Homework 12

Due Monday, December 2

1. Let L be a language with two unary predicates, A and B. Consider the equivalence
$$\forall x(A(x) \vee B(x)) \leftrightarrow \forall x A(x) \vee \forall x B(x).$$
   (a) Show that one direction is valid, using only semantic notions. In particular, your answer should make it clear that you know what "valid" means!
   (b) Show that the other direction is not valid.

2. Find a prenex sentence (i.e. one where all the quantifiers occur up front) equivalent to
$$\neg(\exists x A(x) \rightarrow \forall y B(y)).$$

3. Give natural deduction proofs of the following validities (using the 4 quantifier rules, and not defining $\exists\varphi$ as $\neg\forall\neg\varphi$!).
   (a) $\neg\exists x\varphi(x) \rightarrow \forall x\neg\varphi(x)$
   (b) $\exists x\neg\varphi(x) \rightarrow \neg\forall x\varphi(x)$
   (c) $(\exists x\varphi \rightarrow \psi) \rightarrow \forall x(\varphi \rightarrow \psi)$, where x is not free in ψ.

4. Determine whether the following syllogism is valid (justify your answer).

   Some Greeks are not slaves.
   No slaves are women.
   Therefore, some women are not Greek.

5. The language of *monoids* has a constant symbol 1 and a binary function symbol, written $x \cdot y$. The axioms for monoids are associativity, $\forall x, y, z\ (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$, and 1 is a (two-sided) unit, $\forall x\ (1 \cdot x = x)$ and $\forall x\ (x \cdot 1 = x)$.
   (a) Use natural deduction to show that every monoid can be ordered by defining $x \leq y$ iff $\exists z(x \cdot z = y)$, i.e. show that this relation is reflexive and transitive.
   $\star$ (b) Is it always a partial ordering? That is, is it necessarily antisymmetric, in the sense that $x \leq y$ and $y \leq x$ implies $x = y$?