Constructive Logic
Fall 2000

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Assignment 5

Out: Friday Oct 13

Due: Thursday Oct 19

The use of the tutch proof checker is recommended for the formal proofs, as soon as it is available for first-order logic and arithmetic. Please check the course home page `http://www.cs.cmu.edu/~fp/courses/logic/` or the course bulletin board for an announcement.

1. First-Order Logic (40 Points)

For each of the following three propositions give a formal proof, using either a tree or a linear notation. In each of the problems we assume τ *type*, $\vdash A$ *prop* and $x{\in}\tau \vdash B(x)$ *prop*. In particular this means that x does not occur in A while it may occur in $B(x)$. Note that implication binds more tightly than universal or existential quantification, so $\forall x{\in}\tau.\ A \supset B(x)$ is the same as $\forall x{\in}\tau.\ (A \supset B(x))$.

1. $(A \supset \forall x{\in}\tau.\ B(x)) \supset (\forall x{\in}\tau.\ A \supset B(x))$.
2. $(\forall x{\in}\tau.\ A \supset B(x)) \supset (A \supset \forall x{\in}\tau.\ B(x))$.
3. $(\exists x{\in}\tau.\ A \supset B(x)) \supset (A \supset \exists x{\in}\tau.\ B(x))$.

For the following proposition, show that it does not have a normal proof (and therefore no proof), unless more about A and B is known.

4. $(A \supset \exists x{\in}\tau.\ B(x)) \supset (\exists x{\in}\tau.\ A \supset B(x))$.

2. Arithmetic (60 Points)

For each of the following propositions, first give an *informal* proof that clearly exhibits the inductive structure of your argument without necessarily showing all the details. You may use propositions proved earlier in later proofs.
Then give a formal proof (a linear form is recommended here). Previously proved propositions may be used by simply repeating them within a derivation. You may hand in that portion of the homework electronically when `tutch` is available for arithmetic.
You will need the inference rules for $n < m$ *true* and $n =_N m$ *true* summarized on Page 57 of the course notes. In this exercise we use the following notational definitions.

$$\begin{array}{rcl} \mathit{plus} & = & \lambda x{\in}\mathbf{nat}.\ \mathbf{rec}\ x \\ & & \quad\quad \mathbf{of}\ p(\mathbf{0}) \Rightarrow \lambda y{\in}\mathbf{nat}.\ y \\ & & \quad\quad\ \mid p(\mathbf{s}(x')) \Rightarrow \lambda y{\in}\mathbf{nat}.\ \mathbf{s}(p(x')\ y) \\ \mathit{less}\ x\ y & = & \exists z{\in}\mathbf{nat}.\ \mathbf{s}(\mathit{plus}\ x\ z) =_N y \end{array}$$

Note that steps of computation occur in the informal, but not the formal proof (they are performed by the proof checker).

1. $\forall x{\in}\mathbf{nat}.\ \forall y{\in}\mathbf{nat}.\ \neg(x < y \wedge y < x)$.
2. $\forall x{\in}\mathbf{nat}.\ 0 < x \supset \exists y{\in}\mathbf{nat}.\ \mathbf{s}(y) =_N x$.
3. $\forall x{\in}\mathbf{nat}.\ \forall y{\in}\mathbf{nat}.\ \mathit{less}\ x\ y \supset x < y$.

3. Alternative Definition of Order (20 Points Extra Credit)

In Problem 2(3) we show one direction of the equivalence between the definition of less-than via introduction and elimination rules, and its definition via addition and existential quantification. In this extra credit problem, you are asked to complete this proof. Again, you should first give an informal proof (exhibiting the inductive structure and possible lemmas you need) and then its formalization.

$$\forall x{\in}\mathbf{nat}.\ \forall y{\in}\mathbf{nat}.\ x < y \supset \mathit{less}\ x\ y$$