Chapter 3

Continuous and Algebraic Domains

(Notes for CS 819 — April 17, 2000 — ©2000 John C. Reynolds)

3.1 Topology

Let P be any dcpo. Then $U \subseteq P$ is said to be *open* iff

1. U is upward closed, i.e., if $x \in U$ and $x \sqsubseteq y$, then $y \in U$;
2. For all directed $D \subseteq P$, if $\bigsqcup D \in U$, then there is an $x \in D$ such that $x \in U$.

On the other hand, $C \subseteq P$ is said to be *closed* iff

1. C is downward closed, i.e., if $y \in C$ and $x \sqsubseteq y$, then $x \in C$;
2. For all directed $D \subseteq P$, if $D \subseteq C$, then $\bigsqcup D \in C$.

(Note that the second conditions in both of these definitions are trivially true for uninteresting directed sets.) It is easily seen that $S \subseteq P$ is open (closed) iff $P - S$ is closed (open), that unions and finite intersections of open sets are open, and that both P and the empty set are both open and closed. If P has a least element or a greatest element, then these are the only subsets that are both open and closed; at the opposite extreme, when P is a discretely ordered poset ($x \sqsubseteq y$ iff $x = y$), then every subset is both open and closed.

For $y \in P$, we define the *lower* and *upper sections* of y by

$$\text{lsect}(y) = \{z \in P \mid z \sqsubseteq y\}$$
$$\text{usect}(y) = \{z \in P \mid y \sqsubseteq z\}.$$

For any $y \in P$, lsect(y) is closed, so that $P - \text{lsect}(y)$ is open. On the other hand, while usect(y) always meets the first condition in the definition of open set, it may or may not meet the second condition.

Proposition 3.1 *For any dcpo P:*

1. *$x \sqsubseteq y$ iff, for all open $U \subseteq P$, $x \in U$ implies $y \in U$.*
2. *If $x \neq y$, then there is either an open set containing x but not y, or an open set containing y but not x (or both).*

PROOF

1. If $x \sqsubseteq y$ and U is open, then, by the first part of the definition of open, $x \in U$ implies $y \in U$. On the other hand, if $\neg\, x \sqsubseteq y$ then $D - \text{lsect}(y)$ is an open set containing x but not y.
2. If $x \neq y$, then either $\neg\, x \sqsubseteq y$, in which case $D - \text{lsect}(y)$ in an open set containing x but not y, or $\neg\, y \sqsubseteq x$, in which case $D - \text{lsect}(x)$ in an open set containing y but not x.

END OF PROOF

In topology, a space P satisfying the second part of the above proposition is called a *T_0-space*.

If f is a function from P to P', and $S' \subset P'$, then the set

$$f^{-1}S' \stackrel{\text{def}}{=} \{x \in P \mid fx \in S'\}$$

is called the *preimage of S' under f*.

Proposition 3.2 *A function f from a dcpo P to a dcpo P' is continuous iff it is "topologically" continuous, i.e., if the preimage $f^{-1}U' \subseteq P$ of every open $U' \subseteq P'$ is open.*

PROOF Suppose f is continuous and U' is open. To see that $f^{-1}U'$ is open:

1. If $x \in f^{-1}U'$ and $x \sqsubseteq y$, then $fx \in U'$ and $fx \sqsubseteq' fy$, so that $fy \in U'$ and $y \in f^{-1}U'$.

2. If D is a directed set such that $\bigsqcup D \in f^{-1}U'$, then $f(\bigsqcup D) \in U'$, and the continuity of f gives $\bigsqcup'\{fx \mid x \in D\} \in U'$. Then, since $\{fx \mid x \in D\}$ is directed, there is an $x \in D$ such that $fx \in U'$, and thus $x \in f^{-1}U'$.

On the other hand, suppose that, for all open U', $f^{-1}U'$ is open. Then:

1. To see that f is monotone, suppose $x \sqsubseteq y$. For any open U' containing fx, we have $x \in f^{-1}U'$, so that $y \in f^{-1}U'$, and $fy \in U'$. Thus, by the first part of Proposition 3.1, $fx \sqsubseteq fy$.

2. Suppose D is directed. For any open U' such that $f(\bigsqcup D) \in U'$, we have $\bigsqcup D \in f^{-1}U'$, so that there is an $x \in D$ such that $x \in f^{-1}U'$. Then $fx \in U'$ and, since $fx \sqsubseteq' \bigsqcup'\{fx \mid x \in D\}$, we have $\bigsqcup'\{fx \mid x \in D\} \in U'$. Thus, by the first part of Proposition 3.1, $f(\bigsqcup D) \sqsubseteq \bigsqcup'\{fx \mid x \in D\}$, and by Proposition 2.6, f is continuous.

END OF PROOF

As a special case, suppose f is a function from P to the two-element dcpo $\{\bot, \top\}$. The open subsets of $\{\bot, \top\}$ are the empty set, $\{\top\}$, and $\{\bot, \top\}$, and the preimages $f^{-1}\{\} = \{\}$ and $f^{-1}\{\bot, \top\} = P$ are always open. Thus f is continuous iff

$$f^{-1}\{\top\} = \{x \in P \mid fx = \top\} \text{ is open,}$$

or equivalently, iff

$$P - f^{-1}\{\top\} = \{x \in P \mid fx = \bot\} \text{ is closed.}$$

3.2 The Way-Below Relation

When x and y belong to a dcpo (directed-complete poset) P, we write $x \prec y$, and say that x is *way-below* y, iff, for all directed $D \subseteq P$, if $y \sqsubseteq \bigsqcup D$, then there is a $z \in D$ such that $x \sqsubseteq z$.

