A Simple Proof of Call-by-Value Standardization

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Abstract

We give a simple proof of the Standardization Theorem for call-by-value based on Takahashi's method of parallel reduction. The proof is formalized in Twelf.

1 Introduction

Contextual equivalence, written $M \cong M'$, can be defined as the statement for all contexts $\mathcal{C}$, $\mathcal{C}[M]$ halts if and only if $\mathcal{C}[M']$ halts. Equivalently, we may define contextual equivalence as the largest adequate congruence, where we say that a relation R is adequate when $M\ R\ M'$ implies that M halts if and only if M' halts. Our purpose is to show that, in the call-by-value lambda calculus, contextual equivalence respects evaluation. Moreover, we wish to do so directly, that is, without first proving that contextual equivalence coincides with some other equivalence.

Let us write (small-step, call-by-value) evaluation as $M \mapsto M'$, and define reduction ($M \longrightarrow M'$) as the compatible closure of evaluation. We wish to show that reduction is adequate. It then follows that contextual equivalence respects evaluation. Using the former definition, let $\mathcal{C}$ be arbitrary and suppose $M \mapsto M'$. Then $\mathcal{C}[M] \longrightarrow \mathcal{C}[M']$, and by adequacy of reduction, $\mathcal{C}[M]$ halts iff $\mathcal{C}[M']$ halts. Using the latter definition, it is easy to show that the reflexive, transitive closure[1] of reduction is an adequate congruence, and is therefore contained in contextual equivalence.

The adequacy of reduction follows from the Standardization Theorem [2, 1], which says that any multi-step reduction can be carried out by a sequence whose redices are selected in a standard order. In a call-by-name setting, standard order is simply left-to-right (in the fully parenthesized representation). Standard reductions always begin with evaluation steps and continue with internal reductions, so it follows that if $M \longrightarrow^* \lambda x.N$, then there exists N' such that $M \mapsto^* \lambda x.N'$. From that, and confluence of reduction, it is easy to show that reduction is adequate.

Plotkin [5] adapts the Standardization Theorem to the call-by-value setting. In call-by-value, standard order is more delicate to define, since function applications can be contracted only after their arguments have been evaluated, but the essence is the same: standard reduction begins with evaluation steps, and then continues with internal reductions selected in standard order. As in call-by-name, it follows that if $M \longrightarrow^* V$ for a value V, then there exists a value V' such that $M \mapsto^* V'$, as desired.

Paolini and Ronchi Della Rocca [3] further generalize the Standardization Theorem to their λV-calculus, which is parameterized over a set V of terms that may be taken as inputs to functions. In the λV-calculus, call-by-value may be obtained by choosing V to be the set of values, and call-by-name by choosing V to be the set of all terms.

Takahashi [6] gives a particularly elegant proof of the Standardization Theorem for call-by-name using parallel reduction. In this paper, we adapt Takahashi's method to call-by-value. All the proofs are formalized in Twelf [4]; the code is available on-line at:

`www.cs.cmu.edu/~crary/papers/2009/standard.elf`

[1] For any relation R, we write its reflexive, transitive closure as R^*.

$\boxed{\text{values}}$

$$\frac{}{x\ \texttt{value}} \qquad \frac{}{\lambda x.M\ \texttt{value}} \qquad \frac{}{M N\ \texttt{nonvalue}}$$

$\boxed{\text{evaluation}}$

$$\frac{M \mapsto N}{MN \mapsto M'N} \qquad \frac{M\ \texttt{value} \quad N \mapsto N'}{MN \mapsto MN'}$$

$$\frac{N\ \texttt{value}}{(\lambda x.M)N \mapsto [N/x]M}$$

$\boxed{\text{parallel reduction}}$

$$\frac{}{M \Longrightarrow M} \qquad \frac{M \Longrightarrow N}{\lambda x.M \Longrightarrow \lambda x.N}$$

$$\frac{M \Longrightarrow M' \quad N \Longrightarrow N'}{MN \Longrightarrow M'N'}$$

$$\frac{N\ \texttt{value} \quad M \Longrightarrow M' \quad N \Longrightarrow N'}{(\lambda x.M)N \Longrightarrow [N'/x]M'}$$

$\boxed{\text{internal parallel reduction}}$

$$\frac{}{M \stackrel{\mathit{int}}{\Longrightarrow} M} \qquad \frac{M \Longrightarrow N}{\lambda x.M \stackrel{\mathit{int}}{\Longrightarrow} \lambda x.N}$$

$$\frac{M\ \texttt{nonvalue} \quad M \stackrel{\mathit{int}}{\Longrightarrow} M' \quad N \Longrightarrow N'}{MN \stackrel{\mathit{int}}{\Longrightarrow} M'N'}$$

$$\frac{M \stackrel{\mathit{int}}{\Longrightarrow} M' \quad N \stackrel{\mathit{int}}{\Longrightarrow} N'}{MN \stackrel{\mathit{int}}{\Longrightarrow} M'N'}$$

Figure 1: Definitions

2 Bifurcation

Several definitions are given in Figure 1. Values, evaluation ($M \mapsto N$), and parallel reduction ($M \Longrightarrow N$) are standard.

