Lecture Notes on
Harmony

15-317: Constructive Logic
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1 Introduction

In the verificationist definition of the logical connectives via their introduction rules we have briefly justified the elimination rules. In this section we study the balance between introduction and elimination rules more closely. In order to show that the two are in harmony we establish two properties: *local soundness* and *local completeness*.

Local soundness shows that the elimination rules are not too strong: no matter how we apply elimination rules to the result of an introduction we cannot gain any new information. We demonstrate this by showing that we can find a more direct proof of the conclusion of an elimination than one that first introduces and then eliminates the connective in question. This is witnessed by a *local reduction* of the given introduction and the subsequent elimination.

Local completeness shows that the elimination rules are not too weak: there is always a way to apply elimination rules so that we can reconstitute a proof of the original proposition from the results by applying introduction rules. This is witnessed by a *local expansion* of an arbitrary given derivation into one that introduces the primary connective.

Connectives whose introduction and elimination rules are in harmony in the sense that they are locally sound and complete are properly defined from the verificationist perspective. If not, the proposed connective should be viewed with suspicion. Another criterion we would like to apply uniformly is that both introduction and elimination rules do not refer to other

propositional constants or connectives (besides the one we are trying to define), which could create a dangerous dependency of the various connectives on each other. As we present correct definitions we will occasionally also give some counterexamples to illustrate the consequences of violating the principles behind the patterns of valid inference.

In the discussion of each individual connective below we use the notation

$$\frac{\mathcal{D}}{A \; true} \Longrightarrow_R \frac{\mathcal{D}'}{A \; true}$$

for the local reduction of a deduction $\mathcal{D}$ to another deduction $\mathcal{D}'$ of the same judgment $A \; true$. In fact, $\Longrightarrow_R$ can itself be a higher level judgment relating two proofs, $\mathcal{D}$ and $\mathcal{D}'$, although we will not directly exploit this point of view. Similarly,

$$\frac{\mathcal{D}}{A \; true} \Longrightarrow_E \frac{\mathcal{D}'}{A \; true}$$

is the notation of the local expansion of $\mathcal{D}$ to $\mathcal{D}'$.

Conjunction. We start with local soundness. Since there are two elimination rules and one introduction, it turns out we have two cases to consider. In either case, we can easily reduce.

$$\cfrac{\cfrac{\overset{\mathcal{D}}{A \; true} \quad \overset{\mathcal{E}}{B \; true}}{A \wedge B \; true} \wedge I}{A \; true} \wedge E_L \quad \Longrightarrow_R \quad \overset{\mathcal{D}}{A \; true}$$

$$\cfrac{\cfrac{\overset{\mathcal{D}}{A \; true} \quad \overset{\mathcal{E}}{B \; true}}{A \wedge B \; true} \wedge I}{B \; true} \wedge E_R \quad \Longrightarrow_R \quad \overset{\mathcal{E}}{B \; true}$$

Local completeness requires us to apply eliminations to an arbitrary proof of $A \wedge B \; true$ in such a way that we can reconstitute a proof of $A \wedge B$ from the results.

$$\overset{\mathcal{D}}{A \wedge B \; true} \Longrightarrow_E \cfrac{\cfrac{\overset{\mathcal{D}}{A \wedge B \; true}}{A \; true} \wedge E_L \quad \cfrac{\overset{\mathcal{D}}{A \wedge B \; true}}{B \; true} \wedge E_R}{A \wedge B \; true} \wedge I$$

As an example where local completeness might fail, consider the case where we "forget" the right elimination rule for conjunction. The remaining rule is still locally sound, but not locally complete because we cannot extract a proof of B from the assumption $A \wedge B$. Now, for example, we cannot prove $(A \wedge B) \supset (B \wedge A)$ even though this should clearly be true.

Substitution Principle. We need the defining property for hypothetical judgments before we can discuss implication. Intuitively, we can always substitute a deduction of A *true* for any use of a hypothesis A *true*. In order to avoid ambiguity, we make sure assumptions are labelled and we substitute for all uses of an assumption with a given label. Note that we can only substitute for assumptions that are not discharged in the subproof we are considering. The substitution principle then reads as follows:

> If
>
> $$\frac{}{A\ true}\ u$$
> $$\mathcal{E}$$
> $$B\ true$$
>
> is a hypothetical proof of B *true* under the undischarged hypothesis A *true* labelled u, and
>
> $$\mathcal{D}$$
> $$A\ true$$
>
> is a proof of A *true* then
>
> $$\frac{\mathcal{D}}{A\ true}\ u$$
> $$\mathcal{E}$$
> $$B\ true$$
>
> is our notation for substituting $\mathcal{D}$ for all uses of the hypothesis labelled u in $\mathcal{E}$. This deduction, also sometime written as $[\mathcal{D}/u]\mathcal{E}$ no longer depends on u.