When $x \prec x$, we say that x is *isolated*.

Proposition 3.3 *In a dcpo P:*

1. *If $x \prec y$ and $y \sqsubseteq z$, then $x \prec z$.*
2. *If $x \sqsubseteq y$ and $y \prec z$, then $x \prec z$.*
3. *If $x \prec y$, then $x \sqsubseteq y$.*
4. *$x \prec y$ iff $x \sqsubseteq y$ and, for all interesting directed $D \subseteq P$, if $y \sqsubseteq \bigsqcup D$, then there is a $z \in D$ such that $x \sqsubseteq z$.*
5. *If P has a least element $\bot$, then $\bot \prec y$.*
6. *If x is isolated, then $\{y \in P \mid x \sqsubseteq y\}$ is open.*
7. *If x is isolated, then x is not the lub of any interesting directed set.*
8. *Suppose P is (bounded $\cap$ nonempty)-complete and F is a finite nonempty subset of P. If, for all $x \in F$, $x \prec y$, then $\bigsqcup F \prec y$.*
9. *If P is (bounded $\cap$ nonempty)-complete, then $\{x \in P \mid x \prec y\}$ is directed or empty.*

Proof

1. Straightforward.
2. Straightforward.
3. Take D to be $\{y\}$.
4. Suppose $x \sqsubseteq y$. If D is an uninteresting directed set and $y \sqsubseteq \bigsqcup D$, then $\bigsqcup D$ itself is a member of D such that $x \sqsubseteq z$. Thus, if the condition in the definition of $x \prec y$ is met for interesting directed D, we have $x \prec y$. The converse follows from (3) and the definition of $x \prec y$.
5. Straightforward.
6. Clearly, $\{y \in P \mid x \sqsubseteq y\}$ is upward closed. If it contains the lub of a directed set D, then $x \prec x$ implies that there is a $z \in D$ such that $x \sqsubseteq z$.
7. If $\bigsqcup D \prec \bigsqcup D$, then there is a $z \in D$ such that $\bigsqcup D \sqsubseteq z$. Then $z = \bigsqcup D$, so that D is uninteresting.

8. Let $F = \{x_1, \ldots, x_n\}$, and assume $x_1 \prec y, \ldots, x_n \prec y$. To see that $\bigsqcup F \prec y$, suppose D is directed and $y \sqsubseteq \bigsqcup D$. Then there are $z_1, \ldots, z_n \in D$ such that $x_1 \sqsubseteq z_1, \ldots, x_n \sqsubseteq z_n$. Then, since D is directed, there is a $z \in D$ that is an upper bound on the z_i, and therefore on the x_i. Thus $\bigsqcup F \sqsubseteq z$.

9. Assume $\{x \in P \mid x \prec y\}$ is not empty; then it contains a bound on the empty set. For any other finite subset F of $\{x \in P \mid x \prec y\}$, (8) implies that $\{x \in P \mid x \prec y\}$ contains an upper bound of F, namely $\bigsqcup F$. Thus $\{x \in P \mid x \prec y\}$ is directed.

END OF PROOF

For x, y in a dcpo P, we write $x \prec_{\text{old}} y$ iff there is an open $U \subseteq P$ that contains y and is a subset of $\text{usect}(x) = \{z \in P \mid x \sqsubseteq z\}$.

Proposition 3.4 *In a dcpo P, if $x \prec_{\text{old}} y$, then $x \prec y$.*

PROOF Assume $x \prec_{\text{old}} y$, so that $y \in U$ for some open U whose members are all extensions of x. To show $x \prec y$, assume D is a directed set such that $y \sqsubseteq \bigsqcup D$. Since U is upward closed, $\bigsqcup D \in U$. Then since U is open and D is directed, there is a $z \in D$ such that $z \in U$, which implies $x \sqsubseteq z$.

END OF PROOF

The converse of this proposition is not valid.

Proposition 3.5 *For dcpo's P and P', suppose there are continuous functions $f \in P \to P'$ and $g \in P' \to P$ such that, for all $x \in P$, $g(fx) \sqsubseteq x$. Then*

$$x' \prec' fx \ \textit{ implies } \ gx' \prec x$$

$$x' \prec'_{\text{old}} fx \ \textit{ implies } \ gx' \prec_{\text{old}} x.$$

PROOF Suppose $x' \prec' fx$ and $x \sqsubseteq \bigsqcup D$ for some directed $D \subseteq P$. Then $fx \sqsubseteq f(\bigsqcup D) = \bigsqcup'\{fz \mid z \in D\}$. Since $x' \prec' fx$, there is a member of $\{fz \mid z \in D\}$ that extends x', so that there is a $z \in D$ such that $x' \sqsubseteq' fz$. Then $gx' \sqsubseteq g(fz) \sqsubseteq z$.

