Chapter 2

Linear Natural Deduction

Linear logic, in its original formulation by Girard [Gir87] and many subsequent investigations was presented as a refinement of classical logic. This calculus of *classical linear logic* can be cleanly related to classical logic and exhibits many pleasant symmetries. On the other hand, a number of applications in logic and functional programming can be treated most directly using the intuitionistic version. In this chapter we present a basic system of natural deduction defining intuitionistic linear logic.

Our presentation is a judgmental reconstruction of linear logic in the style of Martin-Löf [ML96]. It follows the traditions of Gentzen [Gen35], who first introduced natural deduction, and Prawitz [Pra65], who thoroughly investigated its theory. A similar development of modal logic is given in [PD01]. The way of combining of linear and unrestricted resources goes back to Andreoli [And92] and Girard [Gir93] and, in an explicitly intuitionistic version, Barber [Bar96].

2.1 Judgments and Propositions

In his Siena lectures from 1983 (finally published in 1996), Martin-Löf provides a foundation for logic based on a clear separation of the notions of judgment and proposition. He reasons that to judge is to know and that an evident judgment is an object of knowledge. A proof is what makes a judgment evident. In logic, we make particular judgments such as "A *is a proposition*" or "A *is true*", presupposing in the latter case that A is already known to be a proposition. To know that "A *is a proposition*" means to know what counts as a verification of A, whereas to know that "A *is true*" means to know how to verify A. In his words [ML96, Page 27]:

> *The meaning of a proposition is determined by [...] what counts as a verification of it.*

This approach leads to a clear conceptual priority: we first need to understand the notions of judgment and evidence for judgments, then the notions of proposition and verifications of propositions to understand truth.

As an example, we consider the explanation of conjunction. We know that $A \wedge B$ is a proposition if both A and B are propositions. As a rule of inference (called conjunction formation):

$$\frac{A\ \mathit{prop} \qquad B\ \mathit{prop}}{A \wedge B\ \mathit{prop}} \wedge F$$

The meaning is given by stating what counts a verification of $A \wedge B$. We say that we have a verification of $A \wedge B$ if we have verifications for both A and B. As a rule of inference:

$$\frac{A\ \mathit{true} \qquad B\ \mathit{true}}{A \wedge B\ \mathit{true}} \wedge I$$

where we presuppose that A and B are already known to be propositions. This is known as an *introduction rule*, a term due to Gentzen [Gen35] who first formulated a system of natural deduction. Conversely, what do we know if we know that $A \wedge B$ is true? Since a verification of $A \wedge B$ consists of verifications for both A and B, we know that A must be true and B must be true. Formulated as rules of inference (called conjunction eliminations):

$$\frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}} \wedge E_L \qquad \frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}} \wedge E_R$$

From the explanation above it should be clear that the two elimination rules are *sound*: if we define the meaning of conjunction by its introduction rule then we are fully justified in concluding that A is true if $A \wedge B$ is true, and similarly for the second rule.

Soundness guarantees that the elimination rules are not too strong. We have sufficient evidence for the judgment in the conclusion if we have sufficient evidence for the judgment in the premise. This is witnessed by a *local reduction* which constructs evidence for the conclusion from evidence for the premise.

$$\frac{\dfrac{\begin{matrix}\mathcal{D} \\ A\ \mathit{true}\end{matrix} \qquad \begin{matrix}\mathcal{E} \\ B\ \mathit{true}\end{matrix}}{A \wedge B\ \mathit{true}} \wedge I}{A\ \mathit{true}} \wedge E_L \quad \longrightarrow \quad \begin{matrix}\mathcal{D} \\ A\ \mathit{true}\end{matrix}$$

A symmetric reduction exists for $\wedge E_R$. We only consider each elimination immediately preceded by an introduction for a connective. We therefore call the property that each such pattern can be reduced *local soundness*.

The dual question, namely if the elimination rules are sufficiently strong, has, as far as we know, not been discussed by Martin-Löf. Of course, we can never achieve "absolute" completeness of rules for inferring evident judgments. But in some situations, elimination rules may be obviously incomplete. For example, we might have overlooked the second elimination rule for conjunction, $\wedge E_R$. This would not contradict soundness, but we would not be able to exploit the knowledge that $A \wedge B$ is true to its fullest. In particular, we cannot recover the knowledge that B is true even if we know that $A \wedge B$ is true.

In general we say that the elimination rules for a connective are *locally complete* if we can apply the elimination rules to a judgment to recover enough knowledge to permit reconstruction of the original judgment. In the case of conjunction, this is only possible if we have both elimination rules.

$$\begin{array}{c}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{array}}{A\ \mathit{true}}\wedge E_L \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{array}}{B\ \mathit{true}}\wedge E_R}{A \wedge B\ \mathit{true}}\wedge I$$

We call this pattern a *local expansion* since we obtain more complex evidence for the original judgment.

An alternative way to understand local completeness is to reconsider our meaning explanation of conjunction. We have said that a verification of $A \wedge B$ consists of a verification of A and a verification of B. Local completeness entails that it is always possible to bring the verification of $A \wedge B$ into this form by a local expansion.

To summarize, logic is based on the notion of judgment where an evident judgment is an object of knowledge. A judgment can be immediately evident or, more typically, mediately evident, in which case the evidence is provided by a proof. The meaning of a proposition is given by what counts as a verification of it. This is written out in the form of introduction rules for logical connectives which allow us to conclude when propositions are true. They are complemented by elimination rules which allow us to obtain further knowledge from the knowledge of compound propositions. The elimination rules for a connective should be locally sound and complete in order to have a satisfactory meaning explanation for the connective. Local soundness and completeness are witnessed by local reductions and expansions of proofs, respectively.

Note that there are other ways to define meaning. For example, we frequently expand our language by *notational definition*. In intuitionistic logic negation is often given as a derived concept, where $\neg A$ is considered a notation for $A \supset \bot$. This means that negation has a rather weak status, as its meaning relies entirely on the meaning of implication and falsehood rather than having an independent explanation. The two should not be mixed: introduction and elimination rules for a connective should rely solely on judgmental concepts and not on other connectives. Sometimes (as in the case of negation) a connective can be explained directly or as a notational definition and we can establish that the two meanings coincide.

2.2 Linear Hypothetical Judgments

So far we have seen two forms of judgment: "*A is a proposition*" and "*A is true*". These are insufficient to explain logical reasoning from assumptions.

