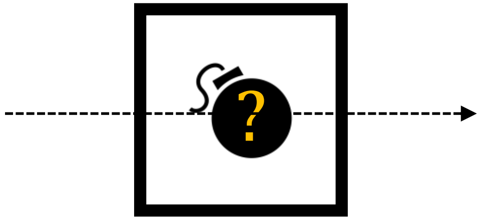
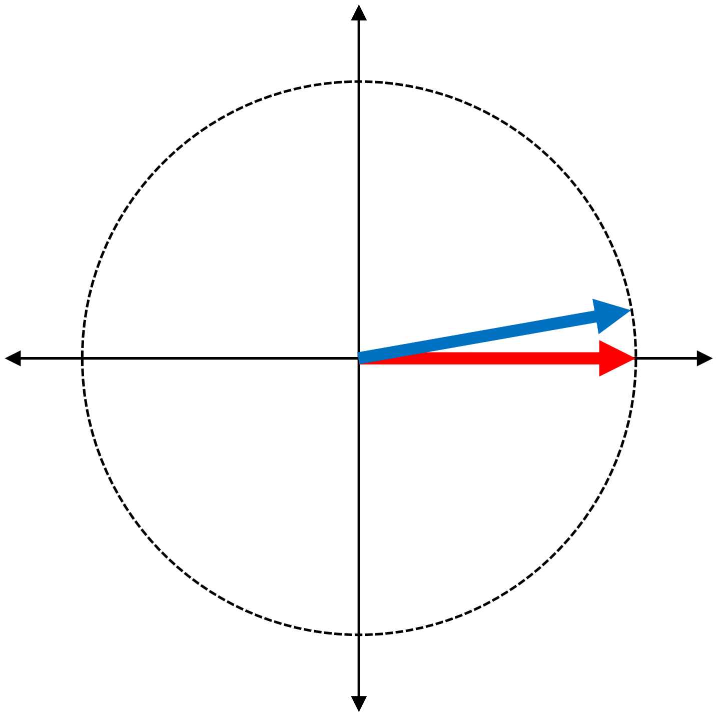
Lecture 4.5:
Discriminating Two Qubits

Input qubit at angle ϵ



Dud: Qubit at angle ϵ

Bomb: Qubit at angle 0 (assuming no explosion)



Discriminating Quantum States:

Given an *unknown* quantum state $|\psi\rangle$.

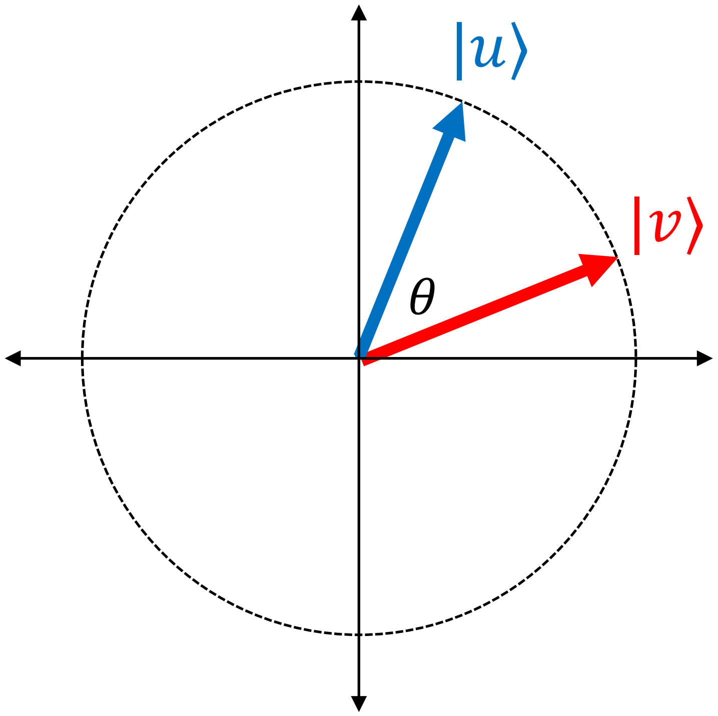
You're ***promised*** it's either $|u\rangle$ or $|v\rangle$.

(These are two states you *know*.)

Must guess whether $|\psi\rangle = |u\rangle$ or $|\psi\rangle = |v\rangle$.

Apply a unitary transformation?

Let's put their bisector at 45°.



