Chapter 4

Retractions, Projections, and Closures

(Notes for CS 819 — April 17, 2000 — ©2000 John C. Reynolds)

4.1 Galois Connections

Let P and P' be posets. Then a *Galois connection* from P to P' is a pair of monotone functions

$$P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$$

such that, for all $x \in P$ and $x' \in P'$,

$$x' \sqsubseteq' \phi x \text{ iff } \psi x' \sqsubseteq x.$$

(When posets are viewed as categories and monotone functions as functors, this is equivalent to saying that ψ is a left adjoint of ϕ, or ϕ is a right adjoint of ψ.)

Proposition 4.1 *For posets P and P' and functions $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$, the following conditions are equivalent:*

1. *$P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is a Galois connection.*
2. *ϕ and ψ are monotone, and, for all $x \in P$ and $x' \in P'$,*

$$\psi(\phi x) \sqsubseteq x \text{ and } \phi(\psi x') \sqsupseteq' x'.$$

3. *ϕ is monotone and, for all $x' \in P'$, $\psi x'$ is the least $x \in P$ such that $\phi x \sqsupseteq' x'$.*

4. *ψ is monotone and, for all $x \in P$, ϕx is the greatest $x' \in P'$ such that $\psi x' \sqsubseteq x$.*

PROOF The equivalence of (1) and (2) are left to the reader.

Assume (2). Then $\phi(\psi x') \sqsupseteq' x'$ establishes that $\psi x'$ is an x such that $\phi x \sqsupseteq' x'$. Moreover, for any x such that $\phi x \sqsupseteq' x'$, $\psi(\phi x) \sqsubseteq x$ and the monotonicity of ψ give $x \sqsupseteq \psi(\phi x) \sqsupseteq \psi x'$, which establishes that $\psi x'$ is the least x such that $\phi x \sqsupseteq' x'$.

Conversely, assume (3). Since $\psi x'$ is an x such that $\phi x \sqsupseteq' x'$, we have $\phi(\psi x') \sqsupseteq' x'$. Since it is the least such x, we have that $\psi x' \sqsubseteq x$ whenever $\phi x \sqsupseteq' x'$; then taking the case where $x' = \phi x$ gives $\psi(\phi x) \sqsubseteq x$. Finally, suppose $x' \sqsubseteq' y'$. Then

$$\{x \mid \phi x \sqsupseteq' x'\} \supseteq \{x \mid \phi x \sqsupseteq' y'\},$$

so that, since $\psi y'$ is the least member of the righthand set, it must be a member of the lefthand set, and therefore an extension of its least member $\psi x'$. Thus ψ is monotone.

The equivalence of (2) and (4) is left to the reader. The proof is the dual of that for (3), with ϕ and ψ interchanged. END OF PROOF

Notice that (3) and (4) show that each function in a Galois connection uniquely determines the other.

Proposition 4.2

1. *If $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is a Galois connection, then ϕ preserves all existing glb's in P.*

2. *If P is a complete lattice, and ϕ preserves all glb's, then ϕ is part of a Galois connection.*

3. *If $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is a Galois connection, then ψ preserves all existing lub's in P'.*

4. *If P' is a complete lattice, and ψ preserves all lub's, then ψ is part of a Galois connection.*

Proof

1. Suppose $X \subseteq P$ possesses a glb. Since ϕ is monotone, $\phi(\bigsqcap X)$ is a lower bound of $\{\phi x \mid x \in X\}$. Suppose x' is a lower bound of $\{\phi x \mid x \in X\}$. Since ψ is monotone, $\psi x'$ is a lower bound of $\{\psi(\phi x) \mid x \in X\}$, and since $\psi(\phi x) \sqsubseteq x$, $\psi x'$ is a lower bound of X. Thus $\psi x' \sqsubseteq \bigsqcap X$. Then $x' \sqsubseteq' \phi(\psi x') \sqsubseteq' \phi(\bigsqcap X)$, so that $\phi(\bigsqcap X)$ is the glb of $\{\phi x \mid x \in X\}$.

2. Suppose P is a complete lattice (and thus has all glb's), and ϕ preserves glb's. To see that ϕ is monotone, suppose $x \sqsubseteq y$. Then $\phi x = \phi(x \sqcap y) = \phi x \sqcap' \phi y \sqsubseteq' \phi y$.