For this we need hypothetical judgments and hypothetical proofs, which are new primitive notions. Since we are primarily interested in linear logic, we begin with *linear hypothetical judgments* and *linear hypothetical proofs*. We will postpone discussion of (unrestricted) hypothetical judgments until Section 2.4.

We write the general form of a linear hypothetical judgment as

$$J_1, \ldots, J_n \Vdash J$$

which expresses "*J assuming J_1 through J_n linearly*" or "*J under linear hypotheses J_1 through J_n*". We also refer to $J_1, \ldots, J_n$ as the *antecedents* and J as the *succedent* of the linear hypothetical judgment. The intent of the qualifier "linear" is to indicate that each hypothesis J_i in the antecedent is to be used exactly once. The order of the linear hypotheses is irrelvant, so we will silently allow them to be exchanged.

We now explain what constitutes evidence for a linear hypothetical judgment, namely a linear hypothetical proof. In a hypothetical proof of the judgment above we can use the hypotheses J_i as if they were available as resources. We can consequently substitute an arbitrary derivation of J_i for the uses of a hypothesis J_i to obtain a judgment which no longer depends on J_i. Thus, at the core, the meaning of hypothetical judgments relies upon substitution on the level of proofs, that is, supplanting the use of a hypothesis by evidence for it.

The first particular form of linear hypothetical judgment we need here is

$$u_1{:}A_1\ \mathit{true}, \ldots, u_n{:}A_n\ \mathit{true} \Vdash A\ \mathit{true}$$

where we presuppose that A_1 through A_n and A are all propositions. Note that the propositions A_i do not need to be distinct. We therefore label them with distinct variables u_i so we can refer to them unambiguously. We will sometimes omit the labels for the sake of brevity, but one should keep in mind that

$$A_1\ \mathit{true}, \ldots, A_n\ \mathit{true} \Vdash A\ \mathit{true}$$

is just a shorthand. We write Δ for a collection of linear hypotheses of the form above. The special case of the substitution principle for such hypotheses has the form

Linear Substitution Principle for Truth
If $\Delta \Vdash A\ \mathit{true}$ and $\Delta', u{:}A\ \mathit{true} \Vdash C\ \mathit{true}$ then $\Delta', \Delta \Vdash C\ \mathit{true}$.

Here we write Δ', Δ for the concatenation of two collections of linear hypotheses with distinct labels. We can always rename some labels in Δ or Δ' in order to satisfy this side condition. We further have the general rule for the use of hypotheses.

Linear Hypothesis Rule

$$\frac{}{u{:}A\ \mathit{true} \Vdash A\ \mathit{true}}\ u$$

We sometimes write hyp as the justification for the hypothesis rule if the label u is omitted or irrelevant.

Note that the substitution principle and the linear hypothesis rule together enforce that assumptions are used exactly once. Viewed from the conclusion, the substitution principle splits its resources, distributing it to the two premises. Therefore each assumption in Δ, Δ' will have to be used in either the proof of A or the proof of C from A, but not in both. The linear hypothesis rule does not allow any additional resources among the assumptions besides A, thereby forcing each resource to be used.

We emphasize that the substitution principle should not be viewed as an inference rule, but a property defining hypothetical judgments which we use in the design of a formal system. Therefore it should hold for any system of connectives and inference rules we devise. The correctness of the hypothesis rule, for example, can be seen from the substitution principle.

One further notation: $[\mathcal{D}/u]\mathcal{E}$ is our notation for the result of an appeal to the substitution principle. That is,

$$\text{If} \quad \begin{array}{c}\mathcal{D}\\ \Delta \Vdash A\ \mathit{true}\end{array} \quad \text{and} \quad \begin{array}{c}\mathcal{E}\\ \Delta', u{:}A \Vdash C\ \mathit{true}\end{array} \quad \text{then} \quad \begin{array}{c}[\mathcal{D}/u]\mathcal{E}\\ \Delta, \Delta' \Vdash C\ \mathit{true}\end{array}$$

2.3 Propositions in Linear Logic

Based on the notion of linear hypothetical judgment, we now introduce the various connectives of linear logic via their introduction and elimination rules. We skip, for now, the obvious formation rules for propositions. For each of the connectives we carefully check the local soundness and completeness of the rules and verify the preservation of resources. Also for purely typographical reasons, we abbreviate "A *true*" by just writing "A" in the linear hypothetical judgments.

Simultaneous Conjunction. Assume we have some resources and we want to achieve goals A and B simultaneously, written as $A \otimes B$ (pronounced "A *and* B" or "A *tensor* B"). We need to split our resources into Δ and Δ' and show that with resources Δ we can achieve A and with Δ' we can achieve B.

$$\frac{\Delta \Vdash A \qquad \Delta' \Vdash B}{\Delta, \Delta \Vdash A \otimes B}\otimes\mathrm{I}$$

Note that the splitting of resources, viewed bottom-up, is a non-deterministic operation.

The elimination rule should capture what we can achieve if we know that we can achieve both A and B simultaneously from some resources Δ. We reason as follows: If with A, B, and additional resources Δ' we could achieve goal C, then we could achieve C from resources Δ and Δ'.

$$\frac{\Delta \Vdash A \otimes B \qquad \Delta', u{:}A, w{:}B \Vdash C}{\Delta, \Delta' \Vdash C}\otimes\mathrm{E}$$

Note that by our general assumption, u and w must be new hypothesis labels in the second premise. The way we achieve C is to commit resources Δ to achieving A and B by the derivation of the left premise and then using the remaining resources Δ' together with A and B to achieve C.

As before, we should check that the rules above are locally sound and complete. First, the local reduction

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1 \Vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta_2 \Vdash B\end{array}}{\Delta_1, \Delta_2 \Vdash A \otimes B}\otimes I \qquad \begin{array}{c}\mathcal{E}\\ \Delta', u{:}A, w{:}B \Vdash C\end{array}}{\Delta_1, \Delta_2, \Delta' \Vdash C}\otimes E \quad \Longrightarrow_R \quad \begin{array}{c}[\mathcal{D}_1/u][\mathcal{D}_2/w]\mathcal{E}\\ \Delta_1, \Delta_2, \Delta' \Vdash C\end{array}$$

which requires two substitutions for linear hypotheses and the application of the substitution principle. The derivation on the right shows that the elimination rules are not too strong.