Implication. To witness local soundness, we reduce an implication introduction followed by an elimination using the substitution operation.

$$\cfrac{\cfrac{\cfrac{\dfrac{}{A\ true}\ u}{\begin{array}{c}\mathcal{E}\\ B\ true\end{array}}}{A \supset B\ true}\supset I^u \qquad \begin{array}{c}\mathcal{D}\\ A\ true\end{array}}{B\ true}\supset E \quad \Longrightarrow_R \quad \begin{array}{c}\dfrac{\mathcal{D}}{A\ true}\ u\\ \mathcal{E}\\ B\ true\end{array}$$

The conditions on the substitution operation is satisfied, because u is introduced at the $\supset I^u$ inference and therefore not discharged in $\mathcal{E}$.

Local completeness is witnessed by the following expansion.

$$
\begin{array}{c}
\mathcal{D} \\
A \supset B\ \textit{true}
\end{array}
\quad \Longrightarrow_E \quad
\cfrac{\cfrac{\begin{array}{cc} \mathcal{D} & \overline{A\ \textit{true}}\ u \\ A \supset B\ \textit{true} & \end{array}}{B\ \textit{true}}\ \supset E}{A \supset B\ \textit{true}}\ \supset I^u
$$

Here u must be chosen fresh: it only labels the new hypothesis A *true* which is used only once.

Disjunction.   For disjunction we also employ the substitution principle because the two cases we consider in the elimination rule introduce hypotheses. Also, in order to show local soundness we have two possibilities for the introduction rule, in both situations followed by the only elimination rule.

$$
\cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\A\ \textit{true}\end{array}}{A \vee B\ \textit{true}}\ \vee I_L \quad \cfrac{\begin{array}{c}\overline{A\ \textit{true}}\ u\\ \mathcal{E}\\ C\ \textit{true}\end{array}}{} \quad \cfrac{\begin{array}{c}\overline{B\ \textit{true}}\ w\\ \mathcal{F}\\ C\ \textit{true}\end{array}}{}}{C\ \textit{true}}\ \vee E^{u,w}
\quad \Longrightarrow_R \quad
\begin{array}{c}\mathcal{D}\\A\ \textit{true}\ u\\ \mathcal{E}\\ C\ \textit{true}\end{array}
$$

$$
\cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\B\ \textit{true}\end{array}}{A \vee B\ \textit{true}}\ \vee I_R \quad \cfrac{\begin{array}{c}\overline{A\ \textit{true}}\ u\\ \mathcal{E}\\ C\ \textit{true}\end{array}}{} \quad \cfrac{\begin{array}{c}\overline{B\ \textit{true}}\ w\\ \mathcal{F}\\ C\ \textit{true}\end{array}}{}}{C\ \textit{true}}\ \vee E^{u,w}
\quad \Longrightarrow_R \quad
\begin{array}{c}\mathcal{D}\\B\ \textit{true}\ w\\ \mathcal{F}\\ C\ \textit{true}\end{array}
$$

An example of a rule that would not be locally sound is

$$
\cfrac{A \vee B\ \textit{true}}{A\ \textit{true}}\ \vee E_L?
$$

and, indeed, we would not be able to reduce

$$
\cfrac{\cfrac{B\ \textit{true}}{A \vee B\ \textit{true}}\ \vee I_R}{A\ \textit{true}}\ \vee E_L?
$$

In fact we can now derive a contradiction from no assumption, which means the whole system is incorrect.

$$\dfrac{\dfrac{\dfrac{}{\top\ true}\ \top I}{\bot \vee \top\ true}\ \vee I_R}{\bot\ true}\ \vee E_L?$$

Local completeness of disjunction distinguishes cases on the known $A \vee B\ true$, using $A \vee B\ true$ as the conclusion.

$$\dfrac{\mathcal{D}}{A \vee B\ true} \implies_E \dfrac{\begin{array}{ccc} & \dfrac{\overline{A\ true}^{\ u}}{A \vee B\ true}\ \vee I_L & \dfrac{\overline{B\ true}^{\ w}}{A \vee B\ true}\ \vee I_R \\ \dfrac{\mathcal{D}}{A \vee B\ true} & & \end{array}}{A \vee B\ true}\ \vee E^{u,w}$$

Visually, this looks somewhat different from the local expansions for conjunction or implication. It looks like the elimination rule is applied last, rather than first. Mostly, this is due to the notation of natural deduction: the above represents the step from using the knowledge of $A \vee B\ true$ and eliminating it to obtain the hypotheses $A\ true$ and $B\ true$ in the two cases.

