Homework 10

Due Thursday, December 8

1. Let L be a language with two unary predicates, A and B. Consider the equivalence

$$\exists x(A(x) \wedge B(x)) \leftrightarrow \exists x A(x) \wedge \exists x B(x).$$

   (a) Show that one direction is valid, using only semantic notions. In particular, your answer should make it clear that you know what "valid" means!

   (b) Show that the other direction is not valid.

2. Find a prenex sentence (i.e. one where all the quantifiers occur up front) equivalent to the following:

$$\neg(\exists x A(x) \rightarrow \forall y B(y))$$

   Prove the equivalence algebraically.

3. Give natural deduction proofs of the following validities (using the 4 quantifier rules, and not defining $\exists\varphi$ as $\neg\forall\neg\varphi$!).

   (a) $\neg\exists x\varphi(x) \rightarrow \forall x\neg\varphi(x)$

   (b) $\exists x\neg\varphi(x) \rightarrow \neg\forall x\varphi(x)$

   (c) $(\exists x\varphi \rightarrow \psi) \rightarrow \forall x(\varphi \rightarrow \psi)$, where x is not free in ψ.

4. Formalize the following argument in first-order logic, and determine whether it is valid (justify your answer).

   Some Greeks are not philosophers.
   No slaves are philosophers.
   Therefore, some Greeks are not slaves.

5. State and prove the Compactness Theorem.

6. (a) State the Model Existence Lemma.

   (b) State the Completeness Theorem.

(c) Assuming the Model Existence Lemma, prove the Completeness Theorem.

7. The language of *linear orders with endpoints* has two constant symbols $0, 1$ and a binary relation symbol, written $x \leq y$. The axioms for linear orders with endpoints are:

$$\begin{aligned}
&\text{reflexivity:} \quad \forall x \ (x \leq x),\\
&\text{transitivity:} \quad \forall x, y, z \ ((x \leq y \wedge y \leq z) \rightarrow x \leq z),\\
&\text{antisymmetry:} \quad \forall x, y \ ((x \leq y \wedge y \leq x) \rightarrow x = y),\\
&\text{linearity:} \quad \forall x, y \ (x \leq y) \vee (y \leq x),\\
&\text{endpoints:} \quad \forall x \ (0 \leq x) \wedge (x \leq 1).
\end{aligned}$$

Consider the following models of the theory of rooted partial orders:

$$\mathcal{I} = ([0,1] \subseteq \mathbb{R}, 0, 1, \leq)$$

(the usual ordering of unit interval in the reals)

$$\mathcal{N} = (\mathbb{N} \cup \{\infty\}, 0, \infty, \leq)$$

(the usual ordering of the natural numbers, but with a new element ∞ added at infinity)

(a) Show that these models are distinguishable in first-order logic by producing a sentence that is satisfied by one but not the other.

(b) Can there be a model that satisfies all the same first-order sentences as $\mathcal{N}$ and is uncountable? Justify your answer!

$\star$ (c) (for Grad Students)
Using compactness, show that there are models of this theory that are strictly larger than $\mathcal{I}$.