For local completeness we have the following expansion.

$$\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A \otimes B\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A \otimes B\end{array} \qquad \dfrac{\overline{u{:}A \Vdash A}\,u \qquad \overline{w{:}B \Vdash B}\,w}{u{:}A, w{:}B \Vdash A \otimes B}\otimes I}{\Delta \Vdash A \otimes B}\otimes E$$

The derivation on the right verifies that the elimination rules are strong enough so that the simultaneous conjunction can be reconstituted from the parts we obtain from the elimination rule.

Alternative Conjunction. Next we come to *alternative conjunction* $A \& B$ (pronounced "*A with B*"). It is sometimes also called *internal choice*. In its introduction rule, the resources are made available in both premises, since we have to make a choice which among A and B we want to achieve.

$$\dfrac{\Delta \Vdash A \qquad \Delta \Vdash B}{\Delta \Vdash A \& B}\& I$$

Consequently, if we have a resource $A \& B$, we can recover either A or B, but not both simultaneously. Therefore we have two elimination rules.

$$\dfrac{\Delta \Vdash A \& B}{\Delta \Vdash A}\& E_L \qquad \dfrac{\Delta \Vdash A \& B}{\Delta \Vdash B}\& E_R$$

The local reductions formalize the reasoning above.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \Delta \Vdash B\end{array}}{\Delta \Vdash A \& B}\& I}{\Delta \Vdash A}\& E_L \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{D}\\ \Delta \Vdash A\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \Delta \Vdash B\end{array}}{\Delta \Vdash A \& B}\,\&\mathrm{I}}{\Delta \Vdash B}\,\&\mathrm{E_R} \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{E}\\ \Delta \Vdash B\end{array}$$

We may recognize these rules from intuitionistic natural deduction, where the assumptions are also available in both premises. The embedding of unrestricted intuitionistic logic in linear logic will therefore map intuitionistic conjunction $A \wedge B$ to alternative conjunction $A \& B$. The expansion is also already familiar.

$$\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A\&B\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A\&B\end{array}}{\Delta \Vdash A}\,\&\mathrm{E_L} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A\&B\end{array}}{\Delta \Vdash B}\,\&\mathrm{E_R}}{\Delta \Vdash A\&B}\,\&\mathrm{I}$$

Linear Implication. The *linear implication* or *resource implication* internalizes the linear hypothetical judgment at the level of propositions. We $A \multimap B$ (pronounced "*A linearly implies B*" or "*A lolli B*") for the goal of achieving B with resource A.

$$\dfrac{\Delta, w{:}A \Vdash B}{\Delta \Vdash A \multimap B}\,\multimap\mathrm{I}$$

If we know $A \multimap B$ we can obtain B from a derivation of A.

$$\dfrac{\Delta \Vdash A \multimap B \qquad \Delta' \Vdash A}{\Delta, \Delta' \Vdash B}\,\multimap\mathrm{E}$$

As in the case for simultaneous conjunction, we have to split the resources, devoting Δ to achieving $A \multimap B$ and Δ' to achieving A.

The local reduction carries out the expected substitution for the linear hypothesis.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta, w{:}A \Vdash B\end{array}}{\Delta \Vdash A \multimap B}\,\multimap\mathrm{I} \qquad \begin{array}{c}\mathcal{E}\\ \Delta' \Vdash A\end{array}}{\Delta, \Delta' \Vdash B}\,\multimap\mathrm{E} \quad \Longrightarrow_R \quad \begin{array}{c}[\mathcal{E}/w]\mathcal{D}\\ \Delta, \Delta' \Vdash B\end{array}$$

The rules are also locally complete, as witnessed by the local expansion.

$$\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A \multimap B\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A \multimap B\end{array} \qquad \dfrac{}{w{:}A \Vdash A}\,w}{\Delta, w{:}A \Vdash B}\,\multimap\mathrm{E}}{\Delta \Vdash A \multimap B}\,\multimap\mathrm{I}$$

Unit. The trivial goal which requires no resources is written as **1**.

$$\frac{}{\cdot \Vdash \mathbf{1}}\ \mathbf{1}\mathrm{I}$$

If we can achieve **1** from some resources Δ we know that we can consume all those resources.

$$\frac{\Delta \Vdash \mathbf{1} \qquad \Delta' \Vdash C}{\Delta, \Delta' \Vdash C}\ \mathbf{1}\mathrm{E}$$

The rules above and the local reduction and expansion can be seen as a case of 0-ary simultaneous conjunction. In particular, we will see that $\mathbf{1} \otimes A$ is equivalent to A.

$$\frac{\dfrac{}{\cdot \Vdash \mathbf{1}}\ \mathbf{1}\mathrm{I} \qquad \begin{array}{c}\mathcal{E}\\ \Delta' \Vdash C\end{array}}{\Delta' \Vdash C}\ \mathbf{1}\mathrm{E} \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{E}\\ \Delta' \Vdash C\end{array}$$

$$\begin{array}{c}\mathcal{D}\\ \Delta \Vdash \mathbf{1}\end{array} \quad \Longrightarrow_E \quad \frac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash \mathbf{1}\end{array} \qquad \dfrac{}{\cdot \Vdash \mathbf{1}}\ \mathbf{1}\mathrm{I}}{\Delta \Vdash \mathbf{1}}\ \mathbf{1}\mathrm{E}$$

Top. There is also a goal which consumes all resources. It is the unit of alternative conjunction and follows the laws of intuitionistic truth.

$$\frac{}{\Delta \Vdash \top}\ \top\mathrm{I}$$

There is no elimination rule for $\top$ and consequently no local reduction (it is trivially locally sound). The local expansion replaces an arbitrary derivation by the introduction rule.

$$\begin{array}{c}\mathcal{D}\\ \Delta \Vdash \top\end{array} \quad \Longrightarrow_E \quad \frac{}{\Delta \Vdash \top}\ \top\mathrm{I}$$

Disjunction. The *disjunction* $A \oplus B$ (also called *external choice*) is characterized by two introduction rules.

$$\frac{\Delta \Vdash A}{\Delta \Vdash A \oplus B}\ \oplus\mathrm{I}_{\mathrm{L}} \qquad \frac{\Delta \Vdash B}{\Delta \Vdash A \oplus B}\ \oplus\mathrm{I}_{\mathrm{R}}$$

As in the case for intuitionistic disjunction, we therefore have to distinguish two cases when we know that we can achieve $A \oplus B$.

$$\frac{\Delta \Vdash A \oplus B \qquad \Delta', u{:}A \Vdash C \qquad \Delta', w{:}B \Vdash C}{\Delta, \Delta' \Vdash C}\ \oplus\mathrm{E}$$

