrel{+}{\Longrightarrow} A$*, and*

2. *if* $\Gamma; \Delta \vdash^{+} A \downarrow$ *then for any formula* C *and derivation of* $\Gamma; (\Delta', A) \stackrel{+}{\Longrightarrow} C$ *there is a derivation of* $\Gamma; (\Delta', \Delta) \stackrel{+}{\Longrightarrow} C$.

Proof: As in the proof of Theorem 3.2 by induction on the structure of the given derivations. In the new case of the $\uparrow\downarrow$ coercion, we use the rule of *Cut*. The other new rule, *Cut*!, is not needed for this proof, but is necessary for the proof of admissibility of cut in the next section. □

3.4 Cut Elimination

We viewed the sequent calculus as a calculus of proof search for natural deduction. The proofs of the soundness theorems 3.2 and 3.7 provide ways to translate cut-free sequent derivations into normal natural deductions, and sequent derivations with cut into arbitrary natural deductions.

This section is devoted to showing that the two rules of cut are redundant in the sense that any derivation in the sequent calculus which makes use of the rules of cut can be translated to one that does not. Taken together with the soundness and completeness theorems for the sequent calculi with and without cut, this has many important consequences.

First of all, a proof search procedure which looks only for cut-free sequent derivations will be complete: any derivable proposition can be proven this way. When the cut rule

$$\frac{\Gamma; \Delta \stackrel{+}{\Longrightarrow} A \qquad \Gamma; \Delta', A \stackrel{+}{\Longrightarrow} C}{\Gamma; \Delta', \Delta \stackrel{+}{\Longrightarrow} C}\,\mathit{Cut}$$

is viewed in the bottom-up direction the way it would be used during proof search, it introduces a new and arbitrary proposition A. Clearly, this introduces a great amount of non-determinism into the search. The cut elimination theorem now tells us that we never need to use this rule. All the remaining rules have the property that the premises contain only instances of propositions in the conclusion, or parts thereof. This latter property is often called the *subformula property*.

Secondly, it is easy to see that the logic is *consistent*, that is, not every proposition is provable. In particular, the sequent $\cdot;\cdot \Longrightarrow \mathbf{0}$ does not have a cut-free derivation, because there is simply no rule which could be applied to infer it! This property clearly fails in the presence of cut: it is *prima facie* quite possible that the sequent $\cdot;\cdot \overset{+}{\Longrightarrow} \mathbf{0}$ is the conclusion of the cut rule.

Along the same lines, we can show that a number of propositions are *not derivable* in the sequent calculus and therefore not true as defined by the natural deduction rules. Examples of this kind are given at the end of this section.

We prove cut elimination by showing that the two cut rules are *admissible rules of inference* in the sequent calculus without cut. An inference rule is admissible if whenever we can find derivations for its premises we can find a derivation of its conclusion. This should be distinguished from a *derived rule of inference* which requires a direct derivation of the conclusion from the premises. We can also think of a derived rule as an evident hypothetical judgment where the premises are (unrestricted) hypotheses.

Derived rules of inference have the important property that they remain evident under any extension of the logic. An admissible rule, on the other hand, represents a global property of the deductive system under consideration and may well fail when the system is extended. Of course, every derived rule is also admissible.

Theorem 3.8 (Admissibility of Cut)

1. *If* $\Gamma;\Delta \Longrightarrow A$ *and* $\Gamma;(\Delta',A) \Longrightarrow C$ *then* $\Gamma;\Delta',\Delta \Longrightarrow C$.
2. *If* $\Gamma;\cdot \Longrightarrow A$ *and* $(\Gamma,A);\Delta' \Longrightarrow C$ *then* $\Gamma;\Delta' \Longrightarrow C$.

Proof: By nested inductions on the structure of the cut formula A and the given derivations, where induction hypothesis (1) has priority over (2). To state this more precisely, we refer to the given derivations as $\mathcal{D} :: (\Gamma;\Delta \Longrightarrow A)$, $\mathcal{D}' :: (\Gamma;\cdot \Longrightarrow A)$, $\mathcal{E} :: (\Gamma;(\Delta,A) \Longrightarrow C)$, and $\mathcal{E}' :: ((\Gamma,A);\Delta' \vdash C)$. Then we may appeal to the induction hypothesis whenever

a. the cut formula A is strictly smaller, or

b. the cut formula A remains the same, but we appeal to induction hypothesis (1) in the proof of (2) (but when we appeal to (2) in the proof of (1) the cut formula must be strictly smaller), or

c. the cut formula A and the derivation $\mathcal{E}$ remain the same, but the derivation $\mathcal{D}$ becomes smaller, or

d. the cut formula A and the derivation $\mathcal{D}$ remain the same, but the derivation $\mathcal{E}$ or $\mathcal{E}'$ becomes smaller.

