

80-310/610:
LOGIC AND COMPUTATION

Midterm Exam

Name: _____

October 19, 2006

Write your answers in the space provided, using the back of the page if necessary. You may use additional scratch paper. Justify your answers, and provide clear, readable explanations.

You may refer to one sheet of notes.

Problem	Points	Score
1	15	
2	20	
3	10	
4	10	
5	25	
6	20	
Total	100	

GOOD LUCK !

Problem 1. Let A be the subset of natural numbers defined inductively by the following clauses:

- $1 \in A$
- If $n \in A$, then $2n \in A$
- If $n \in A$, then $3n + 1 \in A$

Part a) (5 points) Show that 28 is an element of A by giving a construction sequence.

Part b) (5 points) Show that 10 is not an element of A .

Part c) (5 points) Is A freely generated? Justify your answer.

Problem 2. Remember that if φ and ψ are any propositional formulas and p_i is a propositional variable, then $\varphi[\psi/p_i]$ is defined by recursion, as follows:

- If φ is a propositional variable p_j , then

$$\varphi[\psi/p_i] = \begin{cases} \psi & \text{if } i = j \\ p_j & \text{otherwise} \end{cases}$$

- If φ is of the form $(\theta \square \eta)$, then

$$\varphi[\psi/p_i] = (\theta[\psi/p_i] \square \eta[\psi/p_i]).$$

- If φ is of the form $(\neg\theta)$, then

$$\varphi[\psi/p_i] = (\neg\theta[\psi/p_i]).$$

Part a) (5 points) Show that if φ and ψ are any two formulas, and v and v' are truth value assignments such that

$$v'(p_j) = \begin{cases} \llbracket \psi \rrbracket_v & \text{if } i = j \\ v(p_j) & \text{otherwise} \end{cases}$$

then

$$\llbracket \varphi[\psi/p_i] \rrbracket_v = \llbracket \varphi \rrbracket_{v'}.$$

Part b) (5 points) What does it mean for a propositional formula φ to be *valid*?

Part c) (10 points) Show (using parts a and b) that if φ is valid then so is $\varphi[\psi/p_i]$ for any formula ψ and variable p_i .

Problem 3. (10 points) Use algebraic means to put the following formula into conjunctive normal form:

$$\neg((\neg p_0 \wedge p_1) \vee (p_2 \rightarrow p_3))$$

Problem 4. Use truth tables or a semantic argument to show the following.

Part a) (5 points) $(p \wedge r) \rightarrow (q \rightarrow (p \vee r))$ is a tautology.

Part b) (5 points) It is not the case that $p \vee q \models p \wedge q$

Problem 5. Give natural deduction proofs of the following:

Part a) (5 points) $\neg(\varphi \wedge \neg\varphi)$

Part b) (10 points) $(\varphi \rightarrow \psi) \vdash \neg(\varphi \wedge \neg\psi)$

Part c) (10 points) $(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$

Problem 6.

Part a) (5 points) State the Soundness and Completeness Theorems (and say which is which).

Part b) (5 points) Show that the following set of sentences is consistent:

$$\{p \vee q, \neg p \rightarrow r, \neg q\}$$

Part c) (10 points) Suppose that for formulas φ and ψ no truth valuation v makes both of them true. Show that then $\varphi \vdash \neg\psi$.