Note that resources Δ' appear in both branches, since only one of those two derivations will actually be used to achieve C, depending on the derivation of $A \oplus B$. This can be seen from the local reductions.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A\end{array}}{\Delta \Vdash A \oplus B}\oplus\mathrm{I_L} \quad \begin{array}{c}\mathcal{E}\\ \Delta', u{:}A \Vdash C\end{array} \quad \begin{array}{c}\mathcal{F}\\ \Delta', w{:}B \Vdash C\end{array}}{\Delta, \Delta' \Vdash C}\oplus\mathrm{E} \quad \Longrightarrow_R \quad \begin{array}{c}[\mathcal{D}/u]\mathcal{E}\\ \Delta, \Delta' \Vdash C\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash B\end{array}}{\Delta \Vdash A \oplus B}\oplus\mathrm{I_L} \quad \begin{array}{c}\mathcal{E}\\ \Delta', u{:}A \Vdash C\end{array} \quad \begin{array}{c}\mathcal{F}\\ \Delta', w{:}B \Vdash C\end{array}}{\Delta, \Delta' \Vdash C}\oplus\mathrm{E} \quad \Longrightarrow_R \quad \begin{array}{c}[\mathcal{D}/w]\mathcal{F}\\ \Delta, \Delta' \Vdash C\end{array}$$

The local expansion is straightforward.

$$\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A \oplus B\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A \oplus B\end{array} \quad \dfrac{\overline{u{:}A \Vdash A}^{\,u}}{u{:}A \Vdash A \oplus B}\vee\mathrm{I_L} \quad \dfrac{\overline{w{:}B \Vdash B}^{\,w}}{w{:}B \Vdash A \oplus B}\vee\mathrm{I_R}}{\Delta \Vdash A \oplus B}\vee\mathrm{E}$$

Impossibility. The *impossibility* $\mathbf{0}$ is the case of a disjunction between zero alternatives and the unit of $\oplus$. There is no introduction rule. In the elimination rule we have to consider no branches.

$$\dfrac{\Delta \Vdash \mathbf{0}}{\Delta, \Delta' \Vdash C}\mathbf{0}\mathrm{E}$$

There is no local reduction, since there is no introduction rule. However, as in the case of falsehood in intuitionistic logic, we have a local expansion.

$$\begin{array}{c}\mathcal{D}\\ \Delta \Vdash \mathbf{0}\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash \mathbf{0}\end{array}}{\Delta \Vdash \mathbf{0}}\mathbf{0}\mathrm{E}$$

Universal Quantification. Quantifiers do not interact much with linearity. We say $\forall x.\ A$ is true if $[a/x]A$ is true for an arbitrary a. This is an example of a *parametric judgment* that we will discuss in more detail in Section **??**.

$$\dfrac{\Delta \Vdash [a/x]A}{\Delta \Vdash \forall x.\ A}\forall\mathrm{I}^a \qquad \dfrac{\Delta \Vdash \forall x.\ A}{\Delta \Vdash [t/x]A}\forall\mathrm{E}$$

The label a on the introduction rule is a reminder the parameter a must be "new", that is, it may not occur in Δ or $\forall x.\ A$. In other words, the derivation of the premise must *parametric in a*. The local reduction carries out the substitution for the parameter.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Delta \Vdash [a/x]A\end{matrix}}{\Delta \Vdash \forall x.\, A}\,\forall\mathrm{I}^a}{\Delta \Vdash [t/x]A}\,\forall\mathrm{E} \quad \Longrightarrow_R \quad \begin{matrix}[t/a]\mathcal{D}\\ \Delta \Vdash [t/x]A\end{matrix}$$

Here, $[t/a]\mathcal{D}$ is our notation for the result of substituting t for the parameter a throughout the deduction $\mathcal{D}$. For this substitution to preserve the conclusion, we must know that a does not already occur in $\forall x.\, A$ or Δ. The local expansion for universal quantification is even simpler.

$$\begin{matrix}\mathcal{D}\\ \Delta \Vdash \forall x.\, A\end{matrix} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Delta \Vdash \forall x.\, A\end{matrix}}{\Delta \Vdash [a/x]A}\,\forall\mathrm{E}}{\Delta \Vdash \forall x.\, A}\,\forall\mathrm{I}^a$$

Existential Quantification. Again, this does not interact very much with resources.

$$\dfrac{\Delta \Vdash [t/x]A}{\Delta \Vdash \exists x.\, A}\,\exists\mathrm{I} \qquad \dfrac{\Delta \Vdash \exists x.\, A \qquad \Delta', w{:}[a/x]A \Vdash C}{\Delta, \Delta' \Vdash C}\,\exists\mathrm{E}^a$$

The second premise of the elimination rule must be parametric in a, which is indicated by the superscript a. In the local reduction we will substitute for this parameter.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Delta \Vdash [t/x]A\end{matrix}}{\Delta \Vdash \exists x.\, A}\,\exists\mathrm{I} \qquad \begin{matrix}\mathcal{E}\\ \Delta', u{:}[a/x]A \Vdash C\end{matrix}}{\Delta, \Delta' \Vdash C}\,\exists\mathrm{E}^a \quad \Longrightarrow_R \quad \begin{matrix}[\mathcal{D}/u][t/a]\mathcal{E}\\ \Delta, \Delta' \Vdash C\end{matrix}$$

The proviso on occurrences of a guarantees that the conclusion and hypotheses of $[t/a]\mathcal{E}$ have the correct form. The local expansion for existential quantification is also similar to the case for disjunction.

$$\begin{matrix}\mathcal{D}\\ \Delta \Vdash \exists x.\, A\end{matrix} \quad \Longrightarrow_E \quad \dfrac{\begin{matrix}\mathcal{D}\\ \Delta \Vdash \exists x.\, A\end{matrix} \qquad \dfrac{\dfrac{}{u{:}[a/x]A \Vdash [a/x]A}\,u}{u{:}[a/x]A \Vdash \exists x.\, A}\,\exists\mathrm{I}}{\Delta \Vdash \exists x.\, A}\,\exists\mathrm{E}^a$$

This concludes the purely linear operators. Negation and another version of falsehood are postponed to Section **??**, since they may be formally definable, but their interpretation is somewhat questionable in the context we have established so far.

