Homework 9

Due Thursday, December 1

1. Determine whether each of the following arguments is valid. Justify your answer either way.

   (a) All whales are fish.
   Some mammals are not fish.
   No mammals that are whales are fish.
   Therefore, no mammals are whales.

   (b) If you serve in the National Guard, you get paid.
   W got paid for serving in the National Guard.
   Therefore, W served in the National Guard.

2. In the language $\mathcal{L} = \{<\}$ consider the theory $\mathbf{\Delta}$ of the ordering of the Real numbers $\mathbb{R}$,
$$\Delta = \{\sigma \mid \mathbb{R} \models \sigma\}.$$
   Use the downward Löwenheim-Skolem theorem to show that Δ has a countable model $\mathcal{M}$ (i.e. the set $|\mathcal{M}|$ is isomorphic to the natural numbers $\mathbb{N}$). Infer that there is no hope of *fully* axiomatizing the ordering of the Reals in first-order logic (i.e. so that any two models are isomorphic). Would it help to add more relation, function, and constant symbols? Why?

3. In the language $\mathcal{L} = \{0, S, +, \cdot\}$ of Arithmetic (see van Dalen, 2.7, example 6) consider the theory $\mathbf{T}$ of the Natural numbers $\mathbb{N}$,
$$\mathbf{T} = \{\sigma \mid \mathbb{N} \models \sigma\}.$$
   Use the upward Löwenheim-Skolem theorem to show that $\mathbf{T}$ has an uncountable model $\mathcal{N}$ (i.e. the infinite set $|\mathcal{N}|$ is not isomorphic to the natural numbers $\mathbb{N}$). Infer that there is no hope of *fully* axiomatizing the arithmetic of the Naturals in first-order logic (i.e. so that any two models are isomorphic). Would it help to add more relation, function, and constant symbols? Why?

4. In the language $\mathcal{L} = \{0, S, +, \cdot\}$ of Arithmetic consider the theory **PA** of Peano Arithmetic (see van Dalen, 2.7.7). Show that the model $\mathcal{N}$ from the foregoing problem is a "non-standard" model of **PA**, in the sense that (it models **PA** and) there are elements $n \in |\mathcal{N}|$ satisfying all of the infinitely many sentences in the following set Γ:

$$\Gamma = \{0 < n, S(0) < n, S(S(0)) < n, \ldots\}$$

Infer that the set of elements $\{0, S(0), S(S(0)), \ldots\}$ in this model of **PA** is not definable in first-order logic (hint: consider the induction axiom).

5. Use compactness to show that there is a *countable*, non-standard model of **PA** (i.e. one also satisfying Γ above). Can the same line of reasoning be used to find a non-standard model of the theory **T** of "true arithmetic" from problem 3 in place of **PA**? Justify your answer!

$\star$ 6. Do problem 4 on page 134.