Suppose $x' \prec'_{\text{old}} fx$. Then fx belongs to an open subset U' of $\{y' \mid x' \sqsubseteq' y'\}$. Then x belongs to the preimage U of U' under f, which is open since f is continuous. For any $y \in U$, $fy \in U' \subseteq \{y' \mid x' \sqsubseteq' y'\}$, so that $x' \sqsubseteq' fy$, and $gx' \sqsubseteq g(fy) \sqsubseteq y$. Thus $U \subseteq \{y \mid gx' \sqsubseteq y\}$. END OF PROOF

Suppose P and P' are dcpo's, P' has a least element, $x \in P$, and $x' \in P'$. We write $(x \to x')$ for the function from P to P' such that

$$(x \to x')y = \begin{cases} x' & \text{when } x \prec_{\text{old}} y \\ \bot' & \text{otherwise.} \end{cases}$$

Proposition 3.6 *$(x \to x')$ is a continuous function from P to P' that is the greatest lower bound of the set of continuous $f \in P \to P'$ such that $fx = x'$.*

PROOF The function $(x \to x') \in P \to P'$ can be written as the composition of $(x \to \top) \in P \to \{\bot, \top\}$ with an obviously continuous function from $\{\bot, \top\}$ to P'. The function $(x \to \top)$ will be continuous iff the preimage of $\top$ is open. This preimage is the set of y that belong to some open subset of $\{z \in P \mid x \sqsubseteq z\}$; thus the preimage is the union of the open subsets of $\{z \in P \mid x \sqsubseteq z\}$, which is open since the union of a set of open sets is open.

Let F be the set of continuous $f \in P \to P'$ such that $fx = x'$. It is easy to see that $(x \to x')$ is a lower bound of F. On the other hand, suppose g is a lower bound of F. To show that $g \sqsubseteq_{P \Rightarrow P'} (x \to x')$, we first note that, for all $z \in P$, $gz \sqsubseteq x'$, since F contains the constant function that always gives x'. What remains is to show that $gz = \bot'$ when $\neg\, x \prec_{\text{old}} z$.

Suppose z is a member of P such that $\neg\, x \sqsubseteq z$. Let $f \in P \to P'$ be the function such that

$$fy = \begin{cases} \bot' & \text{when } y \sqsubseteq z \\ x' & \text{otherwise.} \end{cases}$$

Since $fx = x'$, to show that it belongs to F, we need only show that it is continuous. This can be established by much the same argument as for $(x \to x')$: It is the composition of a function from P to $\{\bot, \top\}$ with an obviously continuous function, and the function into $\{\bot, \top\}$ will be continuous if the preimage of $\top$ is open. In this case, the preimage is $\{y \in P \mid \neg\, y \sqsubseteq z\}$, which is open by Proposition ???.

Since $f \in F$, we have $g \sqsubseteq f$ and, in particular, $gz \sqsubseteq fz = \bot'$. Then, since this holds whenever $\neg\, x \sqsubseteq z$, we have that $gz \neq \bot'$ implies $x \sqsubseteq z$. Thus

$$\{z \in P \mid gz \neq \bot'\} \subseteq \{z \in P \mid x \sqsubseteq z\}.$$

The set on the left is the preimage of the set $P' - \{\bot'\}$, which is easily seen to be open. Thus, since g is continuous, the set on the left is open. It follows that, when $gz \neq \bot'$, gz belongs to an open subset of $\{z \in P \mid x \sqsubseteq z\}$, i.e. $x \prec_{\text{old}} z$. END OF PROOF

Proposition 3.7 *Let P be a dcpo and P' be a dcpo with a least element, with $x \in P$, $x' \in P$, and a continuous $f \in P \to P'$. Then*

$$\begin{aligned} x' \prec fx \ &\textit{implies}\ (x \to x') \prec f \\ f \prec (x \to x') \ &\textit{implies}\ fx \prec x' \\ x' \prec_{\mathrm{old}} fx \ &\textit{implies}\ (x \to x') \prec_{\mathrm{old}} f \\ f \prec_{\mathrm{old}} (x \to x') \ &\textit{implies}\ fx \prec_{\mathrm{old}} x'. \end{aligned}$$

PROOF Let $\phi_x \in P' \to (P \Rightarrow P')$ and $\psi_x \in (P \Rightarrow P') \to P'$ be such that

$$\phi_x x' = (x \to x') \qquad\qquad \psi_x f = fx.$$

To see that ϕ_x is continuous, let D' be a directed subset of P'. Then

$$\begin{aligned} \phi_x({\textstyle\bigsqcup}' D')y &= \textbf{if } x \prec_{\mathrm{old}} y \textbf{ then } {\textstyle\bigsqcup}' D' \textbf{ else } \bot' \\ &= \textbf{if } x \prec_{\mathrm{old}} y \textbf{ then } {\textstyle\bigsqcup}'\{x' \mid x' \in D'\} \textbf{ else } {\textstyle\bigsqcup}'\{\bot' \mid x' \in D'\} \\ &= {\textstyle\bigsqcup}'\{\textbf{if } x \prec_{\mathrm{old}} y \textbf{ then } x' \textbf{ else } \bot' \mid x' \in D'\} \\ &= {\textstyle\bigsqcup}'\{(x \to x')y \mid x' \in D'\} \\ &= ({\textstyle\bigsqcup}_{P \to P'}\{(x \to x') \mid x' \in D'\})y \\ &= ({\textstyle\bigsqcup}_{P \to P'}\{\phi x' \mid x' \in D'\})y, \end{aligned}$$

and since this holds for all $y \in D$, we have the continuity equation for ϕ_x. The continuity of ψ_x is obvious.