We have no use for ordinary reduction apart from its reflexive, transitive closure, which coincides with the reflexive, transitive closure of parallel reduction. Therefore we omit the definition of ordinary reduction.

Internal parallel reduction ($M \stackrel{int}{\Longrightarrow} N$) is parallel reduction that does not contract the active redex. Thus, internal parallel reduction omits the beta rule. Also, since this is a call-by-value setting, an internal parallel reduction of MN cannot contract the active redex in N unless M is a nonvalue.

The key technical definition is strong parallel reduction ($M \Rrightarrow N$). We say that M strongly reduces to N if $M \Longrightarrow N$ and also $M \mapsto^* P \stackrel{int}{\Longrightarrow} N$, for some P. Note that strong parallel reduction is reflexive.

We require two well-known properties of parallel reduction:

Lemma 1 (Parallel Reduction)

1. *If $M \Longrightarrow M'$ and $N \Longrightarrow N'$ then $[N/x]M \Longrightarrow [N'/x]M'$.*
2. *If $M \Longrightarrow^* P$ and $M \Longrightarrow^* Q$ then there exists N such that $P \Longrightarrow^* N$ and $Q \Longrightarrow^* N$.*

Our main lemma shows that parallel reduction coincides with strong parallel reduction. By definition, strong parallel reduction implies parallel reduction, so we need only to show the other direction. First we establish four lemmas governing strong parallel reduction:

Lemma 2 *If $M \Rrightarrow M'$ and $N \Longrightarrow N'$ and M' is a nonvalue then $MN \Rrightarrow M'N'$.*

Proof

Certainly $MN \Longrightarrow M'N'$. Suppose $M = M_0 \mapsto \cdots \mapsto M_m \stackrel{int}{\Longrightarrow} M'$. Then $MN = M_0N \mapsto \cdots \mapsto M_mN$. Since $M_m \Longrightarrow M'$, M_m is a nonvalue. Therefore, $M_mN \stackrel{int}{\Longrightarrow} M'N'$. $\Box$

Lemma 3 *If $M \Rrightarrow M'$ and $N \Rrightarrow N'$ then $MN \Rrightarrow M'N'$.*

Proof

Certainly $MN \Longrightarrow M'N'$. If M' is a nonvalue then the result follows from Lemma 2. Therefore assume M' is a value. Suppose $M = M_0 \mapsto \cdots \mapsto M_m \stackrel{int}{\Longrightarrow} M'$ and suppose $N = N_0 \mapsto \cdots \mapsto N_n \stackrel{int}{\Longrightarrow} N'$. Observe that M_m is a value. Then $MN = M_0N_0 \mapsto \cdots \mapsto M_mN_0 \mapsto \cdots \mapsto M_mN_n \stackrel{int}{\Longrightarrow} M'N'$. $\Box$

Lemma 4 *If $M \stackrel{int}{\Longrightarrow} M'$ and $N \Rrightarrow N'$ and N is a value then $[N/x]M \Rrightarrow [N'/x]M'$.*

Proof

By Lemma 1, $[N/x]M \Longrightarrow [N'/x]M'$. We proceed by induction on M.

Case 1: Suppose $M = x$. Then $M' = x$. So $[N/x]M = N \Rrightarrow N' = [N'/x]M'$.

Case 2: Suppose $M = y$ where $x \neq y$. Then $M' = y$. So $[N/x]M = y = [N'/x]M'$.

Case 3: Suppose $M = \lambda y.P$. Then $M' = \lambda y.P'$ with $P \Longrightarrow P'$. By Lemma 1, $[N/x]P \Longrightarrow [N'/x]P'$. Therefore $\lambda y.[N/x]P \stackrel{int}{\Longrightarrow} \lambda y.[N'/x]P'$, so $\lambda y.[N/x]P \Rrightarrow \lambda y.[N'/x]P'$.

Case 4: Suppose $M = PQ$ where P is a value. Then $M' = P'Q'$ with $P \stackrel{int}{\Longrightarrow} P'$ and $Q \stackrel{int}{\Longrightarrow} Q'$. By induction, $[N/x]P \Rrightarrow [N'/x]P'$ and $[N/x]Q \Rrightarrow [N'/x]Q'$. By Lemma 3, $[N/x]M \Rrightarrow [N'/x]M'$.