   Define $\psi x' = \bigsqcap\{x \mid \phi x \sqsupseteq' x'\}$. Since ϕ preserves glb's, $\phi(\psi x') = \bigsqcap'\{\phi x \mid \phi x \sqsupseteq' x'\}$. But this is the glb of a set whose members all extend x', so that $\phi(\psi x') \sqsupseteq' x'$. Thus $\psi x'$ is a member of the set $\{\phi x \mid \phi x \sqsupseteq' x'\}$, and since it is also the glb of this set, it is the least member.

3. This is the dual of part 1.

4. This is the dual of part 2.

END OF PROOF

Proposition 4.3 *Suppose $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is a Galois connection in which P and P' are dcpo's and ϕ and ψ are continuous. Then, for all $x', y' \in P'$, if $x' \prec' y'$ then $\psi x' \prec \psi y'$.*

Proof Suppose $x' \prec' y'$. Since $\phi(\psi y') \sqsupseteq' y'$, we have $x' \prec' \phi(\psi y')$. Then by Proposition 3.5, taking f to be ϕ, g to be ψ, and x to be $\psi y'$, we have $\psi x' \prec \psi y'$.

END OF PROOF

4.2 Retraction, Projection, and Closure Pairs

Let P and P' be dcpo's and ϕ and ψ be continuous functions such that

$$P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$$

- If, for all $x \in P$, $\psi(\phi x) = x$, then $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is called a *retraction pair*, and P is called a *retract* of P'.

- If, for all $x' \in P'$, the retraction pair $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ satisfies $\phi(\psi x') \sqsubseteq' x'$, than it is called a *projection pair*, and P is called a *projection* of P'.
- If, for all $x' \in P'$, the retraction pair $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ satisfies $\phi(\psi x') \sqsupseteq' x'$, than it is called a *closure pair*, and P is called a *closure* of P'.

(In a retraction, projection, or closure pair $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$, ϕ is sometimes called an *embedding* and ψ is sometimes called a retraction, projection, or closure.)

Proposition 4.4

1. *Suppose $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is a retraction pair. For all $x, y \in P$,*

$$\phi x \sqsubseteq' \phi y \text{ implies } x \sqsubseteq y$$
$$\phi x \prec' \phi y \text{ implies } x \prec y.$$

2. *Suppose $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is a projection pair. For all $x, y \in P$,*

$$x \prec y \text{ implies } \phi x \prec' \phi y.$$

   Moreover, ϕ preserves all existing lub's in P, and ψ preserves all existing glb's in P'.

3. *Suppose $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is a closure pair. For all $x', y' \in P'$,*

$$x' \prec' y' \text{ implies } \psi x' \prec \psi y'.$$

   Moreover, ϕ preserves all existing glb's in P, and ψ preserves all existing lub's in P'.

PROOF

1. If $\phi x \sqsubseteq' \phi y$, then the monotonicity of ψ gives $\psi(\phi x) \sqsubseteq \psi(\phi y)$, and the equation defining retraction pairs gives $x \sqsubseteq y$.

   If $\phi x \prec' \phi y$ then by Proposition 3.5, taking f to be ϕ, g to be ψ, x' to be ϕx, and x to be y, we have $\psi(\phi x) \prec y$, and the equation defining retraction pairs gives $x \prec y$.

2. When $P \overset{\phi}{\underset{\psi}{\rightleftarrows}} P'$ is a projection pair, $P' \overset{\psi}{\underset{\phi}{\rightleftarrows}} P$ is a Galois connection. The desired properties are then given by Propositions 4.3 and 4.2.

3. A closure pair is a Galois connection. The desired properties are then given by Propositions 4.3 and 4.2.

END OF PROOF

Proposition 4.5 *If c is a color, P' is a c-complete dcpo, and P is a retract of P', then P is a c-complete dcpo.*

PROOF Let $P \overset{\phi}{\underset{\psi}{\rightleftarrows}} P'$ be a retraction pair, and $D \in c_P$. By the first property of colors, $\{\phi x \mid x \in D\}$ belongs to $c_{P'}$ and, since P' is $c-$complete, possesses a lub x'.

Since x' is an upper bound of $\{\phi x \mid x \in D\}$, $\psi x'$ is an upper bound of $\{\psi(\phi x) \mid x \in D\} = D$. On the other hand, suppose y is an upper bound of D. Then ϕy is an upper bound on $\{\phi x \mid x \in D\}$, so that $\phi y \sqsupseteq x'$. Then $y = \psi(\phi y) \sqsupseteq \psi x'$. Thus $\psi x'$ is the lub of D. END OF PROOF

Proposition 4.6 *Suppose $P \overset{\phi}{\underset{\psi}{\rightleftarrows}} P'$ is a retraction pair. Then*

1. *If E' is a basis of P', then*

$$E \overset{\text{def}}{=} \{\psi e' \mid e' \in E'\}$$

is a basis of P.