Morever,

$$\begin{aligned} \psi_x(\phi_x x') &= (x \to x')x \\ &= \textbf{if } x \prec_{\mathrm{old}} x \textbf{ then } x' \textbf{ else } \bot' \\ &\sqsubseteq x', \end{aligned}$$

and, for all $y \in P$,

$$\begin{aligned} \phi_x(\psi_x f)y &= (x \to fx)y \\ &= \textbf{if } x \prec_{\mathrm{old}} y \textbf{ then } fx \textbf{ else } \bot' \\ &\sqsubseteq \textbf{if } x \prec_{\mathrm{old}} y \textbf{ then } fy \textbf{ else } fy \\ &= fy, \end{aligned}$$

so that

$$\phi_x(\psi_x f) = f.$$

The results then follow from Proposition 3.5. END OF PROOF

3.3 Continuous Domains

A dcpo P is said to be *continuous* iff, for all $y \in P$, the set $\{x \in P \mid x \prec y\}$ is directed, and its lub is y.

Proposition 3.8 *Suppose P is a (bounded $\cap$ nonempty)-complete dcpo. Then P is continuous iff, for all $y \in P$, y is the lub of the set $\{x \in P \mid x \prec y\}$.*

PROOF Suppose, for all $y \in P$, y is the lub of $\{x \in P \mid x \prec y\}$. By part 9 of Proposition 3.3, $\{x \in P \mid x \prec y\}$ is directed or empty. But if it were empty, then y would be the lub of the empty set, which is $\perp$, so that part 5 of Proposition 3.3 would give $\perp \prec y$, contradicting the emptyness of $\{x \in P \mid x \prec y\}$. Thus $\{x \in P \mid x \prec y\}$ is directed. END OF PROOF

Proposition 3.9 *In a continuous dcpo P:*

1. *$x \prec y$ iff, for all directed $D \subseteq P$, if $y = \bigsqcup D$, then there is a $z \in D$ such that such that $x \sqsubseteq z$.*
2. *(Density) If $x \prec y$, then there is a z such that $x \prec z$ and $z \prec y$.*
3. *For all directed $D \subseteq P$, $x \prec \bigsqcup D$ iff there is a $y \in D$ such that $x \prec y$.*
4. *$\{y \in P \mid x \prec y\}$ is an open subset of P.*
5. *x is isolated iff it is not the lub of any interesting directed set.*
6. *$x \prec y$ iff $x \prec_{\text{old}} y$.*

PROOF

1. Suppose that, for all directed $D \subseteq P$, if $y = \bigsqcup D$, then there is a $z \in D$ such that such that $x \sqsubseteq z$. Taking D to be $\{x \in P \mid x \prec y\}$, we obtain a z such that $z \prec y$ and $x \sqsubseteq z$. Thus, by part 2 of Proposition 3.3, $x \prec y$. The other half of the implication follows from the definition of $\prec$.

2. From the definition of $\prec$,

$$\begin{aligned} y &= \bigsqcup\{z \in P \mid z \prec y\} \\ &= \bigsqcup\{\bigsqcup\{w \in P \mid w \prec z\} \mid z \prec y\} \\ &= \bigsqcup\{w \in P \mid \exists z \in P.\ w \prec z \text{ and } z \prec y\}. \end{aligned}$$

To see this is the lub of a directed set, suppose $\{w_1, \ldots, w_n\}$ is a finite subset. Then, for each i in 1 to n, there is a z_i such that $w_i \prec z_i$ and $z_i \prec y$. Since $\{z \mid z \prec y\}$ is directed, there is a $z \prec y$ such that, for each i, $z_i \sqsubseteq z$, and therefore $w_i \prec z$. Since $\{w \mid w \prec z\}$ is directed, there is a $w \prec z$ such that, for each i, $w_i \sqsubseteq w$. Thus w is an upper bound of $\{w_1, \ldots, w_n\}$ that belongs to $\{w \mid \exists z \in P.\ w \prec z \text{ and } z \prec y\}$.

Now suppose $x \prec y$. By applying the definition of $\prec$ to the directed set we have constructed, we find there is a w such that $\exists z \in P.\ w \prec z$ and $z \prec y$ and $x \sqsubseteq w$. Then $\exists z \in P.\ x \prec z$ and $z \prec y$.

3. Suppose $x \prec \bigsqcup D$. By density, there is a z such that $x \prec z$ and $z \prec \bigsqcup D$. Then by the definition of $\prec$, there is a $y \in D$ such that $z \sqsubseteq y$. Thus $x \prec y$. The opposite implication is obvious.

4. Clearly, $\{y \mid x \prec y\}$ is upward directed. If this set contains the lub of a directed set D, then $x \prec \bigsqcup D$, so that, by part 3 of this proposition, there is a $y \in D$ such that $x \prec y$.

5. Suppose x is not the lub of any interesting directed set. Then for any directed D such that $x = \bigsqcup D$, there is a $z \in D$, namely $z = \bigsqcup D$, such that $x \sqsubseteq \bigsqcup D$. Then by part 1 of this proposition, $x \prec x$. The opposite implication is given by part 7 of Proposition 3.3.

6. Suppose $x \prec y$. Then y is contained in $\{y \mid x \prec y\}$, which is a subset of $\{z \mid x \sqsubseteq z\}$. Since, by part 4 of this proposition, $\{y \mid x \prec y\}$ is open, we have $x \prec_{\text{old}} y$. The opposite implication is given by Proposition 3.4.

END OF PROOF

Proposition 3.10 *Suppose P is a (bounded $\cap$ nonempty)-complete dcpo. Then P is continuous iff, for all $y \in P$, y is the lub of the set $\{x \in P \mid x \prec_{\text{old}} y\}$.*

Proof If P is continuous, then, by part 6 of Proposition 3.9, $\prec_{\text{old}}$ and $\prec$ are the same relation, so that the desired conclusion comes from the definition of continuous dcpo.