The connectives we have introduced may be classified as to whether the resources are split among the premises or distributed to the premises. Connectives of the former kind are called *multiplicative*, the latter *additive*. For example, we might refer to simultaneous conjunction also as *multiplicative conjunction* and to *alternative conjunction* as *additive conjunction*. When we line up the operators against each other, we notice some gaps. For example, there seems to be only a multiplicative implication, but no additive implication. Dually, there seems to be only an additive disjunction, but no multiplicative disjunction. This is not an accident and is pursued further in Exercise 2.4.

2.4 Unrestricted Hypotheses in Linear Logic

So far, the main judgment permits only linear hypotheses. This means that the logic is too weak to embed ordinary intuitionistic or classical logic, and we have failed so far to design a true extension. In order to accomodate ordinary intuitionistic or classical reasoning, we introduce a new judgment, "*A is valid*", written A *valid*. We say that A is valid if A is true, independently of the any resources. This means we must be able to prove A without any resources. More formally:

Validity
A *valid* if $\cdot \Vdash A$ *true*.

Note that validity is not a primitive, but a notion derived from truth and linear hypothetical judgments. The judgment $\cdot \Vdash A$ *true* is an example of a *categorical judgment* that asserts independence from hypotheses and also arises in modal logic [PD01]. We can see that, for example, $A \multimap A$ *valid* and $(A \& B) \multimap A$ *valid* for any propositions A and B.

Validity by itself is a completely straightforward judgment. But matters become interesting when we admit *hypotheses* about the validity of propositions. What laws should govern such hypotheses? Let us assume A *valid*, which means that $\cdot \Vdash A$ *true*. First note, that obtaining an instance of A can be achieved without requiring any resources. This means we can generate as many copies of the resource A as we wish, or we may decide not to generate any copies at all. In other words, uses of an assumption A *valid* should be *unrestricted* rather than linear. If we use "$\vdash$" to separate unrestricted hypotheses from a judgment we are trying to deduce, then our main judgment would have the form

$$B_1 \textit{ valid}, \ldots, B_m \textit{ valid} \vdash (A_1 \textit{ true}, \ldots, A_n \textit{ true} \Vdash C \textit{ true})$$

which may be read: *under the assumption that $B_1, \ldots, B_m$ are valid and $A_1, \ldots, A_n$ are true we can prove C*. Alternatively, we could say: *with inexhaustible resources $B_1, \ldots, B_m$ and linear resources $A_1, \ldots, A_n$ we can achieve goal C.*

Instead, we will stick with a more customary way of writing this dual hypothetical judgment form by separating the two forms of assumption by a semi-colon ";". As before, we also label assumptions of either kind with distint variables.

$$(v_1{:}B_1\ \mathit{valid}, \ldots, v_m{:}B_m\ \mathit{valid}); (u_1{:}A_1\ \mathit{true}, \ldots, u_n{:}A_n\ \mathit{valid}) \vdash C\ \mathit{true}$$

It is critical to remember that the first collection of assumptions is unrestricted while the second collection is linear. We abbreviate unrestricted assumptions by Γ and linear assumptions by Δ.

The valid assumptions are independent of the state and can therefore be used freely when proving other valid assumptions. That is,

Validity under Hypotheses
$\Gamma \vdash A\ \mathit{valid}$ if $\Gamma; \cdot \vdash A\ \mathit{true}$.

From this definition we can directly derive a new form of the substitution principle.

Substitution Principle for Validity
If $\Gamma; \cdot \vdash A\ \mathit{true}$ and $(\Gamma, v{:}A\ \mathit{valid}); \Delta \vdash C\ \mathit{true}$ then $\Gamma; \Delta \vdash C\ \mathit{true}$.

Note that the same unrestricted hypotheses Γ appear in the first two jugments, which contrasts with the linear substitution principle where the linear hypotheses are disjoint. This reflects the fact that assumptions in Γ may be used arbitrarily many times in a proof. Note also that the first judgment expresses $\Gamma \vdash A\ \mathit{valid}$, which is necessary so we can substitute for the assumption that $A\ \mathit{valid}$. For a counterexample see Exercise 2.1.

We also have a new hypothesis rule which stems from the definition of validity: if A is valid than it definitely must be true.

Unrestricted Hypothesis Rule

$$\frac{}{(\Gamma, v{:}A\ \mathit{valid}); \cdot \vdash A\ \mathit{true}}\, v$$

Note that there may not be any linear hypotheses (which would be unused), but there may be additional unrestricted hypotheses since they need not be used.

We now restate the original substitution principle and hypothesis rules for our more general judgment. Their form is determined by the unrestricted nature of the validity assumptions. We assume that comma binds more tightly than semi-colon, but may still parenthesize hypotheses to make the judgments more easily readable.

Substitution Principle for Truth
If $\Gamma; \Delta \vdash A\ \mathit{true}$ and $\Gamma; (\Delta', u{:}A\ \mathit{true}) \vdash C\ \mathit{true}$ then $\Gamma; (\Delta', \Delta) \vdash C\ \mathit{true}$.

Hypothesis Rule

$$\frac{}{\Gamma; u{:}A\ \mathit{true} \vdash A\ \mathit{true}}\ u$$

All the rules we presented for pure linear logic so far are extended by adding the unrestricted context to premises and conclusion (see the rule summary on page 24). At this point, for example, we can capture the blocks work example completely inside linear logic. The idea is that the proposition stating the legal moves do not depend on the current state and are therefore given in Γ.

Returning to the blocks world example, a planning problem is now represented as judgment

$$\Gamma_0; \Delta_0 \vdash A_0$$

where Γ_0 represent the rules which describe the legal operations, Δ_0 is the initial state represented as a context of the propositions which are true, and A is the goal to be achieved. For example, the initial state considered early would be represented by

$$\Delta_0 = \mathrm{empty}, \mathrm{tb}(a), \mathrm{on}(b,a), \mathrm{clear}(b), \mathrm{tb}(c), \mathrm{clear}(c)$$

where we have omitted labels for the sake of brevity. The rules are represented by unrestricted hypotheses, since they may be used arbitrarily often in the course of solving a problem. We use the following for rules for picking up or putting down an object. We use the convention that simultaneous conjunction $\otimes$ binds more tightly than linear implication $\multimap$.

$$\begin{array}{lll}
\Gamma_0 = & & \\
\mathrm{geton} & : & \forall x.\, \forall y.\, \mathrm{empty} \otimes \mathrm{clear}(x) \otimes \mathrm{on}(x,y) \multimap \mathrm{holds}(x) \otimes \mathrm{clear}(y), \\
\mathrm{gettb} & : & \forall x.\, \mathrm{empty} \otimes \mathrm{clear}(x) \otimes \mathrm{tb}(x) \multimap \mathrm{holds}(x), \\
\mathrm{puton} & : & \forall x.\, \forall y.\, \mathrm{holds}(x) \otimes \mathrm{clear}(y) \multimap \mathrm{empty} \otimes \mathrm{on}(x,y) \otimes \mathrm{clear}(x), \\
\mathrm{puttb} & : & \forall x.\, \mathrm{holds}(x) \multimap \mathrm{empty} \otimes \mathrm{tb}(x) \otimes \mathrm{clear}(x).
\end{array}$$

Each of these represents a particular possible action, assuming that it can be carried out successfully. Matching the left-hand side of one these rules will consume the corresponding resources so that, for example, the proposition *empty* with no longer be available after the *geton* action has been applied.