2. *If P' is a continuous (countably-based) dcpo, then P is a continuous (countably-based) dcpo.*

3. *If $P \overset{\phi}{\underset{\psi}{\rightleftarrows}} P'$ is a closure pair, and all elements of $E' \subseteq P'$ are isolated, then all elements of $E = \{\psi e' \mid e' \in E'\}$ are isolated.*

4. *If $P \overset{\phi}{\underset{\psi}{\rightleftarrows}} P'$ is a closure pair, and P' is an algebraic (countably-based) dcpo, then P is an algebraic (countably-based) dcpo.*

PROOF

1. Suppose $y \in P$. Since E' is a basis of P',

$$\phi y = {\bigsqcup}'\{e' \mid e' \in E' \text{ and } e' \prec' \phi y\},$$

where the set on the right is directed. Then since $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is a retraction and ψ is continuous,

$$y = \psi(\phi y) = \bigsqcup Y$$

where

$$Y = \{\psi e' \mid e' \in E' \text{ and } e' \prec' \phi y\}$$

is directed.

Now consider the set

$$\begin{aligned} X &= \{e \mid e \in E \text{ and } e \prec y\} \\ &= \{\psi e' \mid e' \in E' \text{ and } \psi e' \prec y\}. \end{aligned}$$

To show that E is a basis, we must show that y is the lub of X and that X is directed. Obviously y is an upper bound on X. Suppose z is an upper bound on X. Since, for all $x \in P$, $\psi(\phi x) \sqsubseteq x$, Proposition 3.5 shows that $e' \prec' \phi y$ implies $\psi e' \prec y$, so that $Y \subseteq X$, and thus z is an upper bound of Y. Then $z \sqsupseteq y$, so that y is the lub of X.

To show that X is directed, let $\{e_1, \ldots, e_n\}$ be a finite subset of X. Each e_i satisfies $e_i \prec y = \bigsqcup Y$, so that, since Y is directed, there is an $x_i \in Y$ such that $e_i \sqsubseteq x_i$. Then, again since Y is directed, there is an $x \in Y$ that is an upper bound of $\{x_1, \ldots, x_n\}$ and therefore of $\{e_1, \ldots, e_n\}$. But $Y \subseteq X$, so that x is an upper bound of $\{e_1, \ldots, e_n\}$ that belongs to X.

2. This follows from the previous part since a dcpo is continuous iff it has a basis.

3. Suppose $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is a closure pair and every member of E' is isolated. For any $e \in E$, there is an $e' \in E'$ such that $e = \psi e'$. Since e' is isolated, $e' \prec' e'$. The part 3 of Proposition 4.4 gives $\psi e' \prec \psi e'$, so that e is isolated.

4. This follows since a dcpo is algebraic iff it has a basis whose members are all isolated.

END OF PROOF

Proposition 4.7 *If P is a complete lattice with basis E, then P is a retract of $\mathcal{P}\, E$. If, in addition, E contains only isolated elements, then P is a closure of $\mathcal{P}\, E$.*

PROOF Let $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathcal{P}\, E$ be the functions such that

$$\phi y = \{e \mid e \in E \text{ and } e \prec y\} \qquad\qquad \psi S = \textstyle\bigsqcup_P S.$$

To see that ϕ is continuous, let D be any directed subset of P. Then

$$\begin{aligned}\phi(\textstyle\bigsqcup D) &= \{e \mid e \in E \text{ and } e \prec \textstyle\bigsqcup D\}\\ &= \{e \mid e \in E \text{ and } \exists x \in D.\ e \prec x\}\\ &= \textstyle\bigcup\{\{e \mid e \in E \text{ and } e \prec x\} \mid x \in D\}\\ &= \textstyle\bigcup\{\phi x \mid x \in D\},\end{aligned}$$

where the second equality is justified by part 3 of Proposition 3.9.