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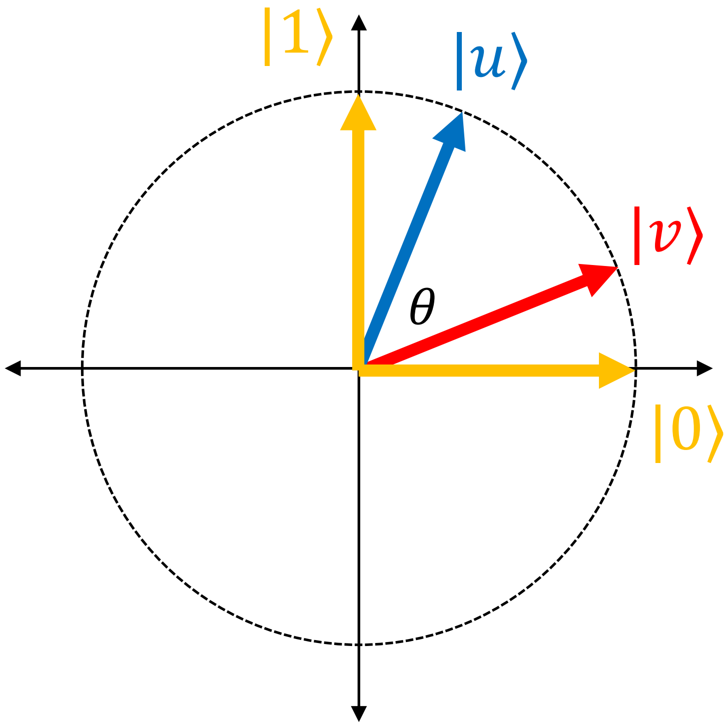
Measure in some basis?



Measure in the **standard basis**…

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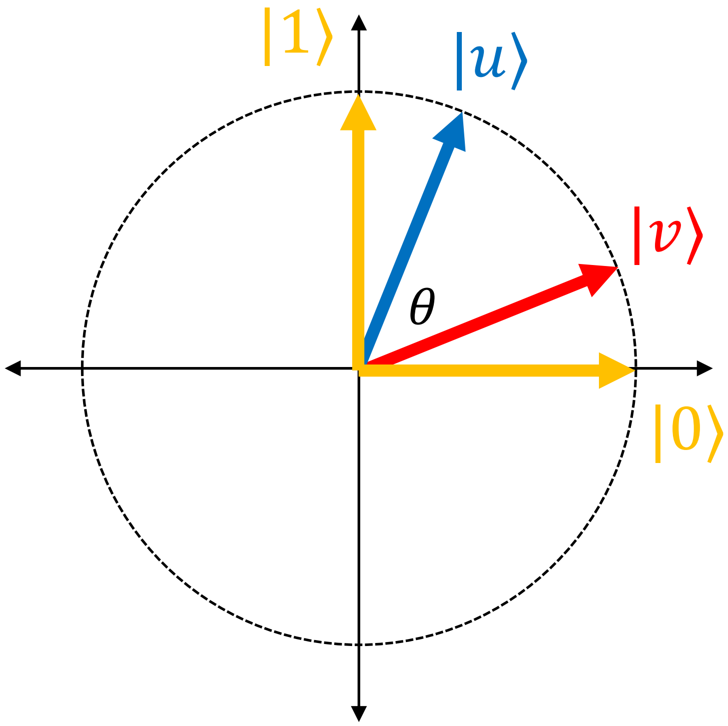


Measure in the **standard basis**...

- If readout $|0\rangle$, guess $|v\rangle$
- If readout $|1\rangle$, guess $|u\rangle$

Error?