Truth. The local constant $\top$ has only an introduction rule, but no elimination rule. Consequently, there are no cases to check for local soundness: any introduction followed by any elimination can be reduced.

However, local completeness still yields a local expansion: Any proof of $\top\ true$ can be trivially converted to one by $\top I$.

$$\dfrac{\mathcal{D}}{\top\ true} \implies_E \dfrac{}{\top\ true}\ \top I$$

Falsehood. As for truth, there is no local reduction because local soundness is trivially satisfied since we have no introduction rule.

Local completeness is slightly tricky. Literally, we have to show that there is a way to apply an elimination rule to any proof of $\bot\ true$ so that we can reintroduce a proof of $\bot\ true$ from the result. However, there will be zero cases to consider, so we apply no introductions. Nevertheless, the following is the right local expansion.

$$\dfrac{\mathcal{D}}{\bot\ true} \implies_E \dfrac{\dfrac{\mathcal{D}}{\bot\ true}}{\bot\ true}\ \bot E$$

Reasoning about situation when falsehood is true may seem vacuous, but is common in practice because it corresponds to reaching a contradiction. In intuitionistic reasoning, this occurs when we prove $A \supset \bot$ which is often abbreviated as $\neg A$. In classical reasoning it is even more frequent, due to the rule of proof by contradiction.

2   Verifications

The verificationist point of view on the meaning of a proposition is that it is determined by its *verifications*. Intuitively, a verification should be a proof that only analyzes the constituents of a proposition. This restriction of the space of all possible proofs is necessary so that the definition is well-founded. For example, if in order to understand the meaning of A, we would have to understand the meaning of $B \supset A$ and B, the whole program of understanding the meaning of the connectives by their proofs is in jeopardy because B could be a proposition containing, say, A. But the meaning of A would then in turn depend on the meaning of A, creating a vicious cycle.

In this section we will make the structure of verifications more explicit. We write $A\uparrow$ for the judgment "*A has a verification*". Naturally, this should mean that A is true, and that the evidence for that has a special form. Eventually we will also establish the converse: if A is true than A has a verification.

Conjunction is easy to understand. A verification of $A \wedge B$ should consist of a verification of A and a verification of B.

$$\frac{A\uparrow \quad B\uparrow}{A \wedge B\uparrow} \wedge I$$

We reuse here the names of the introduction rule, because this rule is strictly analogous to the introduction rule for the truth of a conjunction.

Implication, however, introduces a new hypothesis which is not explicitly justified by an introduction rule but just a new label. For example, in the proof

$$\frac{\dfrac{\overline{A \wedge B \ true}\ u}{A \ true} \wedge E_L}{(A \wedge B) \supset A \ true} \supset I^u$$

the conjunction $A \wedge B$ is not justified by an introduction.

The informal discussion of proof search strategies earlier, namely to use introduction rules from the bottom up and elimination rules from the top down contains the answer. We introduce a second judgment, $A{\downarrow}$ which means "*A may be used*". $A{\downarrow}$ should be the case when either *A true* is a hypothesis, or *A* is deduced from a hypothesis via elimination rules. Our local soundness arguments provide some evidence that we cannot deduce anything incorrect in this manner.

We now go through the connectives in turn, defining verifications and uses.

Conjunction. In summary of the discussion above, we obtain:

$$\frac{A{\uparrow} \quad B{\uparrow}}{A \wedge B{\uparrow}} \wedge I \qquad \frac{A \wedge B{\downarrow}}{A{\downarrow}} \wedge E_L \qquad \frac{A \wedge B{\downarrow}}{B{\downarrow}} \wedge E_R$$

The left elimination rule can be read as: "*If we can use $A \wedge B$ we can use A*", and similarly for the right elimination rule.