To see that ψ is continuous (actually additive), suppose $\mathcal{S} \subseteq \mathcal{P}\, E$.

$$\begin{aligned}\psi(\textstyle\bigcup \mathcal{S}) &= \textstyle\bigsqcup(\bigcup \mathcal{S})\\ &= \textstyle\bigsqcup\{\bigsqcup S \mid S \in \mathcal{S}\}\\ &= \textstyle\bigsqcup\{\psi S \mid S \in \mathcal{S}\}.\end{aligned}$$

To see that $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathcal{P}\, E$ is a retraction pair, suppose $y \in P$. Then

$$\begin{aligned}\psi(\phi y) &= \textstyle\bigsqcup\{e \mid e \in E \text{ and } e \prec y\}\\ &= y.\end{aligned}$$

Finally, suppose all members of E are isolated. To show that $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathcal{P}\, E$ is a closure pair, we must show that, for all $S \in \mathcal{P}\, E$, $S \subseteq \phi(\psi S)$. But if $e \in S$ then $e \prec e$ and $e \sqsubseteq \bigsqcup S$, so that $e \in \{e \mid e \in E \text{ and } e \prec \bigsqcup S\} = \phi(\psi S)$.
END OF PROOF

Proposition 4.8 *If $E \subseteq E'$, then $\mathcal{P}\, E$ is both a projection and a closure of $\mathcal{P}\, E'$.*

PROOF For a projection, take $\phi \in \mathcal{P}\, E \to \mathcal{P}\, E'$ and $\psi \in \mathcal{P}\, E' \to \mathcal{P}\, E$ to be the functions such that

$$\phi S = S \qquad\qquad \psi S' = S' \cap E.$$

For a closure, take

$$\phi S = S \cup (E' - E) \qquad\qquad \psi S' = S' \cap E.$$

END OF PROOF

Then as a consequence of Propositions 4.6, 4.7, and 4.8, we have

Proposition 4.9

1. *A poset is a continuous countably-based lattice iff it is a retract of $\mathcal{P}\,\omega$.*
2. *A poset is an algebraic countably-based lattice iff it is a closure of $\mathcal{P}\,\omega$.*

4.3 Retractions, Projections, and Closures

For a function $\rho \in P \to P$, we write

$$\operatorname{ran}\rho \stackrel{\text{def}}{=} \{x \mid x \in P \text{ and } \exists y \in P.\ x = \rho y\}$$

$$\mathcal{F}\rho \stackrel{\text{def}}{=} \{x \mid x \in P \text{ and } x = \rho x\}$$

for the range and set of fixed points of ρ. It is clear that $\mathcal{F}\rho \subseteq \operatorname{ran}\rho \subseteq P$, and that $\mathcal{F}\rho = \operatorname{ran}\rho$ iff, for all $x \in P$, $\rho x = \rho(\rho x)$.

Proposition 4.10 *Suppose P is c-complete and $\rho \in P \to P$ is c-continuous. Then $\mathcal{F}\rho$ (regarded as a subposet of P) is c-complete, and for all $D \in c_{\mathcal{F}\rho}$, $\bigsqcup_{\mathcal{F}\rho} D = \bigsqcup_P D$.*

PROOF Suppose $D \in c_{\mathcal{F}\rho}$. By the first property of colors, applied to the injection from $\mathcal{F}\rho$ to P, $D \in c_P$, so that $\bigsqcup_P D$ exists. To show that this lub is also the lub in $\mathcal{F}\rho \subseteq P$, it is sufficient to show that it belongs to $\mathcal{F}\rho$. But, since ρ is c-continuous, and the members of D are fixed points of ρ,

$$\rho(\textstyle\bigsqcup_P D) = \bigsqcup_P \{\rho x \mid x \in D\} = \bigsqcup_P D.$$

Thus $\bigsqcup_P D$ is a fixed point of ρ. END OF PROOF

Let P be a dcpo and $\phi \in P \to P$ be continuous. Then

- If, for all $x \in P$, $\rho(\rho x) = \rho x$, then ρ is called a *retraction on P*.

- If, for all $x \in P$, the retraction ρ satisfies $\rho x \sqsubseteq x$, then it is called a *projection* on P.
- If, for all $x \in P$, the retraction ρ satisfies $\rho x \sqsupseteq x$, then it is called a *closure* on P.