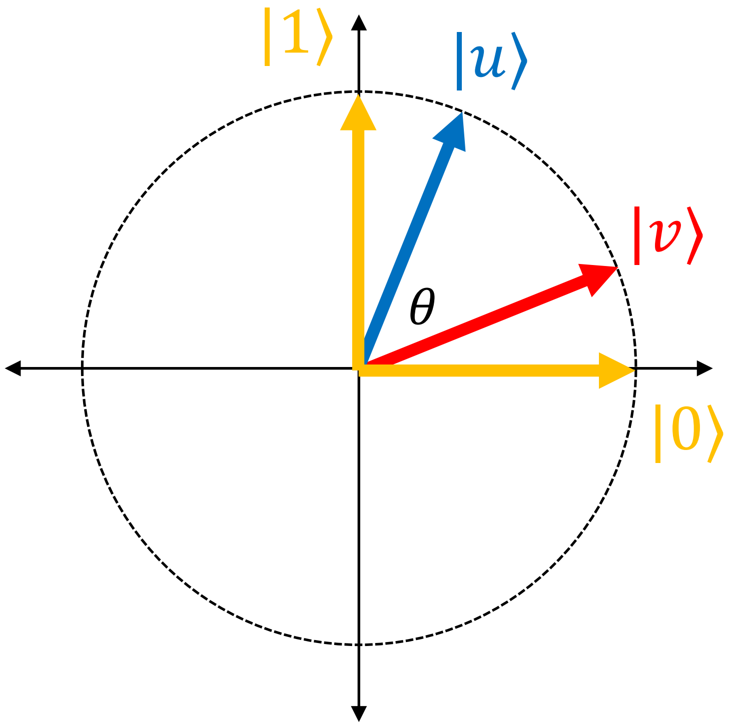
- If $|\psi\rangle = |u\rangle$ then $\mathbf{Pr}[\text{error}] = (\cos\gamma)^2$,

  where γ = angle between $|u\rangle$ and $|0\rangle$

  $$= \cos^2(45° + \theta/2)$$

  $$= \frac{1}{2} - \frac{1}{2}\sin\theta$$

Measure in some basis?



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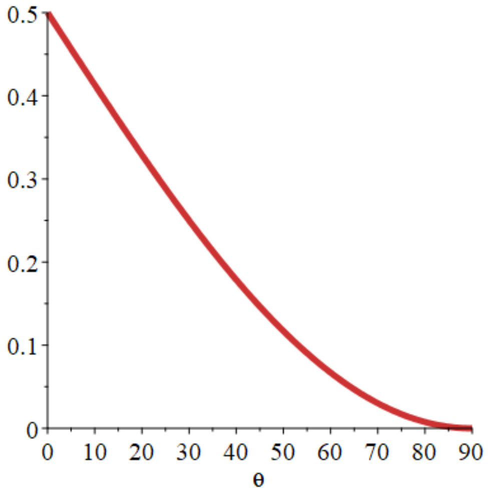
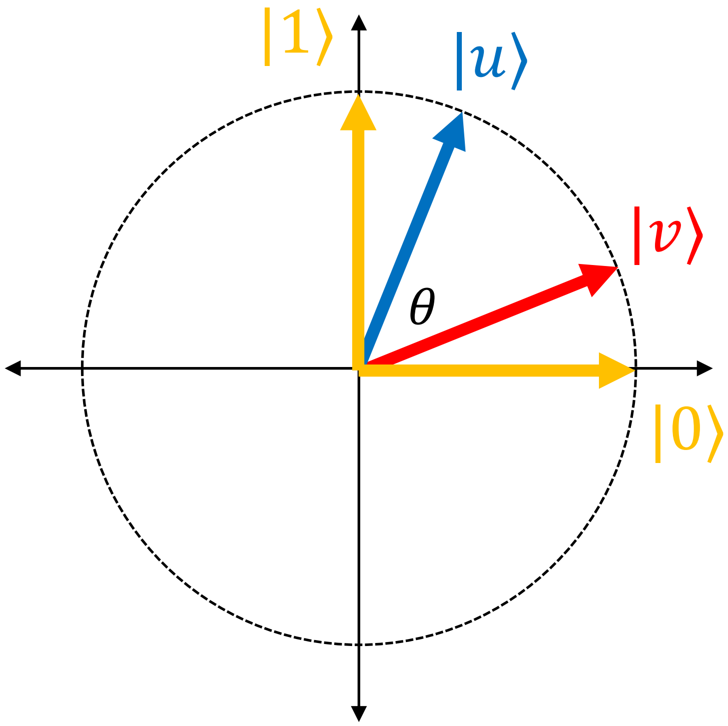


Measure in some basis?