Case 5: Suppose $M = PQ$ where P is a nonvalue. Then $M' = P'Q'$ with $P \stackrel{int}{\Longrightarrow} P'$ and $Q \Longrightarrow Q'$. By induction, $[N/x]P \Rrightarrow [N'/x]P'$. Also, by Lemma 1, $[N/x]Q \Longrightarrow [N'/x]Q'$. Finally, P' is a nonvalue and so $[N'/x]P'$ is a nonvalue. By Lemma 2, $[N/x]M \Rrightarrow [N'/x]M'$. $\Box$

Lemma 5 *If $M \Rrightarrow M'$ and $N \Rrightarrow N'$ and N is a value then $[N/x]M \Rrightarrow [N'/x]M'$.*

Proof

By Lemma 1, $[N/x]M \Longrightarrow [N'/x]M'$. Suppose $M = M_0 \mapsto \cdots \mapsto M_m \stackrel{int}{\Longrightarrow} M'$. Then $[N/x]M = [N/x]M_0 \mapsto \cdots \mapsto [N/x]M_m$. By Lemma 4, $[N/x]M_m \Rrightarrow [N'/x]M'$. Therefore $[N/x]M \Rrightarrow [N'/x]M'$. $\Box$

Now we are ready to prove the main lemma.

Lemma 6 (Main Lemma) *If $M \Longrightarrow M'$ then $M \mapsto^* N$ and $N \stackrel{int}{\Longrightarrow} M'$.*

Proof

We prove that $M \Longrightarrow M'$ implies $M \Rrightarrow M'$, by induction on the derivation of $M \Longrightarrow M'$. The result follows immediately by the definition of strong parallel reduction.

Case 1: Suppose $M = M'$. Then $M \Rrightarrow M'$.

Case 2: Suppose $M = \lambda x.N$ and $M' = \lambda x.N'$ and $N \Longrightarrow N'$. Then $M \stackrel{int}{\Longrightarrow} M'$, so $M \Rrightarrow M'$.

Case 3: Suppose $M = NP$ and $M' = N'P'$ and $N \Longrightarrow N'$ and $P \Longrightarrow P'$. By induction, $N \Rrightarrow N'$ and $P \Rrightarrow P'$. By Lemma 3, $M \Rrightarrow M'$.

Case 4: Suppose $M = (\lambda x.N)P$ and $M' = [P'/x]N'$ and $N \Longrightarrow N'$ and $P \Longrightarrow P'$ and P is a value. By induction, $N \Rrightarrow N'$ and $P \Rrightarrow P'$. By Lemma 5, $[P/x]N \Rrightarrow [P'/x]N'$. Since $M \mapsto P[N/x] \Rrightarrow M'$, $M \Rrightarrow M'$. $\Box$

Next we show that internal parallel reduction can be shifted after evaluation:

Lemma 7 (Postponement) *If $M \stackrel{int}{\Longrightarrow} N$ and $N \mapsto P$ then $M \mapsto N'$ and $N' \Longrightarrow P$.*

Proof

By induction on M.

Case 1: Suppose $M = x$. Then $N = x$ which contradicts $N \mapsto P$.

Case 2: Suppose $M = \lambda x.M'$. Then $N = \lambda x.N'$ which contradicts $N \mapsto P$.

Case 3: Suppose $M = M_1M_2$ and M_1 is a nonvalue. Then $N = N_1N_2$ and $M_1 \stackrel{int}{\Longrightarrow} N_1$ and $M_2 \Longrightarrow N_2$. Also, since N_1 is a nonvalue and $N_1N_2 \mapsto P$, $P = P_1N_2$ where $N_1 \mapsto P_1$. By induction, $M_1 \mapsto N_1' \Longrightarrow P_1$. Then $M \mapsto N_1'M_2 \Longrightarrow P$.

Case 4: Suppose $M = M_1 M_2$ and M_1 is a value and M_2 is a nonvalue. Then $N = N_1 N_2$ and $M_1 \overset{int}{\Longrightarrow} N_1$ and $M_2 \overset{int}{\Longrightarrow} N_2$. Also, since N_1 is a value and N_2 is a nonvalue and $N_1 N_2 \mapsto P$, $P = N_1 P_2$ where $N_2 \mapsto P_2$. By induction, $M_2 \mapsto N_2' \Longrightarrow P_2$. Then $M \mapsto M_1 N_2' \Longrightarrow P$.