It is evident that if $P \underset{\psi}{\overset{\phi}{\rightleftarrows}} P'$ is a retraction (projection, closure) pair than $\psi\,;\phi$ is a retraction (projection, closure). On the other hand,

Proposition 4.11 *Suppose ρ is a retraction on a dcpo P. Let ϕ be the identity injection from $\mathcal{F}\rho$ to P, and let ψ be the corestriction of ρ to a function from P to $\mathcal{F}\rho$. Then $\mathcal{F}\rho \underset{\psi}{\overset{\phi}{\rightleftarrows}} P$ is a retraction pair such that $\psi\,;\phi = \rho$. Moreover, if ρ is a projection (closure), then $\mathcal{F}\rho \underset{\psi}{\overset{\phi}{\rightleftarrows}} P$ is a projection (closure) pair. Also, if $\hat{P} \underset{\psi'}{\overset{\phi'}{\rightleftarrows}} P$ is any retraction pair such that $\psi'\,;\phi' = \rho$, then*

$$\hat{P} \underset{\phi\,;\,\psi'}{\overset{\phi'\,;\,\psi}{\rightleftarrows}} \mathcal{F}\rho$$

is an isomorphism.

Thus every retract (projection, closure) *of* P is isomorphic to the poset of fixed points of a retract (projection, closure) *on* P.

Proof We first note that the definition of ψ makes sense because, for all $x \in P$, $\rho(\rho x) = \rho x$ insures that ρx is a fixed point of ρ.

To see that ϕ is continuous, let D be a directed subset of $\mathcal{F}\rho$. Then, by Proposition 4.10,

$$\begin{aligned} \phi(\textstyle\bigsqcup_{\mathcal{F}\rho} D) &= \textstyle\bigsqcup_{\mathcal{F}\rho} D \\ &= \textstyle\bigsqcup_P D \\ &= \textstyle\bigsqcup_P \{\phi x \mid x \in D\}. \end{aligned}$$

To see that ψ is continuous, let D be a directed subset of P. Then

$$\begin{aligned} \psi(\textstyle\bigsqcup_P D) &= \rho(\textstyle\bigsqcup_P D) \\ &= \textstyle\bigsqcup_P \{\rho x \mid x \in D\}, \end{aligned}$$

since ρ is continuous. But $\{\rho x \mid x \in D\}$ is a directed subset of $\mathcal{F}\rho$. Thus, by Proposition 4.10,

$$\begin{aligned} \textstyle\bigsqcup_P \{\rho x \mid x \in D\} &= \textstyle\bigsqcup_{\mathcal{F}\rho} \{\rho x \mid x \in D\} \\ &= \textstyle\bigsqcup_{\mathcal{F}\rho} \{\psi x \mid x \in D\}. \end{aligned}$$

If $x \in \mathcal{F}\rho$, then $\psi(\phi x) = \rho x = x$, so that $\mathcal{F}\rho \underset{\psi}{\overset{\phi}{\rightleftarrows}} P$ is a retraction pair. On the other hand, if $x \in P$ then $\phi(\psi x) = \rho x$, or $\psi\,;\phi = \rho$. It follows that, if ρ is a projection (closure), then $\mathcal{F}\rho \underset{\psi}{\overset{\phi}{\rightleftarrows}} P$ is a projection (closure) pair.

Finally, suppose $\hat{P} \underset{\psi'}{\overset{\phi'}{\rightleftarrows}} P$ is any retraction pair such that $\psi';\phi' = \rho$. Then:

$$\begin{aligned} \phi'\,;\psi\,;\phi\,;\psi' &= \phi'\,;\rho\,;\psi' \\ &= \phi'\,;\psi'\,;\phi'\,;\psi' \\ &= I_{\hat{P}}\,;I_{\hat{P}} \\ &= I_{\hat{P}} \end{aligned} \qquad \begin{aligned} \phi\,;\psi'\,;\phi'\,;\psi &= \phi\,;\rho\,;\psi \\ &= \phi\,;\psi\,;\phi\,;\psi \\ &= I_{\mathcal{F}\rho}\,;I_{\mathcal{F}\rho} \\ &= I_{\mathcal{F}\rho}. \end{aligned}$$

Proposition 4.12 *If ρ and ρ' are projections on P, then $\rho \sqsubseteq \rho'$ iff $\mathcal{F}\rho \subseteq \mathcal{F}\rho'$. If ρ and ρ' are closures on P, then $\rho \sqsupseteq \rho'$ iff $\mathcal{F}\rho \subseteq \mathcal{F}\rho'$.*

PROOF Suppose $\rho \sqsubseteq \rho'$, and let x be a fixed point of ρ. Then $x = \rho x \sqsubseteq \rho' x \sqsubseteq x$, where the last inequality holds since ρ' is a projection. Thus $x = \rho' x$.

On the other hand, suppose $\mathcal{F}\rho \subseteq \mathcal{F}\rho'$. Then, for any $x \in P$, since ρx is a fixed point of ρ, it is also a fixed point of ρ', so that $\rho x \sqsubseteq \rho'(\rho x) \sqsubseteq \rho' x$, where the last inequality holds since ρ is a projection and ρ' is monotone.