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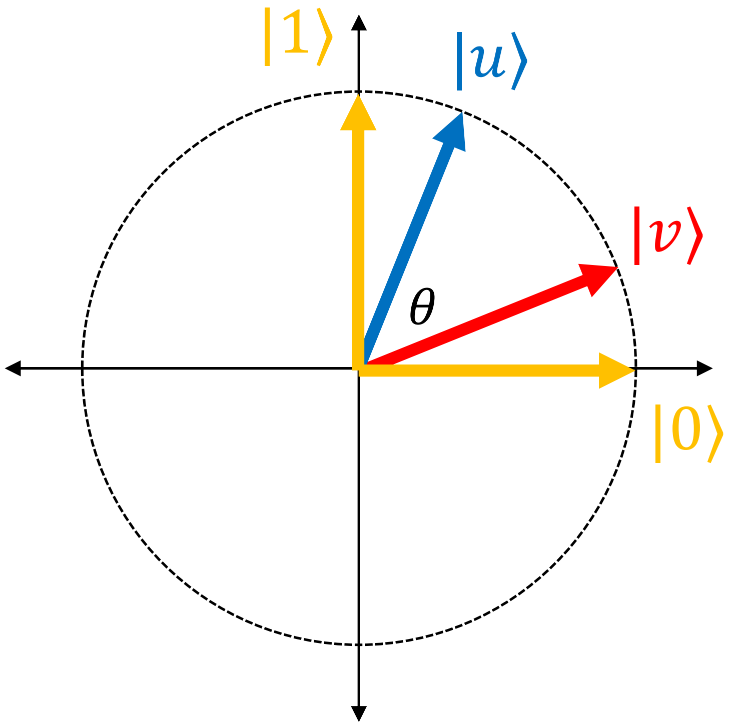
Measure in some basis?



Measure in the **standard basis**...

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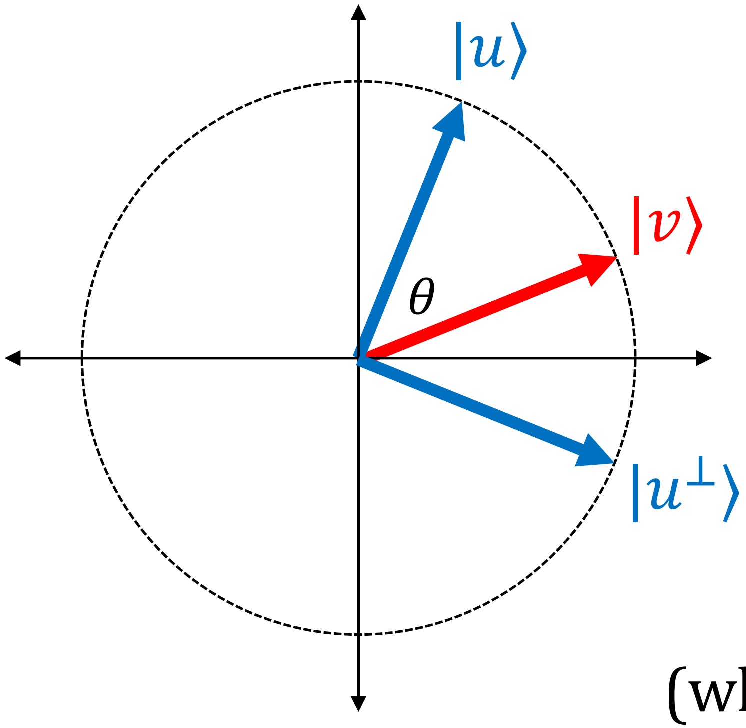
- If $|\psi\rangle = |u\rangle$ then $\mathbf{Pr}[\text{error}] = \frac{1}{2} - \frac{1}{2}\sin\theta$
- If $|\psi\rangle = |v\rangle$ then $\mathbf{Pr}[\text{error}] = \frac{1}{2} - \frac{1}{2}\sin\theta$

This is a **"two-sided error"** algorithm.

A "**one**-sided error" algorithm?



Measure in the $\{|u\rangle, |u^{\perp}\rangle\}$ basis...

- If readout $|u\rangle$, guess $|u\rangle$
- If readout $|u^{\perp}\rangle$, guess $|v\rangle$

Error?