The goal that we would like to achieve $\mathrm{on}(a,b)$, for example, is represented with the aid of using $\top$.

$$A_0 = \mathrm{on}(a,b) \otimes \top$$

Any derivation of the judgment

$$\Gamma_0; \Delta_0 \vdash A_0$$

represents a plan for achieving the goal A_0 from the initial situation state Δ_0.

We now go through a derivation of the particular example above, omitting the unrestricted resources Γ_0 which do not change throughout the derivation. Our first goal is to derive

$$\mathrm{empty}, \mathrm{tb}(a), \mathrm{on}(b,a), \mathrm{clear}(b), \mathrm{tb}(c), \mathrm{clear}(c), \mathrm{empty} \vdash \mathrm{on}(a,b) \otimes \top$$

By using $\otimes$I twice we can prove

$$\mathrm{empty}, \mathrm{on}(b,a), \mathrm{clear}(b) \vdash \mathrm{empty} \otimes \mathrm{clear}(b) \otimes \mathrm{on}(b,a)$$

Using the unrestricted hypothesis rule for *geton* followed by $\forall$E twice and $\multimap$E we obtain

$$\mathrm{empty}, \mathrm{clear}(b), \mathrm{on}(b,a) \vdash \mathrm{holds}(b) \otimes \mathrm{clear}(a)$$

Now we use $\otimes$E with the derivation above as our left premise, to prove our overall goal, leaving us with the goal to derive

$$\mathrm{tb}(a), \mathrm{tb}(c), \mathrm{clear}(c), \mathrm{holds}(b), \mathrm{clear}(a) \vdash \mathrm{on}(a,b) \otimes \top$$

as our right premise. Observe how the original resources Δ_0 have been split between the two premises, and the results from the left premise derivation, $\mathrm{holds}(b)$ and $\mathrm{clear}(a)$ have been added to the description of the situation. The new subgoal has exactly the same form as the original goal (in fact, the conclusion has not changed), but applying the unrestricted assumption *geton* has changed our state.

Proceeding in the same manner, using the rule *puttb* next leaves us with the subgoal

$$\mathrm{tb}(a), \mathrm{tb}(c), \mathrm{clear}(c), \mathrm{clear}(a), \mathrm{empty}, \mathrm{clear}(b), \mathrm{tb}(b) \vdash \mathrm{on}(a,b) \otimes \top$$

We now apply *gettb* using a for x and proceeding as above which gives us a derivation of $\mathrm{holds}(a)$. Instead of $\otimes$E, we now use the substitution principle yielding the subgoal

$$\mathrm{tb}(c), \mathrm{clear}(c), \mathrm{clear}(b), \mathrm{tb}(b), \mathrm{holds}(a) \vdash \mathrm{on}(a,b) \otimes \top$$

With same technique, this time using *puton*, we obtain the subgoal

$$\mathrm{tb}(c), \mathrm{clear}(c), \mathrm{tb}(b), \mathrm{empty}, \mathrm{on}(a,b), \mathrm{clear}(a) \vdash \mathrm{on}(a,b) \otimes \top$$

Now we can conclude the derivation with the $\otimes$I rule, distributing resource $\mathrm{on}(a,b)$ to the left premise, which follows immediately as hypothesis, and distributing the remaining resources to the right premise, where $\top$ follows by $\top$I, ignoring all resources.

Note that different derivations of the original judgment represent different sequences of actions (see Exercise 2.5).

Even though it is not necessary in the blocks world example, in order to embed full intuitionistic (or classical) logic into linear logic, we need connectives that allows us to make unrestricted assumptions. We show two operators of this form. The first is *unrestricted implication*, the second a modal operator expressing validity as a proposition.

Unrestricted Implication. The proof of an unrestricted implication $A \supset B$ allows an unrestricted assumption A *valid* while proving that B is true.

$$\frac{(\Gamma, u{:}A); \Delta \vdash B}{\Gamma; \Delta \vdash A \supset B} \supset\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash A \supset B \qquad \Gamma; \cdot \vdash A}{\Gamma; \Delta \vdash B} \supset\mathrm{E}$$

In the elimination we have to be careful to postulate the *validity* of A rather than just its truth, expressed by requiring that there are no linear hypotheses. The local reduction uses the substitution principle for unrestricted hypotheses.

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\(\Gamma, u{:}A); \Delta \vdash B\end{array}}{\Gamma; \Delta \vdash A \supset B} \supset\mathrm{I}^u \qquad \begin{array}{c}\mathcal{E}\\\Gamma; \cdot \vdash A\end{array}}{\Gamma; \Delta \vdash B} \supset\mathrm{E} \quad \Longrightarrow_R \quad \begin{array}{c}[\mathcal{D}/u]\mathcal{E}\\\Gamma; \Delta \vdash B\end{array}$$

In Exercise 2.2 you are asked to show that the rules would be locally unsound (that is, local reduction is not possible), if the second premise in the elimination rule would be allowed to depend on linear hypotheses. The local expansion requires "weakening", that is, adding unused, unrestricted hypotheses.

$$\begin{array}{c}\mathcal{D}\\\Gamma; \Delta \vdash A \supset B\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\(\Gamma, u{:}A); \Delta \vdash A \supset B\end{array} \qquad \dfrac{}{(\Gamma, u{:}A); \cdot \vdash A}\,u}{(\Gamma, u{:}A); \Delta \vdash B} \supset\mathrm{E}}{\Gamma; \Delta \vdash A \supset B} \supset\mathrm{I}^u$$

Here, $\mathcal{D}'$ is constructed from $\mathcal{D}$ by adjoining the unused hypothesis u to every judgment, which does not affect the structure of the derivation.