Here, we consider a formula smaller it is an immediate subformula, where $[t/x]A$ is considered a subformula of $\forall x.\ A$, since it contains fewer quantifiers and logical connectives. A derivation is smaller if it is an immediate subderivation, where we allow weakening by additional unrestricted hypothesis in one case (which does not affect the structure of the derivation).

The cases we have to consider fall into 5 classes:

Initial Cuts: One of the two premises is an initial sequent. In these cases the cut can be eliminated directly.

Principal Cuts: The cut formula A was just inferred by a right rule in $\mathcal{D}$ and by a left rule in $\mathcal{E}$. In these cases we appeal to the induction hypothesis (possibly several times) on smaller cut formulas (item (a) above).

Dereliction Cut: The cases for the *Cut*! rule are treated as right commutative cuts (see below), except for the rule of dereliction which requires an appeal to induction hypothesis (1) with the same cut formula (item (b) above).

Left Commutative Cuts: The cut formula A is a side formula of the last inference in $\mathcal{D}$. In these cases we may appeal to the induction hypotheses with the same cut formula, but smaller derivation $\mathcal{D}$ (item (c) above).

Right Commutative Cuts: The cut formula A is a side formula of the last inference in $\mathcal{E}$. In these cases we may appeal to the induction hypotheses with the same cut formula, but smaller derivation $\mathcal{E}$ or $\mathcal{E}'$ (item (d) above).

[*Some cases to be filled in later.*]

□

Using the admissibility of cut, the cut elimination theorem follows by a simple structural induction.

Theorem 3.9 (Cut Elimination)
If $\Gamma; \Delta \stackrel{+}{\Longrightarrow} C$ *then* $\Gamma; \Delta \Longrightarrow C$.

Proof: By induction on the structure of $\mathcal{D} :: (\Gamma; \Delta \stackrel{+}{\Longrightarrow} C)$. In each case except *Cut* or *Cut*! we simply appeal to the induction hypothesis on the derivations of the premises and use the corresponding rule in the cut-free sequent calculus. For the *Cut* and *Cut*! rules we appeal to the induction hypothesis and then admissibility of cut (Theorem 3.8) on the resulting derivations. □

3.5 Consequences of Cut Elimination

As a first consequence, we see that linear logic is *consistent*: not every proposition can be proved. A proof of consistency for both intuitionistic and classical logic was Gentzen's original motivation for the development of the sequent calculus and his proof of cut elimination.

Theorem 3.10 (Consistency of Intuitionistic Linear Logic)
$\cdot;\cdot \vdash \mathbf{0}$ *is not derivable.*

Proof: If the judgment were derivable, by Theorems 3.3, 3.7, and 3.9, there must be a cut-free sequent derivation of $\cdot;\cdot \Longrightarrow \mathbf{0}$. But there is no rule with which we could infer this sequent (there is no right rule for $\mathbf{0}$), and so it cannot be derivable. □

A second consequence is that every natural deduction can be translated to a normal natural deduction. The necessary construction is implicit in the proofs of the soundness and completeness theorems for sequent calculi and the proofs of admissibility of cut and cut elimination. In Chapter **??** we will see a much more direct, but in other respects more complicated proof.

Theorem 3.11 (Normalization for Natural Deductions)
If $\Gamma; \Delta \vdash A$ *then* $\Gamma; \Delta \vdash A \uparrow$.

Proof: Directly, using theorems from this chapter. Assume $\Gamma; \Delta \vdash A$. Then

$\Gamma; \Delta \vdash^+ A$ by Theorem 3.3,
$\Gamma; \Delta \stackrel{+}{\Longrightarrow} A$ by completeness of sequent derivations with cut (Theorem 3.7),
$\Gamma; \Delta \Longrightarrow A$ by cut elimination (Theorem 3.9), and
$\Gamma; \Delta \vdash A \uparrow$ by soundness of cut-free sequent derivations (Theorem 3.1).

□

3.6 Exercises

Exercise 3.1 Consider if $\otimes$ and $\&$ can be distributed over $\oplus$ or *vice versa.* There are four different possible equivalences based on eight possible entailments. Give sequent derivations for the entailments that hold.