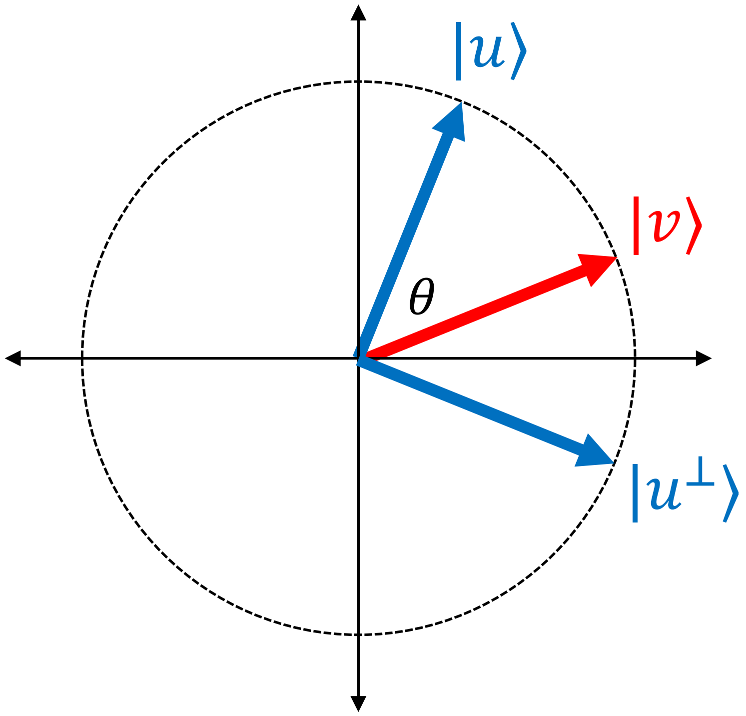
- If $|\psi\rangle = |u\rangle$ then $\mathbf{Pr}[\text{error}] = 0$
- If $|\psi\rangle = |v\rangle$ then $\mathbf{Pr}[\text{error}] = (\cos\theta)^2$

$$= 1 - (\sin\theta)^2$$

"No false positives"
(where $|v\rangle$ = bomb = 'positive')

same as prob. of explosion

A “**one**-sided error” algorithm?



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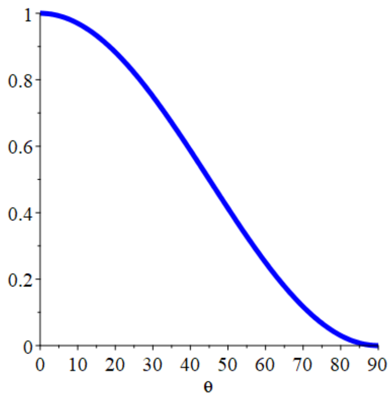
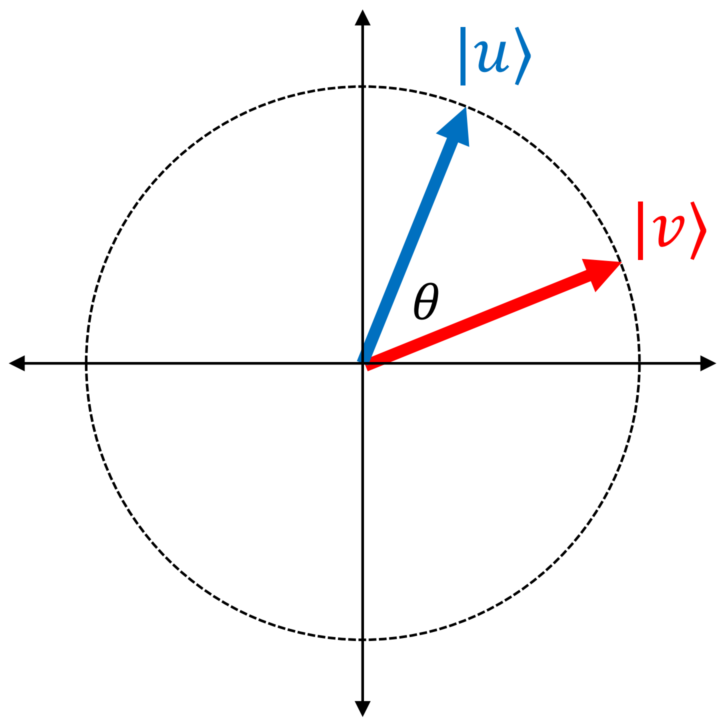


A “**zero**-sided error” algorithm?



We have a “no false positives” algorithm.

