Homework 1

Due Thursday 15 September

Undergraduates are to do only the unstarred problems. Graduate students should also do the starred problem.

1. Prove that there are infinitely many prime numbers.
   (Hint: suppose $p_0, p_1, \ldots, p_k$ is a list of all the primes, and consider the product $q = p_1 \cdot p_2 \cdot \ldots \cdot p_k + 1$. Show that q is either prime or is divisible by a prime different from any of the p_i.)

2. Use the least number principle to prove the induction principle, and vice-versa.

3. Prove by induction that:

   $$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1 .$$

   (Recall that $\sum_{i=0}^{n} 2^i$ is an abbrevation for $2^0 + 2^1 + 2^2 + \ldots + 2^n$.)

4. Define the set of "babble-strings" inductively, as follows:

   - "ba" is a babble-string
   - if s is a babble-string, so is "ab"$\hat{\ }s$
   - if s and t are babble-strings, so is $s\hat{\ }t$

   Prove by induction that every babble-string has the same number of a's and b's, and that every babble-string ends with an "a".

5. Referring to the previous problem, show that the set B of babble-strings is not freely generated. Give a different specification of B such that B is freely generated. Use that specification to define a length function $f : B \to \mathbb{N}$ giving the number of letters in the string.

6. Write down explicit definitions of the functions f and g, defined recursively by:

   a. $f(0) = 0$, and $f(n+1) = 3 + f(n)$,

   b. $g(0) = 1$, and $g(n+1) = (n+1)^2 g(n)$.
   (Hint: use "factorial" notation: $n! = 1 \cdot 2 \cdot \ldots \cdot n$.)

- $\star$ 7. Suppose g is a function from $\mathbb{N}$ to $\mathbb{N}$. Write down a recursive definition of the function $f(n)$, defined by $f(n) = \sum_{i=0}^{n} g(i)$.

- $\star$ 8. Prove by induction that if $n \geq 5$, then $2^n > n^2$.