On the other hand, suppose, for all $y \in P$, y is the lub of the set $\{x \mid x \prec_{\text{old}} y\}$. From Proposition 3.4,

$$\{x \mid x \prec_{\text{old}} y\} \subseteq \{x \mid x \prec y\}.$$

But the set on the right is still bounded by y. Thus y is also the lub of the set on the right. Then it follows from Proposition 3.8 that P is continuous.

END OF PROOF

A subset E of a dcpo P is called a *basis* of P iff, for all $y \in P$, $\{e \in E \mid e \prec y\}$ is directed and its lub is y.

Obviously, a dcpo is continuous iff it is its own basis. Moreover:

Proposition 3.11 *If P is a dcpo with basis E and $E \subseteq E' \subseteq P$, then E' is also a basis of P.*

PROOF Since

$$\{e \in E \mid e \prec y\} \subseteq \{e \in E' \mid e \prec y\},$$

and y is the lub of the set on the left and an upper bound of the set on the right, y is the lub of the set on the right.

To see that $\{e \in E' \mid e \prec y\}$ is a directed set, suppose that it has a finite subset $\{e'_1, \ldots, e'_n\}$. Then each e'_i satisfies $e'_i \prec y$, where y is the lub of the set

$$D = \{e \in E \mid e \prec y\}.$$

But D is directed, so that, by the definition of $\prec$, $e'_i \prec \bigsqcup D$ implies that there is an e_i in D such that $e'_i \sqsubseteq e_i$. Moreover, since D is directed, it contains an e that is an upper bound of $\{e_1, \ldots, e_n\}$, and therefore of $\{e'_1, \ldots, e'_n\}$. But then e is also contained in the larger set $\{e \in E' \mid e \prec y\}$, which is therefore directed.

END OF PROOF

It follows that a dcpo with a basis is continuous.

Proposition 3.12 *If E is the basis of a dcpo P, then E contains the isolated elements of P.*

PROOF Suppose $y \notin E$. Then the equation $y = \bigsqcup\{e \in E \mid e \prec y\}$ expresses y as the lub of an interesting directed set, and by part 7 of Proposition 3.3, y is not isolated.

END OF PROOF

When the set of isolated elements of a dcpo P is a basis of P (and therefore the least basis), we say that P is *algebraic*.

3.4 Products

Proposition 3.13 *For dcpo's P_1 and P_2,*

$$\langle x_1, x_2\rangle \prec_{P_1 \times P_2} \langle y_1, y_2\rangle \text{ iff } x_1 \prec_{P_1} y_1 \text{ and } x_2 \prec_{P_2} y_2.$$

PROOF Assume $x_1 \prec_{P_1} y_1$ and $x_2 \prec_{P_2} y_2$. To show $\langle x_1, x_2\rangle \prec_{P_1\times P_2} \langle y_1, y_2\rangle$, suppose D is a directed subset of $P_1 \times P_2$ such that $\langle y_1, y_2\rangle \sqsubseteq \bigsqcup D$. Then by the equation in Section 2.3 for the lub of a directed subset of a product,

$$\langle y_1, y_2\rangle \sqsubseteq \langle \bigsqcup\{\pi_1 z \mid z \in D\}, \bigsqcup\{\pi_2 z \mid z \in D\}\rangle.$$

Thus $y_1 \sqsubseteq \bigsqcup\{\pi_1 z \mid z \in D\}$, where $\{\pi_1 z \mid z \in D\}$ is a directed set by Proposition 2.3, so that $x_1 \prec y_1$ implies that there is a $z_1 \in D$ such that $x_1 \sqsubseteq \pi_1 z_1$.

Similarly, there is a $z_2 \in D$ such that $x_2 \sqsubseteq \pi_2 z_2$. Then, since D is directed, there is a $z \in D$ that is an upper bound of z_1 and z_2, so that $x_1 \sqsubseteq \pi_1 z$ and $x_2 \sqsubseteq \pi_2 z$, and therefore $\langle x_1, x_2\rangle \sqsubseteq z$.

On the other hand, assume $\langle x_1, x_2\rangle \prec_{P_1\times P_2} \langle y_1, y_2\rangle$. To show $x_1 \prec_{P_1} y_1$, suppose D_1 is a directed subset of P_1 such that $y_1 \sqsubseteq \bigsqcup D_1$. Then

$$\begin{aligned}
\langle y_1, y_2\rangle &\sqsubseteq \langle \bigsqcup D_1, y_2\rangle \\
&= \langle \bigsqcup\{\pi_1\langle x, y_2\rangle \mid x \in D_1\}, \bigsqcup\{\pi_2\langle x, y_2\rangle \mid x \in D_1\}\rangle \\
&= \bigsqcup\{\langle x, y_2\rangle \mid x \in D_1\}
\end{aligned}$$

(where the first equality depends on the fact that D_1, being directed, is not empty). Then there is a $z \in D_1$ such that $\langle x_1, x_2\rangle \sqsubseteq \langle z, y_2\rangle$, and therefore $x_1 \sqsubseteq z$.

The proof that $\langle x_1, x_2\rangle \prec_{P_1\times P_2} \langle y_1, y_2\rangle$ implies $x_2 \prec_{P_2} y_2$ is similar.