Case 5: Suppose $M = M_1 M_2$ and M_1 and M_2 are values. Then $N = N_1 N_2$ and $M_1 \overset{int}{\Longrightarrow} N_1$ and $M_2 \overset{int}{\Longrightarrow} N_2$. Since N_1, and N_2 are values and $N_1 N_2 \mapsto P$, N_1 has the form $\lambda x.N'$ and $P = [N_2/x]N'$. Then $M_1 = \lambda x.M'$ and $M' \Longrightarrow N'$. By Lemma 1, $[M_2/x]M' \Longrightarrow [N_2/x]N'$. Then $M \mapsto [M_2/x]M' \Longrightarrow P$. □

Corollary 8 *If $M \overset{int}{\Longrightarrow} N$ and $N \mapsto P$ then $M \mapsto^* N'$ and $N' \overset{int}{\Longrightarrow} P$.*

Proof

Immediate by Lemmas 7 and 6. □

Now we obtain our first main result: any parallel reduction sequence can be bifurcated into an evaluation sequence followed by an internal parallel reduction sequence.

Lemma 9 (Bifurcation) *If $M \Longrightarrow^* N$ then $M \mapsto^* P$ and $P \overset{int}{\Longrightarrow}{}^* N$.*

Proof

By induction on the length of $M \Longrightarrow^* N$. When $M = N$ the result is trivial. Therefore, suppose $M \Longrightarrow Q \Longrightarrow^* N$. By induction, $Q \mapsto^* R$ and $R \overset{int}{\Longrightarrow}{}^* N$. By Lemma 6, $M \mapsto^* S \overset{int}{\Longrightarrow} Q$. By Corollary 8 and induction on the length of $Q \mapsto^* R$, we have $S \mapsto^* Q' \overset{int}{\Longrightarrow} R$. Therefore $M \mapsto^* S \mapsto^* Q' \overset{int}{\Longrightarrow} R \overset{int}{\Longrightarrow}{}^* N$, so let $P = Q'$. □

3 Adequacy and Standardization

We can use bifurcation to prove our results of interest. First is the result that motivated this work, the adequacy of reduction:

Theorem 10 (Adequacy of Reduction) *If $M \Longrightarrow^* N$ then M halts if and only if N halts.*

Proof

Suppose $N \mapsto^* P$ for some value P. Then $M \Longrightarrow^* P$. By Lemma 9, $M \mapsto^* P' \overset{int}{\Longrightarrow}{}^* P$. Since P is a value, so is P'. Thus M halts.

Conversely, suppose $M \mapsto^* P$ for some value P. Then $M \Longrightarrow^* P$. By Lemma 1, there exists Q such that $N \Longrightarrow^* Q$ and $P \Longrightarrow^* Q$. By Lemma 9, $N \mapsto^* Q' \overset{int}{\Longrightarrow}{}^* Q$. Since P is a value, so is Q, and then so is Q'. Thus N halts. □

Note that the bifurcation lemma was sufficient to prove adequacy of reduction. We did not require the full Standardization Theorem.

However, the bifurcation lemma does suffice to prove the Standardization Theorem without much additional work, so we will complete it. First we define standard reduction, borrowing from Plotkin [5]. A standard reduction begins with zero or more evaluation steps, then continues with internal reductions that are selected in standard order.

Definition 11 *A standard reduction is a sequence of terms defined as follows:*

1. *M is a standard reduction.*
2. *If $M_0 \mapsto M_1$ and $M_1, \ldots, M_m$ is a standard reduction, then $M_0, M_1, \ldots, M_m$ is a standard reduction.*
3. *If $M_1, \ldots, M_m$ is a standard reduction, then $\lambda x.M_1, \ldots, \lambda x.M_m$ is a standard reduction.*
4. *If $M_1, \ldots M_m$ and $N_1, \ldots, N_n$ are standard reductions, then $M_1 N_1, \ldots, M_m N_1, \ldots M_m, N_n$ is a standard reduction.*

We can now prove the theorem by a simple induction.

Theorem 12 (Standardization) *If $M \Longrightarrow^* N$ then there exists a standard reduction from M to N.*

Proof

By induction on N. By Lemma 9, $M = M_0 \mapsto \cdots \mapsto M_m \overset{int}{\Longrightarrow}{}^* N$. We proceed by cases on N.

Case 1: Suppose $N = x$. Then $M_m = x$. So $M = M_0, \ldots, M_m = N$ is a standard reduction.

Case 2: Suppose $N = \lambda x.Q$. Then $M_m = \lambda x.P$ and $P \Longrightarrow^* Q$. By induction, there exists a standard reduction $P{=}P_0, \ldots, P_p{=}Q$. So $M{=}M_0, \ldots, M_m{=}\lambda x.P_0, \ldots, \lambda x.P_p{=}N$ is a standard reduction.

Case 3: Suppose $N = RS$. Then $M_m = PQ$ and $P \Longrightarrow^* R$ and $Q \Longrightarrow^* S$. By induction, there exist standard reductions $P{=}P_0, \ldots, P_p{=}R$ and $Q{=}Q_0, \ldots, Q_q{=}S$. So $M{=}M_0, \ldots, M_m{=}P_0 Q_0, \ldots, P_p Q_0, \ldots P_p Q_q{=}N$ is a standard reduction. □

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