Implication. The introduction rule creates a new hypothesis, which we may use in a proof. The assumption is therefore of the judgment $A{\downarrow}$

$$\frac{\overline{\quad}}{A{\downarrow}}{\,u}$$
$$\vdots$$
$$\frac{B{\uparrow}}{A \supset B{\uparrow}} \supset^u$$

In order to use an implication $A \supset B$ we require a verification of A. Just requiring that A may be used would be too weak, as can be seen when trying to prove $((A \supset A) \supset B) \supset B{\uparrow}$. It should also be clear from the fact that we are not eliminating a connective from A.

$$\frac{A \supset B{\downarrow} \quad A{\uparrow}}{B{\downarrow}} \supset E$$

Disjunction. The verifications of a disjunction immediately follow from their introduction rules.

$$\frac{A{\uparrow}}{A \vee B{\uparrow}} \vee I_L \qquad \frac{B{\uparrow}}{A \vee B{\uparrow}} \vee I_R$$

A disjunction is used in a proof by cases, called here $\vee E$. This introduces two new hypotheses, and each of them may be used in the corresponding subproof. Whenever we set up a hypothetical judgment we are trying to find a verification of the conclusion, possibly with uses of hypotheses. So the conclusion of $\vee E$ should be a verification.

$$
\cfrac{A \vee B \downarrow \quad \overset{\displaystyle \overline{A \downarrow}\; u}{\underset{\displaystyle C \uparrow}{\vdots}} \quad \overset{\displaystyle \overline{B \downarrow}\; w}{\underset{\displaystyle C \uparrow}{\vdots}}}{C \uparrow} \; \vee E^{u,w}
$$

Truth.   The only verification of truth is the trival one.

$$
\frac{}{\top \uparrow} \; \top I
$$

A hypothesis $\top \downarrow$ cannot be used because there is no elimination rule for $\top$.

Falsehood.   There is no verification of falsehood because we have no introduction rule.

We can use falsehood, signifying a contradiction from our current hypotheses, to verify any conclusion. This is the zero-ary case of a disjunction.

$$
\frac{\bot \downarrow}{C \uparrow} \; \bot E
$$

Atomic propositions.   How do we construct a verification of an atomic proposition P? We cannot break down the structure of P because there is none, so we can only proceed if we already know P is true. This can only come from a hypothesis, so we have a rule that lets us use the knowledge of an atomic proposition to construct a verification.

$$
\frac{P \downarrow}{P \uparrow} \; \downarrow \uparrow
$$

This rule has a special status in that it represents a change in judgments but is not tied to a particular local connective. We call this a *judgmental rule* in order to distinguish it from the usual introduction and elimination rules that characterize the connectives.

Global soundness. Local soundness is an intrinsic property of each connective, asserting that the elimination rules for it are not too strong given the introduction rules. Global soundness is its counterpart for the whole system of inference rules. It says that if an arbitrary proposition A has a verification then we may use A without gaining any information. That is, for arbitrary propositions A and C:

$$A\downarrow$$
$$\vdots$$
If $\quad A\uparrow \quad$ and $\quad C\uparrow \quad$ then $\quad C\uparrow.$

We would want to prove this using a substitution principle, except that the judgment $A\uparrow$ and $A\downarrow$ do not match. In the end, the arguments for local soundness will help use carry out this proof later in this course.

Global completeness. Local completeness is also an intrinsic property of each connective. It asserts that the elimination rules are not too weak, given the introduction rule. Global completeness is its counterpart for the whole system of inference rules. It says that if we may use A then we can construct from this a verification of A. That is, for arbitrary propositions A:

$$A\downarrow$$
$$\vdots$$
$$A\uparrow.$$

Global completeness follows from local completeness rather directly by induction on the structure of A.

Global soundness and completeness are properties of whole deductive systems. Their proof must be carried out in a mathematical *metalanguage* which makes them a bit different than the formal proofs that we have done so far within natural deduction. Of course, we would like them to be correct as well, which means they should follow the same principles of valid inference that we have laid out so far.

There are two further properties we would like, relating truth, verifications, and uses. The first is that if A has a verification or A may be used, then A is true. This is rather evident since we have just specialized the introduction and elimination rules, except for the judgmental rule $\downarrow\uparrow$. But under the interpretation of verification and use as truth, this inference becomes redundant.