Exercise 3.2 Prove that the rule

$$\frac{(\Gamma, A\&B, A, B); \Delta \Longrightarrow C}{(\Gamma, A\&B); \Delta \Longrightarrow C}\ \&\mathrm{L}!$$

is admissible in the linear sequent calculus. Further prove that the rule

$$\frac{(\Gamma, A \otimes B, A, B); \Delta \Longrightarrow C}{(\Gamma, A \otimes B); \Delta \Longrightarrow C}\ \otimes\mathrm{L}!$$

is *not* admissible.

Determine which other connectives and constants have similar or analogous admissible rules directly on resource factories and which ones do not. You do not need to formally prove admissibility or unsoundness of your proposed rules.

Bibliography

[ABCJ94] D. Albrecht, F. Bäuerle, J. N. Crossley, and J. S. Jeavons. Curry-Howard terms for linear logic. In ??, editor, *Logic Colloquium '94*, pages ??–?? ??, 1994.

[Abr93] Samson Abramsky. Computational interpretations of linear logic. *Theoretical Computer Science*, 111:3–57, 1993.

[And92] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):197–347, 1992.

[Bar96] Andrew Barber. Dual intuitionistic linear logic. Technical Report ECS-LFCS-96-347, Department of Computer Science, University of Edinburgh, September 1996.

[Bib86] Wolfgang Bibel. A deductive solution for plan generation. *New Generation Computing*, 4:115–132, 1986.

[Bie94] G. Bierman. On intuitionistic linear logic. Technical Report 346, University of Cambridge, Computer Laboratory, August 1994. Revised version of PhD thesis.

[Doš93] Kosta Došen. A historical introduction to substructural logics. In Peter Schroeder-Heister and Kosta Došen, editors, *Substructural Logics*, pages 1–30. Clarendon Press, Oxford, England, 1993.

[Gen35] Gerhard Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935. Translated under the title *Investigations into Logical Deductions* in [Sza69].

[Gir87] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[Gir93] J.-Y. Girard. On the unity of logic. *Annals of Pure and Applied Logic*, 59:201–217, 1993.

[Lin92] P. Lincoln. Linear logic. *ACM SIGACT Notices*, 23(2):29–37, Spring 1992.

[Mil92] D. Miller. The π-calculus as a theory in linear logic: Preliminary results. In E. Lamma and P. Mello, editors, *Proceedings of the Workshop on Extensions of Logic Programming*, pages 242–265. Springer-Verlag LNCS 660, 1992.

[ML96] Per Martin-Löf. On the meanings of the logical constants and the justifications of the logical laws. *Nordic Journal of Philosophical Logic*, 1(1):11–60, 1996.

[MOM91] N. Martí-Oliet and J. Meseguer. From Petri nets to linear logic through categories: A survey. *Journal on Foundations of Computer Science*, 2(4):297–399, December 1991.

[PD01] Frank Pfenning and Rowan Davies. A judgmental reconstruction of modal logic. *Mathematical Structures in Computer Science*, 11:511–540, 2001. Notes to an invited talk at the *Workshop on Intuitionistic Modal Logics and Applications* (IMLA'99), Trento, Italy, July 1999.

[Pra65] Dag Prawitz. *Natural Deduction.* Almquist & Wiksell, Stockholm, 1965.

[Sce93] A. Scedrov. A brief guide to linear logic. In G. Rozenberg and A. Salomaa, editors, *Current Trends in Theoretical Computer Science*, pages 377–394. World Scientific Publishing Company, 1993. Also in Bulletin of the European Association for Theoretical Computer Science, volume 41, pages 154–165.

[SHD93] Peter Schroeder-Heister and Kosta Došen, editors. *Substructural Logics.* Number 2 in Studies in Logic and Computation. Clarendon Press, Oxford, England, 1993.

[Sza69] M. E. Szabo, editor. *The Collected Papers of Gerhard Gentzen.* North-Holland Publishing Co., Amsterdam, 1969.

[Tro92] A. S. Troelstra. *Lectures on Linear Logic.* CSLI Lecture Notes 29, Center for the Study of Language and Information, Stanford, California, 1992.

[Tro93] A. S. Troelstra. Natural deduction for intuitionistic linear logic. Prepublication Series for Mathematical Logic and Foundations ML-93-09, Institute for Language, Logic and Computation, University of Amsterdam, 1993.