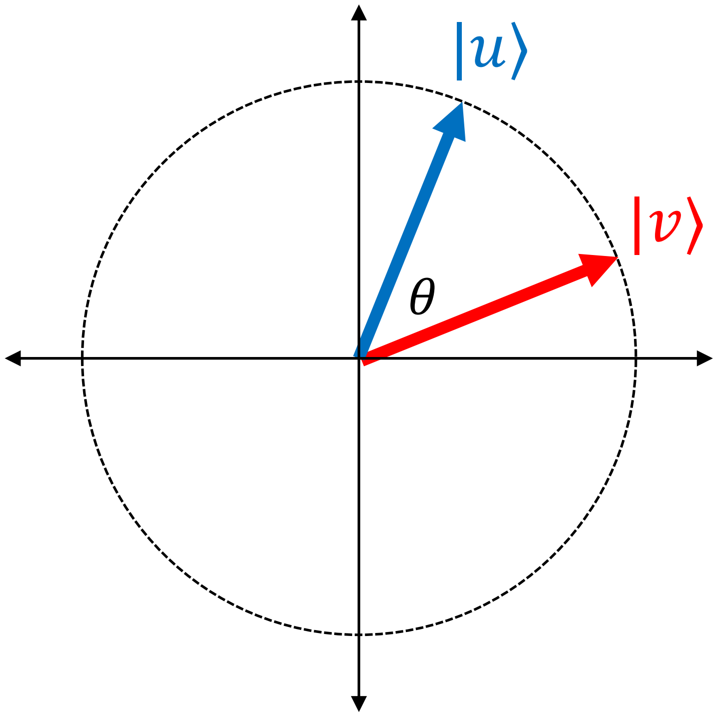
By symmetry, we have an equally good
“no false negatives” algorithm.

For a **zero**-sided error algorithm:

- With probability ½, do no-false-positives test;
  With probability ½, do no-false-negatives test
- If you get the answer you’re “sure of”, guess it;
  Otherwise, output “don’t know”

$$\mathbf{Pr}[\text{don't know}] = \frac{1}{2} + \frac{1}{2}\mathbf{Pr}[\text{error in one-sided alg.}] = 1 - \frac{(\sin\theta)^2}{2}$$

A “**zero**-sided error” algorithm?



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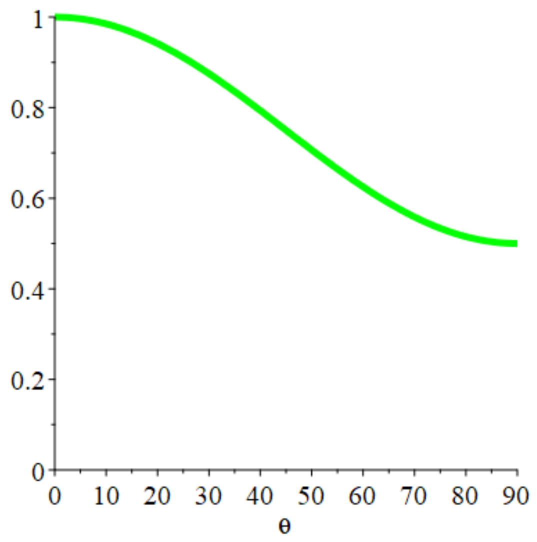
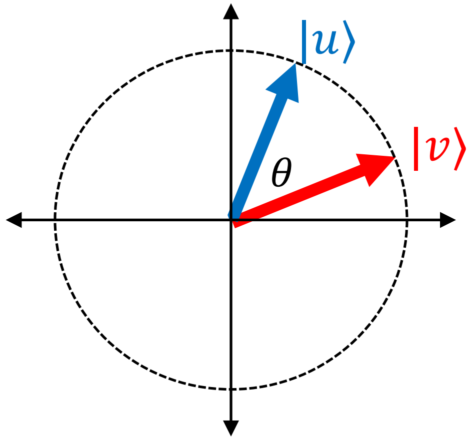


$$\mathbf{Pr}[\text{don't know}] = \frac{1}{2} + \frac{1}{2}\mathbf{Pr}[\text{error in one-sided alg.}] = 1 - \frac{(\sin\theta)^2}{2}$$