Significantly more difficult is the property that if A is true then A has a verification. Since we justified the meaning of the connectives from their

verifications, a failure of this property would be devastating to the verificationist program. Fortunately it holds and can be proved by exhibiting a process of *proof normalization* that takes an arbitrary proof of A true and constructs a verification of A.

All these properties in concert show that our rules are well constructed, locally as well as globally. Experience with many other logical systems indicates that this is not an isolated phenomenon: we can employ the verificationist point of view to give coherent sets of rules not just for constructive logic, but for classical logic, temporal logic, spatial logic, modal logic, and many other logics that are of interest in computer science. Taken together, these constitute strong evidence that separating judgments from propositions and taking a verificationist point of view in the definition of the logical connectives is indeed a proper and useful foundation for logic.

3   Derived Rules of Inference

One popular device for shortening derivations is to introduce *derived rules of inference*. For example,

$$\frac{A \supset B \ true \quad B \supset C \ true}{A \supset C \ true}$$

is a derived rule of inference. Its derivation is the following:

$$\frac{B \supset C \ true \quad \dfrac{A \supset B \ true \quad \overline{A \ true}^{\,u}}{B \ true} \supset E}{\dfrac{C \ true}{A \supset C \ true} \supset I^u} \supset E$$

Note that this is simply a hypothetical deduction, using the premises of the derived rule as assumptions. In other words, a derived rule of inference is nothing but an evident hypothetical judgment; its justification is a hypothetical deduction.

We can freely use derived rules in proofs, since any occurrence of such a rule can be expanded by replacing it with its justification.

4   Logical Equivalences

We now consider several classes of logical equivalences in order to develop some intuitions regarding the truth of propositions. Each equivalence has

the form $A \equiv B$, but we consider only the basic connectives and constants ($\wedge$, $\supset$, $\vee$, $\top$, $\bot$) in A and B. Later on we consider negation as a special case. We use some standard conventions that allow us to omit some parentheses while writing propositions. We use the following operator precedences

$$\neg > \wedge > \vee > \supset > \equiv$$

where $\wedge$, $\vee$, and $\supset$ are right associative. For example

$$\neg A \supset A \vee \neg\neg A \supset \bot$$

stands for

$$(\neg A) \supset ((A \vee (\neg(\neg A))) \supset \bot)$$

In ordinary mathematical usage, $A \equiv B \equiv C$ stands for $(A \equiv B) \wedge (B \equiv C)$; in the formal language we do not allow iterated equivalences without explicit parentheses in order to avoid confusion with propositions such as $(A \equiv A) \equiv \top$.

Commutativity. Conjunction and disjunction are clearly commutative, while implication is not.

(C1) $A \wedge B \equiv B \wedge A$ *true*

(C2) $A \vee B \equiv B \vee A$ *true*

(C3) $A \supset B$ is not commutative

Idempotence. Conjunction and disjunction are idempotent, while self-implication reduces to truth.

(I1) $A \wedge A \equiv A$ *true*

(I2) $A \vee A \equiv A$ *true*

(I3) $A \supset A \equiv \top$ *true*

Interaction Laws. These involve two interacting connectives. In principle, there are left and right interaction laws, but because conjunction and disjunction are commutative, some coincide and are not repeated here.

(L1) $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ *true*

(L2) $A \wedge \top \equiv A$ *true*

 (L3) $A \wedge (B \supset C)$ do not interact

 (L4) $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ *true*

 (L5) $A \wedge \bot \equiv \bot$ *true*

 (L6) $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ *true*

 (L7) $A \vee \top \equiv \top$ *true*

 (L8) $A \vee (B \supset C)$ do not interact

 (L9) $A \vee (B \vee C) \equiv (A \vee B) \vee C$ *true*

(L10) $A \vee \bot \equiv A$ *true*

(L11) $A \supset (B \wedge C) \equiv (A \supset B) \wedge (A \supset C)$ *true*

(L12) $A \supset \top \equiv \top$ *true*

(L13) $A \supset (B \supset C) \equiv (A \wedge B) \supset C$ *true*

(L14) $A \supset (B \vee C)$ do not interact

(L15) $A \supset \bot$ do not interact

(L16) $(A \wedge B) \supset C \equiv A \supset (B \supset C)$ *true*

(L17) $\top \supset C \equiv C$ *true*

(L18) $(A \supset B) \supset C$ do not interact

(L19) $(A \vee B) \supset C \equiv (A \supset C) \wedge (B \supset C)$ *true*

(L20) $\bot \supset C \equiv \top$ *true*

References