"Of Course" Modality. Girard [Gir87] observed that there is an alternative way to connect unrestricted and linear hypotheses by internalizing the notion of validity via a modal operator $!A$, pronounced "*of course* A" or "*bang* A".

$$\frac{\Gamma; \cdot \vdash A}{\Gamma; \cdot \vdash\, !A}\, !\mathrm{I}$$

The elimination rule states that if we can derive $\vdash\, !A$ than we are allowed to use A as an unrestricted hypothesis.

$$\frac{\Gamma; \Delta \vdash\, !A \qquad (\Gamma, v{:}A); \Delta' \vdash C}{\Gamma; (\Delta, \Delta') \vdash C}\, !\mathrm{E}$$

This pair of rules is locally sound and complete via substitution for a valid assumption.

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\\Gamma; \cdot \vdash A\end{array}}{\Gamma; \cdot \vdash\, !A}\, !\mathrm{I} \qquad \begin{array}{c}\mathcal{E}\\(\Gamma, v{:}A); \Delta' \vdash C\end{array}}{\Gamma; \Delta' \vdash C}\, !\mathrm{E} \quad \Longrightarrow_R \quad \begin{array}{c}[\mathcal{D}/v]\mathcal{E}\\\Gamma; \Delta' \vdash C\end{array}$$

$$\begin{array}{c}\mathcal{D}\\ \Gamma;\Delta\vdash\,!A\end{array} \quad\Longrightarrow_E\quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma;\Delta\vdash\,!A\end{array} \qquad \dfrac{\overline{(\Gamma,v{:}A);\cdot\vdash A}\;v}{(\Gamma,v{:}A);\cdot\vdash\,!A}\;!\mathrm{I}}{\Gamma;\Delta\vdash\,!A}\;!\mathrm{E}$$

Using the *of course* modality, one can define the unrestricted implication $A \supset B$ as $(!A) \multimap B$. It was this observation which gave rise to Girard's development of linear logic. Under this interpretation, the introduction and elimination rules for unrestricted implication are *derived rules of inference* (see Exercise 2.3).

We now summarize the rules of intuitionistic linear logic. A very similar calculus was developed and analyzed in the categorical context by Barber [Bar96]. It differs from more traditional treatments by Abramsky [Abr93], Troelstra [Tro93], Bierman [Bie94] and Albrecht et al. [ABCJ94] in that structural rules remain completely implicit. The logic we consider here comprises the following logical operators.

$$\begin{array}{lllll}
\text{Propositions} & A & ::= & P & \text{Atoms}\\
 & & & \mid A_1 \multimap A_2 \mid A_1 \otimes A_2 \mid \mathbf{1} & \text{Multiplicatives}\\
 & & & \qquad \mid A_1 \& A_2 \mid \top \mid A_1 \oplus A_2 \mid \mathbf{0} & \text{Additives}\\
 & & & \mid \forall x.\ A \mid \exists x.\ A & \text{Quantifiers}\\
 & & & \mid A \supset B \mid\ !A & \text{Exponentials}
\end{array}$$

Recall that the order of both linear and unrestricted hypotheses is irrelevant, and that all hypothesis label in a judgment must be distinct.

Hypotheses.

$$\dfrac{}{\Gamma;u{:}A\vdash A}\;u \qquad \dfrac{}{(\Gamma,v{:}A);\cdot\vdash A}\;v$$

Multiplicative Connectives.

$$\dfrac{\Gamma;\Delta_1\vdash A \qquad \Gamma;\Delta_2\vdash B}{\Gamma;(\Delta_1,\Delta_2)\vdash A\otimes B}\;\otimes\mathrm{I} \qquad \dfrac{\Gamma;\Delta\vdash A\otimes B \quad \Gamma;(\Delta',u{:}A,w{:}B)\vdash C}{\Gamma;(\Delta,\Delta')\vdash C}\;\otimes\mathrm{E}$$

$$\dfrac{\Gamma;(\Delta,u{:}A)\vdash B}{\Gamma;\Delta\vdash A\multimap B}\;\multimap\mathrm{I} \qquad \dfrac{\Gamma;\Delta\vdash A\multimap B \qquad \Gamma;\Delta'\vdash A}{\Gamma;(\Delta,\Delta')\vdash B}\;\multimap\mathrm{E}$$

$$\dfrac{}{\Gamma;\cdot\vdash\mathbf{1}}\;\mathbf{1}\mathrm{I} \qquad \dfrac{\Gamma;\Delta\vdash\mathbf{1} \qquad \Gamma;\Delta'\vdash C}{\Gamma;(\Delta,\Delta')\vdash C}\;\mathbf{1}\mathrm{E}$$

Additive Connectives.

$$\frac{\Gamma; \Delta \vdash A \qquad \Gamma; \Delta \vdash B}{\Gamma; \Delta \vdash A \& B}\,\&\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash A \& B}{\Gamma; \Delta \vdash A}\,\&\mathrm{E_L} \qquad \frac{\Gamma; \Delta \vdash A \& B}{\Gamma; \Delta \vdash B}\,\&\mathrm{E_R}$$

$$\frac{}{\Gamma; \Delta \vdash \top}\,\top\mathrm{I} \qquad \textit{no } \top \textit{ elimination}$$

$$\frac{\Gamma; \Delta \vdash A}{\Gamma; \Delta \vdash A \oplus B}\,\oplus\mathrm{I_L} \qquad \frac{\Gamma; \Delta \vdash B}{\Gamma; \Delta \vdash A \oplus B}\,\oplus\mathrm{I_R} \qquad \frac{\Gamma; \Delta \vdash A \oplus B \quad \Gamma; (\Delta', u{:}A) \vdash C \quad \Gamma; (\Delta', w{:}B) \vdash C}{\Gamma; (\Delta, \Delta') \vdash C}\,\oplus\mathrm{E}$$

$$\textit{no } \mathbf{0} \textit{ introduction} \qquad \frac{\Gamma; \Delta \vdash \mathbf{0}}{\Gamma; (\Delta, \Delta') \vdash C}\,\mathbf{0}\mathrm{E}$$

Quantifiers.