The proof for closures is dual. END OF PROOF

4.4 Equalizers

Proposition 4.13 *Suppose*

$$P \underset{g}{\overset{f}{\rightrightarrows}} Q,$$

where P and Q are dcpo's with glb's of all lower-bounded subsets, and f and g are continuous functions that preserve such glb's. Let $\rho \in P \to P$ be such that

$$\rho x = \bigsqcap\{y \mid fy = gy \text{ and } x \sqsubseteq y\},$$

and let ϕ be the identity injection from $\mathcal{F}\rho$ to P. Then

$$\mathcal{F}\rho \xrightarrow{\phi} P \underset{g}{\overset{f}{\rightrightarrows}} Q$$

is an equalizer, where $\mathcal{F}\rho$ has glb's of all lower-bounded subsets, and ϕ is a continuous function that preserves such glb's. If P is continuous (countably based, algebraic), then $\mathcal{F}\rho$ is continuous (countably based, algebraic).

PROOF

1. For all $x \in P$, since ρx is the glb of a set that is lower-bounded (by x),

$$\begin{aligned} f(\rho x) &= \bigsqcap\{fy \mid fy = gy \text{ and } x \sqsubseteq y\} \\ &= \bigsqcap\{gy \mid fy = gy \text{ and } x \sqsubseteq y\} \\ &= g(\rho x). \end{aligned}$$

2. For all $x \in P$, since x is a lower bound on $\{y \mid fy = gy \text{ and } x \sqsubseteq y\}$, $\rho x \sqsupseteq x$.

3. For all $x \in P$, if $fx = gx$, then $x \in \{y \mid fy = gy \text{ and } x \sqsubseteq y\}$, so that $\rho x \sqsubseteq x$ and, from part 2, $\rho x = x$.

4. For all $x \in P$, from parts 1 and 3, $\rho(\rho x) = \rho x$.

5. ρ is obviously monotone.

6. Let D be a directed subset of P. Since $\{\rho x \mid x \in D\}$ is directed and f and g are continuous, part 1 gives

$$\begin{aligned} f(\bigsqcup\{\rho x \mid x \in D\}) &= \bigsqcup\{f(\rho x) \mid x \in D\} \\ &= \bigsqcup\{g(\rho x) \mid x \in D\} \\ &= g(\bigsqcup\{\rho x \mid x \in D\}). \end{aligned}$$

   Moreover, from part 2, $\bigsqcup\{\rho x \mid x \in D\} \sqsupseteq \bigsqcup D$. Thus

$$\bigsqcup\{\rho x \mid x \in D\} \in \{y \mid fy = gy \text{ and } \bigsqcup D \sqsubseteq y\},$$

   so that

$$\begin{aligned} \bigsqcup\{\rho x \mid x \in D\} &\sqsupseteq \bigsqcap\{y \mid fy = gy \text{ and } \bigsqcup D \sqsubseteq y\} \\ &= \rho(\bigsqcup D). \end{aligned}$$

   Then, since ρ is monotone, it is continuous.

7. From parts 6, 4, and 2, ρ is a closure. From parts 1 and 3 the fixed points of ρ are the $x \in P$ such that $fx = gx$.

8. Let ϕ be the identity injection from $\mathcal{F}\rho$ to P, and let ψ be the corestriction of ρ to $\mathcal{F}\rho$. Since $\mathcal{F}\rho = \{x \mid x \in P \text{ and } fx = gx\}$,

$$\mathcal{F}\rho \xrightarrow{\phi} P \overset{f}{\underset{g}{\rightrightarrows}} Q$$

is an equalizer. Moreover, by Proposition 4.11, $\mathcal{F}\rho \overset{\phi}{\underset{\psi}{\rightleftarrows}} P$ is a closure pair.

9. By Proposition **??***, P is nonempty complete. Then by Proposition 4.10, $\mathcal{F}\rho$ is nonempty complete, and by Proposition **??***, $\mathcal{F}\rho$ has glb's of all lower-bounded sets.

10. By part 3 of Proposition 4.4, ϕ preserves glb's of all lower-bounded sets in $\mathcal{F}\rho$.

11. By parts 2 and 4 of Proposition 4.6, if P is continuous (countably based, algebraic), then $\mathcal{F}\rho$ is continuous (countably based, algebraic).

$*$ These propositions have been omitted.