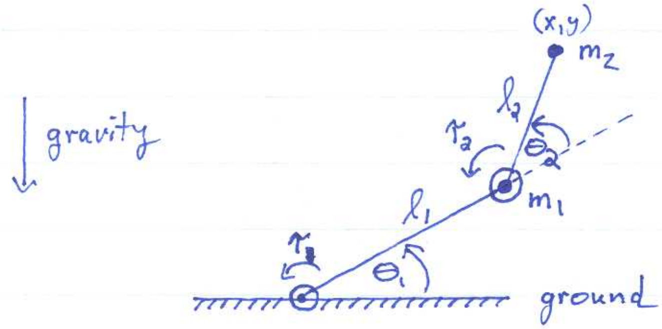
Manipulator Dynamics Summary



Two joints, with joint angles θ_1 & θ_2 as shown and motor torques τ_1 & τ_2.

Kinetic Energy: $T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

Potential Energy: $V = -m_1 g h_1 - m_2 g h_2$ $(g = -9.8\ldots\ m/s^2)$

v_i is the magnitude of mass m_i's velocity;
h_i is the height of mass m_i above the ground, $i = 1, 2$.

Want to derive the Lagrangian dynamics for the manipulator with θ_1 & θ_2 as generalized coordinates and τ_1 & τ_2 generalized forces.

Kinematics:

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

So

$$\dot{x} = -l_1 s_1 \dot{\theta}_1 - l_2 s_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$
$$\dot{y} = l_1 c_1 \dot{\theta}_1 + l_2 c_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$

[Here we use abbreviations: $\dot{x}$ means $\frac{dx}{dt}$, $\dot{y}$ means $\frac{dy}{dt}$, etc., $c_i = \cos(\theta_i)$, $s_i = \sin(\theta_i)$, $c_{12} = \cos(\theta_1 + \theta_2)$, $s_{12} = \sin(\theta_1 + \theta_2)$.]

So, in T & V:

$$h_1 = l_1 s_1$$
$$h_2 = l_1 s_1 + l_2 s_{12}$$

$$v_1^2 = l_1^2 \dot{\theta}_1^2$$
$$v_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 l_2 c_2 (\dot{\theta}_1 + \dot{\theta}_2) \dot{\theta}_1 .$$

Lagrange says:

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1}$$

$$\tau_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2},$$

with $L = T - V$

$$= \frac{1}{2}(m_1+m_2)\,l_1^2\,\dot{\theta}_1^2 + g\,(m_1+m_2)\,l_1 s_1$$

$$+ \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1+\dot{\theta}_2)^2 + m_2 l_1 l_2 c_2 (\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_1$$

$$+ m_2 g\, l_2 s_{12}.$$

In control theory one often writes equations like this in the form

$$\tau = M(\theta)\ddot{\theta} + C(\theta,\dot{\theta}) + G(\theta),$$

with $\tau, \theta, \dot{\theta}, \ddot{\theta}$ vectors,

$M(\theta)$ an inertia matrix (depending on configuration θ),

$C(\theta,\dot{\theta})$ a vector modeling Coriolis & centripetal terms,

$G(\theta)$ a vector modeling gravity terms.

$$\frac{\partial L}{\partial \theta_1} = g(m_1+m_2)\,l_1 c_1 + g m_2 l_2 c_{12}$$

$$\frac{\partial L}{\partial \dot\theta_1} = (m_1+m_2) l_1^2 \dot\theta_1 + m_2 l_2^2 (\dot\theta_1+\dot\theta_2) + 2 m_2 l_1 l_2 c_2 \dot\theta_1 + m_2 l_1 l_2 c_2 \dot\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) =$$

$$\left[(m_1+m_2) l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 c_2\right] \ddot\theta_1$$ inertial term

$$+ \quad \left[m_2 l_2^2 + m_2 l_1 l_2 c_2\right] \ddot\theta_2$$ coupling inertial term

$$+ \quad \left[-2 m_2 l_1 l_2 s_2\right] \dot\theta_1 \dot\theta_2$$ Coriolis term

$$+ \quad \left[-m_2 l_1 l_2 s_2\right] \dot\theta_2^2$$ centripetal term

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1 + g m_2 l_2 c_{12}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + m_2 l_1 l_2 c_2 \dot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \left[m_2 l_2^2 + m_2 l_1 l_2 c_2\right] \ddot{\theta}_1$$ coupling inertial term

$$+ \left[m_2 l_2^2\right] \ddot{\theta}_2$$ inertial term

$$+ \left[-m_2 l_1 l_2 s_2\right] \dot{\theta}_1 \dot{\theta}_2$$ Coriolis term

Thus $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} =$

$$\begin{bmatrix} (m_1+m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 c_2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ m_2 l_2^2 + m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

(effective inertia matrix)

$$+$$

$$\begin{bmatrix} 0 & -2m_2 l_1 l_2 s_2 & -m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix}$$

(velocity-dependent terms)

$$+$$

$$(-g) \begin{bmatrix} (m_1+m_2)l_1 c_1 + m_2 l_2 c_{12} \\ m_2 l_2 c_{12} \end{bmatrix}$$

(gravity-dependent terms)