$$\frac{\Gamma; \Delta \vdash [a/x]A}{\Gamma; \Delta \vdash \forall x.\ A}\,\forall\mathrm{I}^a \qquad \frac{\Gamma; \Delta \vdash \forall x.\ A}{\Gamma; \Delta \vdash [t/x]A}\,\forall\mathrm{E}$$

$$\frac{\Gamma; \Delta \vdash [t/x]A}{\Gamma; \Delta \vdash \exists x.\ A}\,\exists\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash \exists x.\ A \qquad \Gamma; (\Delta', u{:}[a/x]A) \vdash C}{\Gamma; (\Delta, \Delta') \vdash C}\,\exists\mathrm{E}^a$$

Exponentials.

$$\frac{(\Gamma, v{:}A); \Delta \vdash B}{\Gamma; \Delta \vdash A \supset B}\,\supset\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash A \supset B \qquad \Gamma; \cdot \vdash A}{\Gamma; \Delta \vdash B}\,\supset\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash A}{\Gamma; \cdot \vdash\, !A}\,!\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash\, !A \qquad (\Gamma, v{:}A); \Delta' \vdash C}{\Gamma; (\Delta', \Delta) \vdash C}\,!\mathrm{E}$$

We close this section with another example that exploits the connectives of linear logic. The first example is a *menu* consisting of various courses which can be obtained for 200 French Francs.

Menu A: FF 200	$\mathrm{FF}(200) \multimap$
Onion Soup or *Clear Broth*	$((\mathrm{OS} \& \mathrm{CB})$
Honey-Glazed Duck	$\otimes\, \mathrm{HGD}$
Peas or *Red Cabbage* (according to season)	$\otimes\, (\mathrm{P} \oplus \mathrm{RC})$
New Potatoes	$\otimes\, \mathrm{NP}$
Chocolate Mousse (FF 30 extra)	$\otimes\, ((\mathrm{FF}(30) \multimap \mathrm{CM}) \& \mathbf{1})$
Coffee (unlimited refills)	$\otimes\, \mathrm{C}$ $\otimes\, (!\mathrm{C}))$

Note the two different informal uses of "or", one modelled by an alternative conjunction and one by a disjunction. The option of ordering chocolate mousse is also represented by an alternative conjunction: we can choose $(\mathrm{FF}(30) \multimap \mathrm{CM}) \& \mathbf{1}$ to obtain nothing ($\mathbf{1}$) or pay another 30 francs to obtain the mousse.

2.5 Exercises

Exercise 2.1 Give a counterexample that shows that the restriction to empty linear hypotheses in the substitution principle for validity is necessary.

Exercise 2.2 Give a counterexample which shows that the elimination $\supset$E would be locally unsound if its second premise were allowed to depend on linear hypotheses.

Exercise 2.3 If we *define* unrestricted implication $A \supset B$ in linear logic as an abbreviation for $(!A) \multimap B$, then the given introduction and elimination rules become *derived rules of inference*. Prove this by giving a derivation for the conclusion of the $\supset$E rule from its premises under the interpretation, and similarly for the $\supset$I rule.

For the other direction, show how $!A$ could be defined from unrestricted implication or speculate why this might not be possible.

Exercise 2.4 Speculate bout the "missing connectives" of multiplicative disjunction, multiplicative falsehood, and additive implication. What would the introduction and elimination rules look like? What is the difficulty? any ideas for how these difficulties might be overcome?

Exercise 2.5 In the blocks world example, sketch the derivation for the same goal A_0 and initial situation Δ_0 in which block b is put on block c, rather than the table.

Exercise 2.6 Model the *Towers of Hanoi* in linear logic in analogy with our modelling of the blocks world.

1. Define the necessary atomic propositions and their meaning.
2. Describe the legal moves in *Towers of Hanoi* as unrestricted hypotheses Γ_0 independently from the number of towers or disks.
3. Represent the initial situation of three towers, where two are empty and one contains two disks in a legal configuration.
4. Represent the goal of legally stacking the two disks on some arbitrary other tower.
5. Sketch the proof for the obvious 3-move solution.

Exercise 2.7 Consider if $\otimes$ and $\&$ can be distributed over $\oplus$ or *vice versa*. There are four different possible equivalences based on eight possible entailments. Give natural deductions for the entailments which hold.

Exercise 2.8 In this exercise we explore distributive and related *interaction laws* for linear implication. In intuitionistic logic, for example, we have the following $(A \wedge B) \supset C \dashv\vdash A \supset (B \supset C)$ and $A \supset (B \wedge C) \dashv\vdash (A \supset B) \wedge (A \supset C)$, where $\dashv\vdash$ is mutual entailment as in Exercise **??**.

In linear logic, we now write $A \dashv\vdash A'$ for linear mutual entailment, that is, A' follows from linear hypothesis A and *vice versa*. Write out appropriate interaction laws or indicate none exists, for each of the following propositions.

1. $A \multimap (B \otimes C)$
2. $(A \otimes B) \multimap C)$
3. $A \multimap \mathbf{1}$
4. $\mathbf{1} \multimap A$
5. $A \multimap (B \& C)$
6. $(A \& B) \multimap C$
7. $A \multimap \top$
8. $\top \multimap A$
9. $A \multimap (B \oplus C)$
10. $(A \oplus B) \multimap C$

11. $A \multimap \mathbf{0}$

12. $\mathbf{0} \multimap A$

13. $A \multimap (B \multimap C)$

14. $(A \multimap B) \multimap C$

Note that an interaction law exists only if there is a mutual linear entailment—we are not interested if one direction holds, but not the other.

Give the derivations in both directions for one of the interaction laws of a binary connective $\otimes$, $\&$, $\oplus$, or $\multimap$, and for one of the interaction laws of a logical constant $\mathbf{1}$, $\top$, or $\mathbf{0}$.

Exercise 2.9 Consider three forms of equivalence of propositions in linear logic.

- $A \multimapboth B$ which should be true if A linearly implies B and vice versa.
- $A \simeq B$ which should be true if, independently of any linear hypotheses, A linearly implies B and vice versa.
- $A \equiv B$ which should be true if A implies B and B implies A, where both implications are unrestricted.

1. For each of the these connectives, give introduction and elimination rules and show local soundness and completeness of your rules. If it is not possible, argue why. Be careful that your rules do *not* refer to other connectives, but rely entirely on judgmental concepts.

2. Discuss if the specification above is unambiguous or if interpretations essentially different from yours may be possible.

3. Using your rules, prove each linear entailment $A \ \mathrm{op}_1 \ B \ \mathit{true} \Vdash A \ \mathrm{op}_2 \ B \ \mathit{true}$ that holds where op_i are equivalence operators.

4. [Extra Credit] Give counterexamples for the entailments that do not hold.

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