END OF PROOF

Proposition 3.14 *If P_1 and P_2 are dcpo's with (countable) bases E_1 and E_2 respectively, then $P_1 \times P_2$ is a dcpo with (countable) basis $E_1 \times E_2$.*

PROOF Suppose that $\langle y_1, y_2\rangle \in P_1 \times P_2$, and consider the set

$$\begin{aligned}
D &= \{e \in E_1 \times E_2 \mid e \prec \langle y_1, y_2\rangle\} \\
&= \{\langle e_1, e_2\rangle \mid e_1 \in E_1 \text{ and } e_2 \in E_2 \text{ and } e_1 \prec y_1 \text{ and } e_2 \prec y_2\}.
\end{aligned}$$

To see that this set is directed, assume that it has some finite subset $\{\langle e_{11}, e_{12}\rangle, \ldots, \langle e_{n1}, e_{n2}\rangle\}$. Then $\{e_{11}, \ldots, e_{n1}\}$ is a finite subset of $\{e_1 \in E_1 \mid e_1 \prec y_1\}$, and since this set is directed, it contains an upper bound e_1 of $\{e_{11}, \ldots, e_{n1}\}$.

Similarly, $\{e_1 \in E_1 \mid e_1 \prec y_1\}$ contains an upper bound e_2 of $\{e_{12}, \ldots, e_{n2}\}$. Then $\langle e_1, e_2\rangle$ is a member of D that is an upper bound of $\{\langle e_{11}, e_{12}\rangle, \ldots, \langle e_{n1}, e_{n2}\rangle\}$.

From the equation in Section 2.3 for the lub of a directed subset of a product, we have

$$\bigsqcup D = \langle \bigsqcup D_1, \bigsqcup D_2\rangle,$$

where

$$\begin{aligned} D_1 &= \{\pi_1\langle e_1, e_2\rangle \mid e_1 \in E_1 \text{ and } e_2 \in E_2 \text{ and } e_1 \prec y_1 \text{ and } e_2 \prec y_2\} \\ D_2 &= \{\pi_2\langle e_1, e_2\rangle \mid e_1 \in E_1 \text{ and } e_2 \in E_2 \text{ and } e_1 \prec y_1 \text{ and } e_2 \prec y_2\}. \end{aligned}$$

Then

$$\begin{aligned} \bigsqcup D_1 &= \bigsqcup\{e_1 \mid e_1 \in E_1 \text{ and } e_1 \prec y_1 \text{ and } \exists e_2 \in E_2.\ e_2 \prec y_2\} \\ &= \bigsqcup\{e_1 \mid e_1 \in E_1 \text{ and } e_1 \prec y_1\} \\ &= y_1, \end{aligned}$$

where the second equality is justified since $\{e_2 \mid e_2 \in E_2 \text{ and } e_2 \prec y_2\}$ is directed, and therefore nonempty.

Similarly, $\bigsqcup D_2 = y_2$, so that $\bigsqcup D = \langle y_1, y_2\rangle$. END OF PROOF

Then, since a dcpo is continuous iff it has a basis:

Proposition 3.15 *If P_1 and P_2 are continuous dcpo's, then $P_1 \times P_2$ is a continous dcpo.*

Less trivially,

Proposition 3.16 *If P_1 and P_2 are algebraic dcpo's, then $P_1 \times P_2$ is a algebraic dcpo.*

PROOF If P_1 and P_2 are algebraic dcpo's, then they have bases E_1 and E_2 consisting of isolated elements. By Proposition 3.14, $E_1 \times E_2$ is a basis of $P_1 \times P_2$.

We must show that the elements of $E_1 \times E_2$ are isolated. But if $\langle e_1, e_2\rangle$ is such an element, then $e_1 \in E_1$, so that $e_1 \prec e_1$, and $e_2 \in E_2$, so that $e_2 \prec e_2$, and by Proposition 3.13, $\langle e_1, e_2\rangle \prec \langle e_1, e_2\rangle$. END OF PROOF

3.5 Exponentiation

Henceforth, we write $P \Rightarrow P'$ for $P \xRightarrow{\text{directed}} P'$, i.e. for the pointwise ordered domain of continuous functions from P to P'.

Proposition 3.17 *Suppose P is a dcpo with a (countable) basis E, and P' is a bounded-complete dcpo with a least element and a (countable) basis E'. Let*

$$E_{\text{step}} = \{(e \to e') \mid e \in E \text{ and } e' \in E'\}$$
$$E'' = \{\textstyle\bigsqcup_{P \Rightarrow P'} F \mid F \overset{\text{fin}}{\subseteq} E_{\text{step}} \text{ and } F \text{ is bounded in } P \Rightarrow P'\}.$$

Then $P \Rightarrow P'$ is a bounded-complete dcpo with a least element and (countable) basis E''.

Proof Let P'' be $P \Rightarrow P'$. As discussed in Section 2.5, P'' is a bounded-complete dcpo with a least element. This insures that the lub's needed to define E'' exist.

Suppose that $f \in P''$, and consider $\{e'' \in E'' \mid e'' \prec f\}$. To see that this set is directed, suppose that it has some finite subset F. Since F is bounded (by f), it possess a lub; if F is nonempty, then $\bigsqcup F \prec f$ by part 8 of Proposition 3.3, while if F is empty, then $\bigsqcup F = \bot_{P''} \prec f$ by part 5 of Proposition 3.3. Moreover, since F is a finite subset of E'', there are finite subsets $F_1, \ldots, F_n$ of E_{step} such that

$$\begin{aligned} \textstyle\bigsqcup F &= \textstyle\bigsqcup\{\bigsqcup F_1, \ldots, \bigsqcup F_n\} \\ &= \textstyle\bigsqcup(F_1 \cup \cdots \cup F_n), \end{aligned}$$

which belongs to E'' since $F_1 \cup \cdots \cup F_n$ is a finite bounded subset of E_{step}. Thus $\bigsqcup F$ is a bound on F that belongs to $\{e'' \in E'' \mid e'' \prec f\}$.

