Defs

- State j is <u>reachable</u> from state i if $(P^n)_{ij} > 0$ for some $n \geq 0$.

- A set C of states is <u>closed</u> if no state outside C is reachable from any state within C.

- A single state that constitutes a closed set is called <u>absorbing</u>.

- A Markov chain is <u>irreducible</u> if it contains no nonempty proper closed subset of states.

- A state is <u>persistent</u> if the probability of eventually returning to the state from itself is 1. It is called <u>transient</u> if this probability is less than 1.

- For a persistent state j, one speaks of the <u>mean recurrence time μ_j</u>, i.e., the expected time to return to j from itself.

- The <u>period</u> of state j is $\gcd\{n > 0 \mid (P^n)_{jj} > 0\}$. If the period is 1 or the set is ϕ, one says j is <u>aperiodic</u>.

- The <u>type</u> of state j is given by the following matrix:

	transient	persistent	
		$\mu_j < \infty$	$\mu_j = \infty$
aperiodic		also called "ergodic"	also called "null"
periodic with period t			

Facts & Theorems

- A closed set C of states in a Markov chain constitutes a Markov chain in its own right, obtained by deleting all rows & columns of states outside C.

- A Markov chain is irreducible iff every state is reachable from every other state.

- All states in an irreducible Markov chain have the same type (including period t, if relevant).

- The states of a Markov chain can be partitioned in a unique manner into (nonoverlapping) sets

$$T, C_1, C_2, \ldots$$

such that

(i) T consists of all transient states,

& (ii) each C_i is closed and irreducible in its own right, containing only persistent states of the same type, with the probability of eventually reaching state k from state j being 1 for any pair of states j & k in C_i.

The $\{C_i\}$ are sometimes called <u>recurrent classes</u>.

Facts & Theorems (cont)

- In an irreducible Markov chain with only ergodic states, the limits

$$\pi_k = \lim_{n \to \infty} (P^n)_{jk}$$

exist and are independent of j.

Furthermore, $\pi_k = \frac{1}{\mu_k}$

& the vector $(\pi_1, \pi_2, \dots)$

constitutes the invariant distribution of the chain.

aka "resting" or "stationary"

- In a finite Markov chain there exist no null states and not all states are transient.

- For a finite Markov chain, the multiplicity of the eigenvalue 1 of the chain's stochastic matrix P is equal to the number of recurrent classes.

- Moreover: (i) Any eigenvalue of magnitude $\underline{1}$ is actually a root of unity, e.g., $\lambda = e^{\frac{2\pi i}{d}}$, d an integer.
(ii) The d^{th} roots of unity are eigenvalues iff the Markov chain has a recurrent class whose states have period d. The multiplicity of each d^{th} root of unity is the number of recurrent classes with period d.