Partial Differential Equations

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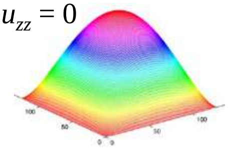
Differential Equation Classes 1

- dimension of unknown:
  - *ordinary differential equation* (ODE) – unknown is a function of one variable, e.g. $y(t)$
  - *partial differential equation* (PDE) – unknown is a function of multiple variables, e.g. $u(t,x,y)$
- number of equations:
  - *single* differential equation, e.g. $y'=y$
  - *system* of differential equations (*coupled*), e.g. $y_1'=y_2,\ \ y_2'=-g$
- *order*
  - *n*th order DE has *n*th derivative, and no higher, e.g. $y''=-g$

Differential Equation Classes 2

- linear & nonlinear:
  - *linear* differential equation: all terms linear in unknown and its derivatives
  - e.g.
    - $x''+ax'+bx+c=0$ – linear
    - $x'=t^2x$ – linear
    - $x''=1/x$ – nonlinear

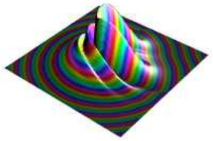
PDE's in Science & Engineering 1



- Laplace's Equation: $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$
  - unknown: $u(x,y,z)$
  - gravitational / electrostatic potential
- Heat Equation: $u_t = a^2\nabla^2 u$
  - unknown: $u(t,x,y,z)$
  - heat conduction
- Wave Equation: $u_{tt} = a^2\nabla^2 u$
  - unknown: $u(t,x,y,z)$
  - wave propagation

PDE’s in Science & Engineering 2

- Schrödinger Wave Equation
  - quantum mechanics
  - (electron probability densities)



- Navier-Stokes Equation
  - fluid flow (fluid velocity & pressure)

2^{nd} Order PDE Classification

- We classify conic curve $ax^2+bxy+cy^2+dx+ey+f=0$ as ellipse/parabola/hyperbola according to sign of discriminant b^2-$4ac$.
- Similarly we classify 2^{nd} order PDE $au_{xx}+bu_{xy}+cu_{yy}+du_x+eu_y+fu+g=0$:

b^2-$4ac < 0$	– elliptic	(e.g. equilibrium)
b^2-$4ac = 0$	– parabolic	(e.g. diffusion)
b^2-$4ac > 0$	– hyperbolic	(e.g. wave motion)

For general PDE’s, class can change from point to point

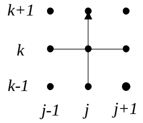
Example: Wave Equation

- $u_{tt} = c\, u_{xx}$ for $0 \le x \le 1$, $t \ge 0$
- initial cond.: $u(0,x)=\sin(\pi x)+x+2$, $u_t(0,x)=4\sin(2\pi x)$
- boundary cond.: $u(t,0) = 2$, $u(t,1)=3$
- $c=1$
- unknown: $u(t,x)$

- simulated using Euler’s method in t

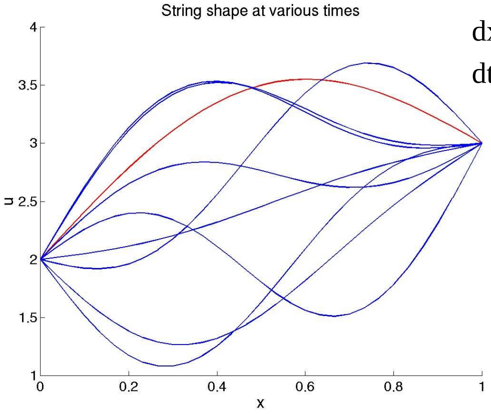
Wave Equation: Numerical Solution

$$u_j^{k+1} = 2u_j^k - u_j^{k-1} + c\frac{\Delta t^2}{\Delta x^2}\left(u_{j+1}^k - 2u_j^k + u_{j-1}^k\right)$$



```
u0 = ...
u1 = ...
for t = 2*dt:dt:endt
   u2(2:n) = 2*u1(2:n)-u0(2:n)
     +c*(dt/dx)^2*(u1(3:n+1)-2*u1(2:n)+u1(1:n-1));
   u0 = u1;
   u1 = u2;
end
```

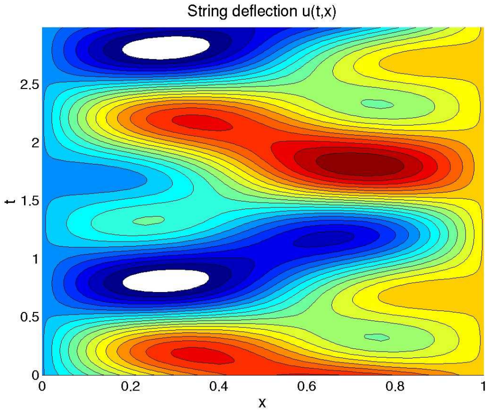
Wave Equation Results



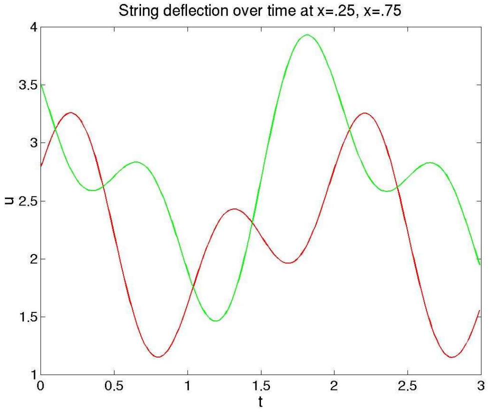
dx=1/30

dt=.01

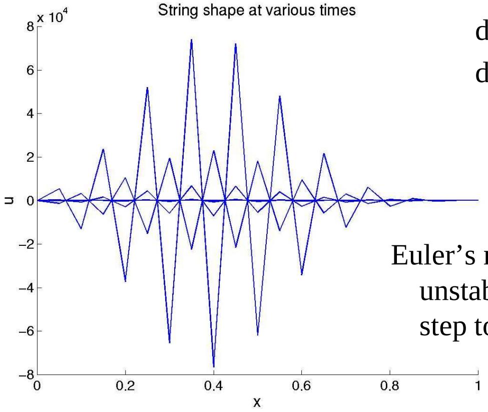
Wave Equation Results



Wave Equation Results



Poor results when dt too big



dx=.05

dt=.06

Euler’s method
unstable when
step too large

PDE Solution Methods

- Discretize in space, transform into system of IVP's
- Discretize in space and time, finite difference method.
- Discretize in space and time, finite element method.
  - Latter methods yield sparse systems.

- Sometimes the geometry and boundary conditions are simple (e.g. rectangular grid);
- Sometimes they're not (need mesh of triangles).