Now suppose $y \in P$. Since f is continuous,

$$\begin{aligned} fy &= f(\textstyle\bigsqcup\{e \in E \mid e \prec y\}) \\ &= \textstyle\bigsqcup'\{fe \mid e \in E \text{ and } e \prec y\} \\ &= \textstyle\bigsqcup'\{\bigsqcup'\{e' \in E' \mid e' \prec' fe\} \mid e \in E \text{ and } e \prec y\} \\ &= \textstyle\bigsqcup'\{e' \in E' \mid \exists e \in E.\, e' \prec fe \text{ and } e \prec y\}. \end{aligned}$$

For each $(e \to e') \in E_{\mathrm{step}}$, E'' contains the lub of the singleton set $\{(e \to e')\}$, which is $(e \to e')$. Thus

$$\{e''y \mid e'' \in E'' \text{ and } e'' \prec f\}$$
$$\supseteq \{(e \to e')y \mid e \in E \text{ and } e' \in E' \text{ and } (e \to e') \prec f\},$$

and by the first part of Proposition 3.7,

$$\cdots \supseteq \{(e \to e')y \mid e \in E \text{ and } e' \in E' \text{ and } e' \prec fe\},$$

and by the definition of $(e \to e')$,

$$\cdots \supseteq \{e' \in E' \mid \exists e \in E.\ e' \prec fe \text{ and } e \prec y\}.$$

Thus, we have a decreasing sequence of sets where fy is the lub of the last set, and is also an upper bound of the first set; this implies that fy is a lub of the first set. Along with the equation for lub's of continuous functions in Section 2.5, this gives

$$(\bigsqcup{}''\{e'' \in E'' \mid e'' \prec f\})y = \bigsqcup{}'\{e''y \mid e'' \in E'' \text{ and } e'' \prec f\} = fy,$$

and since this equation holds for all $y \in P$,

$$(\bigsqcup{}''\{e'' \in E'' \mid e'' \prec f\}) = f.$$

It is obvious that E'' is countable if E and E' are countable.

END OF PROOF

Then

Proposition 3.18 *Suppose P is a continuous dcpo, and P' is a continuous bounded-complete dcpo with a least element. Then $P \Rightarrow P'$ is a continuous bounded-complete dcpo with a least element.*

and

Proposition 3.19 *Suppose P is an algebraic dcpo, and P' is an algebraic bounded-complete dcpo with a least element. Then $P \Rightarrow P'$ is an algebraic bounded-complete dcpo with a least element.*

PROOF Let E and E' be the sets of isolated elements of P and P' respectively, and let E_{step} and E'' be as defined in Proposition 3.17. Since E and E' are bases of P and P', Proposition 3.17 shows that $P \Rightarrow P'$ is a bounded-complete dcpo with a least element and basis E''. To complete our proof, we will show that the members of E_{step} and E'' are isolated.

Any member of E_{step} will have the form $(e \to e')$, where e and e' are isolated. Thus

$$\begin{aligned} e' &\prec e' \\ &= \textbf{if } e \prec e \textbf{ then } e' \textbf{ else } \bot' \\ &= (e \to e')e. \end{aligned}$$

Then the first part of Proposition 3.7 gives $(e \to e') \prec (e \to e')$.

Then any member of E'' will have the form $\bigsqcup F$, where F is a finite bounded set of isolated functions in $P \Rightarrow P'$. For each $f \in F$, we have $f \prec f \sqsubseteq \bigsqcup F$. Thus, by part 8 of Proposition 3.3, $\bigsqcup F \prec \bigsqcup F$.

END OF PROOF

3.6 Disjoint Unions, Lifting, and Topping

With the disjoint union, as in Section 2.6, we can, without loss of generality, assume that P_1 and P_2 are disjoint, and define $P_1 + P_2 \stackrel{\text{def}}{=} P_1 \cup P_2$, with

$$x \sqsubseteq_{P_1+P_2} y \text{ iff } (x, y \in P_1 \text{ and } x \sqsubseteq_{P_1} y) \text{ or } (x, y \in P_2 \text{ and } x \sqsubseteq_{P_2} y).$$

Then

Proposition 3.20

$$x \prec_{P_1+P_2} y \textit{ iff } (x, y \in P_1 \textit{ and } x \prec_{P_1} y) \textit{ or } (x, y \in P_2 \textit{ and } x \prec_{P_2} y).$$

PROOF Assume the condition on the right, and suppose $y \in P_1$. To show $x \prec_{P_1+P_2} y$, let $D \subseteq P_1 + P_2$ be a directed set such that $y \sqsubseteq \bigsqcup_{P_1+P_2} D$. Then $\bigsqcup_{P_1+P_2} D \in P_1$, $D \subseteq P_1$, and, as shown in Section 2.6, $\bigsqcup_{P_1+P_2} D = \bigsqcup_{P_1} D$. Then, since $x \prec_{P_1} y$, there is a $z \in D$ such that $x \sqsubseteq z$. The case where $y \in P_2$ is similar.