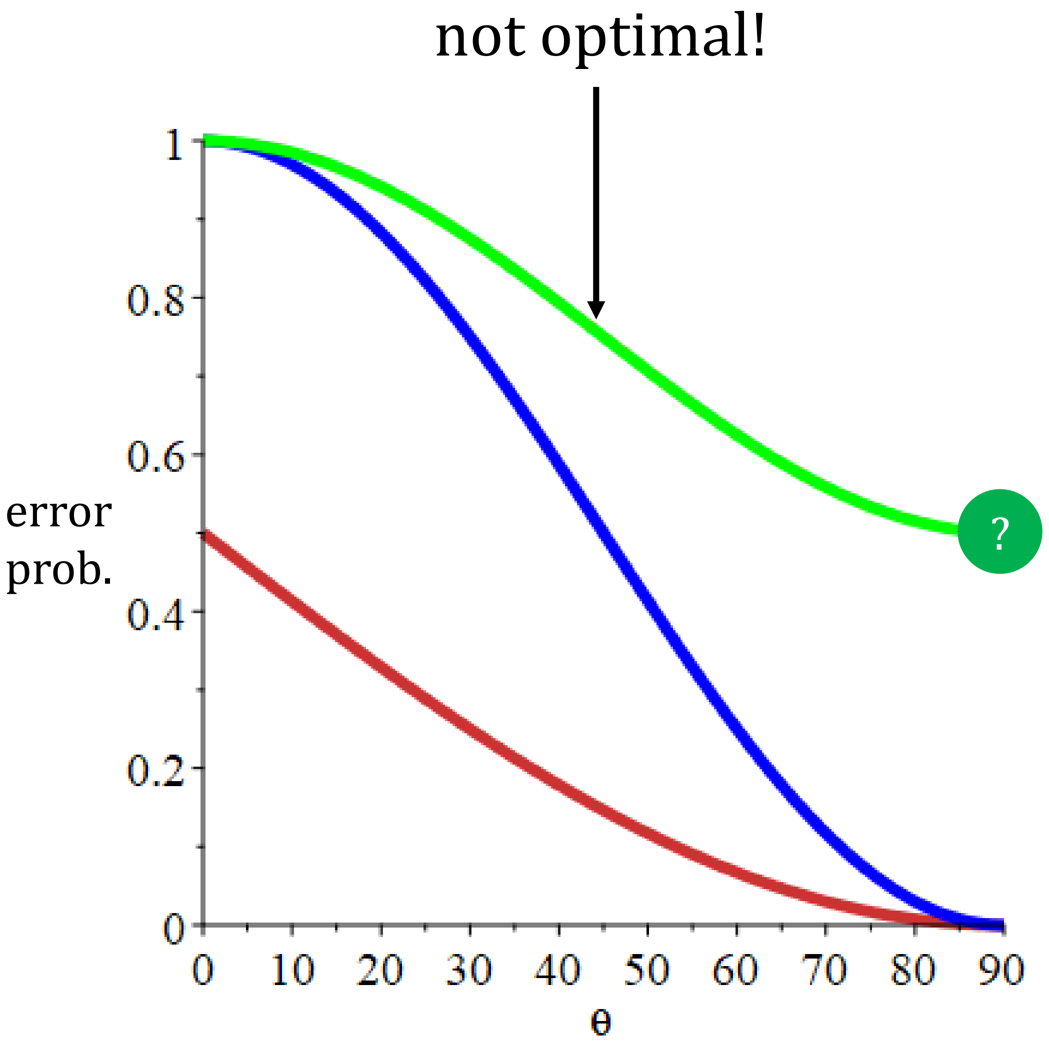
Discriminating Two Quantum States at Angle θ



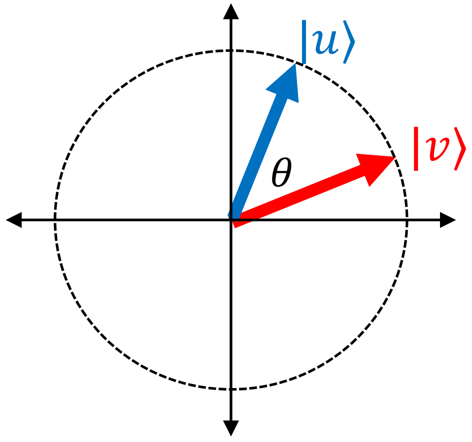
Two-sided error: $\frac{1}{2} - \frac{1}{2}\sin\theta$

One-sided error: $1 - (\sin\theta)^2$

Zero-sided error: $1 - \frac{(\sin\theta)^2}{2}$



Discriminating Two Quantum States at Angle θ



Two-sided error: $\frac{1}{2} - \frac{1}{2}\sin\theta$

One-sided error: $1 - (\sin\theta)^2$

Zero-sided error: $1 - \frac{(\sin\theta)^2}{2}$