On the other hand, assume $x \prec_{P_1+P_2} y$, and suppose $y \in P_1$. To show the condition on the right, let $D \subseteq P_1$ be a directed set such that $y \sqsubseteq \bigsqcup_{P_1} D$. Then $\bigsqcup_{P_1+P_2} D = \bigsqcup_{P_1} D$, so that $y \sqsubseteq \bigsqcup_{P_1+P_2} D$. Since $x \prec_{P_1+P_2} y$, there is a $z \in D$ such that $x \sqsubseteq z$. Thus $x \prec_{P_1} y$ and also $x \in P_1$. The case where $y \in P_2$ is similar.

END OF PROOF

Proposition 3.21 *If P_1 is a dcpo with basis E_1 and P_2 is a dcpo with basis E_2, then $P_1 + P_2$ is a dcpo with basis $E_1 + E_2$.*

PROOF Consider $\{e \mid e \in E_1 + E_2 \text{ and } e \prec_{P_1+P_2} y\}$, and suppose $y \in P_1$. By Proposition 3.20, $e \prec_{P_1+P_2} y$ iff $e \in P_1$ and $e \prec_{P_1} y$, so that

$$\{e \mid e \in E_1 + E_2 \text{ and } e \prec_{P_1+P_2} y\} = \{e \mid e \in E_1 \text{ and } e \prec_{P_1} y\}.$$

Thus this set is directed and its lub in P_1 is y, but this is also the lub in $P_1 + P_2$. The case where $y \in P_2$ is similar. END OF PROOF

The reader may verify that:

Proposition 3.22 *If P_1 and P_2 are continuous dcpo's, then $P_1 + P_2$ is a continuous dcpo.*

Proposition 3.23 *If P_1 and P_2 are algebraic dcpo's, then $P_1 + P_2$ is a algebraic dcpo.*

With lifting, as in Section 2.7, we can, without loss of generality, assume that P does not contain $\perp$, and define $P_\perp = \{\perp\} \cup P$, with the ordering

$$x \sqsubseteq_{P_\perp} y \text{ iff } x = \perp \text{ or } (x, y \in P \text{ and } x \sqsubseteq_P y).$$

Proposition 3.24

$$x \prec_{P_\perp} y \textit{ iff } x = \perp \textit{ or } (x, y \in P \textit{ and } x \prec_P y).$$

PROOF Assume the condition on the right. If $x = \perp$, part 5 of Proposition 3.3 gives $x \prec_{P_\perp} y$. Otherwise, suppose $D \subseteq P_\perp$ is a directed set such that $y \sqsubseteq \bigsqcup_{P_\perp} D$. Since y is not $\perp$, $D - \{\perp\}$ is nonempty and contains a member of P. Then, as we have seen in Section 2.7, $D - \{\perp\}$ is directed and $\bigsqcup_{P_\perp} D = \bigsqcup_P (D - \{\perp\})$. Then, since $x \prec_P y$, there is a $z \in D - \{\perp\} \subseteq D$ such that $x \sqsubseteq z$. Thus $x \prec_{P_\perp} y$.

On the other hand, assume $x \prec_{P_\perp} y$ and $x \neq \perp$. Then $x, y \in P$. To show $x \prec_P y$, suppose $D \subseteq P$ is a directed set such that $y \sqsubseteq \bigsqcup_P D$. Since $\bigsqcup_P D = \bigsqcup_{P_\perp} D$, we have $y \sqsubseteq \bigsqcup_{P_\perp} D$, and since $x \prec_{P_\perp} y$, there is a $z \in D$ such that $x \sqsubseteq z$. END OF PROOF

Proposition 3.25 *If P is a dcpo with basis E, then $P_\perp$ is a cpo with basis $\{\perp\} \cup E$.*

PROOF Consider $\{e \mid e \in \{\bot\} \cup E \text{ and } e \prec P_\bot y\}$. If $y = \bot$, then this set is $\{\bot\}$, which is directed and has the lub $\bot = y$. Otherwise,

$$\{e \mid e \in \{\bot\} \cup E \text{ and } e \prec P_\bot y\} = \{\bot\} \cup \{e \mid e \in E \text{ and } e \prec_P y\}.$$

We know $\{e \mid e \in E \text{ and } e \prec_P y\}$ is directed and has lub y. It is easy to see that adding $\bot$ to a directed set leaves it directed and leaves its lub unchanged. END OF PROOF

The reader may verify that:

Proposition 3.26 *If P is a continuous dcpo, then $P_\bot$ is a continuous cpo.*

Proposition 3.27 *If P is a algebraic dcpo, then $P_\bot$ is a algebraic cpo.*

With topping, as in Section 2.8, we can, without loss of generality, assume that P does not contain $\top$, and define $P_\top = \{\top\} \cup P$, with the ordering

$$x \sqsubseteq_{P_\top} y \text{ iff } y = \top \text{ or } (x, y \in P \text{ and } x \sqsubseteq_P y).$$

We leave the following propositions to the reader:

Proposition 3.28

$$x \prec_{P_\top} y \textit{ iff } y = \top \textit{ or } (x, y \in P \textit{ and } x \prec_P y).$$

Proposition 3.29 *If P is a dcpo with basis E, then $P_\top$ is a dcpo with basis $\{\top\} \cup E$.*

Proposition 3.30 *If P is a continuous dcpo, then $P_\top$ is a continuous dcpo.*

Proposition 3.31 *If P is a algebraic dcpo, then $P_\top$ is a algebraic dcpo.*