Chapter 2

Some Varieties of Domains

(Notes for CS 819 — February 16, 2000 — ©2000 John C. Reynolds)

There is a good deal of overlap between this chapter and the previous one. Here, however, we will be more systematic in exploring the variety of domains, and less formal about category-theoretic concepts.

2.1 Some Basic Subsets of Posets

A poset is *uninteresting* if it contains an upper bound of itself (which must necessarily be its least upper bound). Otherwise it is *interesting*.

A *chain* is a countable increasing sequence of (not necessarily distinct) members of a poset:

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots .$$

When the context makes it clear that a set is meant, we will also use the term *chain* to denote the set of elements of a chain; this set may be finite or countably infinite, but it is never empty.

A chain is interesting just when it (i.e. the set of its elements) is infinite.

A poset D is said to be *directed* iff, for all finite $F \subseteq D$, there exists a $z \in D$ that is an upper bound of F. Equivalently, D is directed iff D is nonempty and, for all $x, y \in D$, there is a $z \in D$ that is an upper bound of $\{x, y\}$. (Proof by induction on the size of the finite set.)

Uninteresting sets are always directed, and finite directed sets are always uninteresting. But infinite directed sets may be either uninteresting or interesting.

When viewed as a set, a chain is a directed set. On the other hand, there are directed sets (e.g. the set of finite sets of natural numbers, ordered by inclusion) than are not chains.

Whenever we regard a subset of a poset P as a poset itself — as when we speak of an interesting or directed subset of P, or a chain of P — unless we explicitly indicate otherwise, we will assume that the subset inherits the ordering of P. Whenever we refer to the existence of the least upper bound (lub) of such a subset, unless we explicitly indicate otherwise, we mean its existence in P, not necessarily in the subset.

Proposition 2.1 *If every chain of P has a lub, then every countable directed subset of P has a lub.*

PROOF Let D be a countable directed set, and let x_0, x_1, ... be an enumeration of its members. Since D is nonempty and contains the upper bounds of pairs, we can construct a sequence of members of D as follows:

$$\begin{aligned} y_0 &= \text{any member of } D \\ y_{n+1} &= \text{any upper bound in } D \text{ of } y_n \text{ and } x_n. \end{aligned}$$

Then $y_0 \sqsubseteq y_1 \sqsubseteq \cdots$ is a chain such that $x_i \sqsubseteq y_i$, so that the lub of this chain is an upper bound on D. Moreover, since this chain is a subset of D, every upper bound of D is an upper bound of the chain, and thus an extension of the lub of the chain. Thus the lub of the chain is the lub of D.

END OF PROOF

A subset B of a poset P is said to be a *bounded* subset of P iff P contains an upper bound of B.

Proposition 2.2 *If every directed subset of P has a lub, and every bounded finite nonempty subset of P has a lub, then every bounded nonempty subset of P has a lub.*

PROOF Let B be a bounded nonempty subset of P. Every finite subset of B is bounded (by the same bound as B), so that we can form the set

$$D = \{\bigsqcup F \mid F \subseteq B \text{ and } F \text{ finite and nonempty}\}.$$

Any nonempty set has a nonempty finite subset, so that D is nonempty. Moreover, if D contains $\bigsqcup F_1$ and $\bigsqcup F_2$, then F_1 and F_2, and therefore $F_1 \cup F_2$, are finite nonempty subsets of B, so that $\bigsqcup(F_1 \cup F_2)$ exists, belongs to B, and is an upper bound of $\bigsqcup F_1$ and $\bigsqcup F_2$. Thus D is a directed, which insures the existence of $\bigsqcup D$. Then by Proposition 1.3,

$$\begin{aligned}\bigsqcup D &= \bigsqcup\{\bigsqcup F \mid F \subseteq B \text{ and } F \text{ finite and nonempty}\}\\ &= \bigsqcup\bigcup\{F \mid F \subseteq B \text{ and } F \text{ finite and nonempty}\}\\ &= \bigsqcup B.\end{aligned}$$

END OF PROOF

Proposition 2.3 *If f is a monotone function from P to P', and D is a directed subset of P, then $D' = \{fx \mid x \in D\}$ is a directed subset of P'.*

PROOF Any finite subset of D' will have the form $\{f_1x_1, \ldots, f_nx_n\}$, where $\{x_1, \ldots, x_n\}$ is some finite subset of D. Since D is directed, there is a $y \in D$ that is an upper bound of $\{x_1, \ldots, x_n\}$. Then, since f is monotone, fy is an upper bound of $\{f_1x_1, \ldots, f_nx_n\}$. END OF PROOF

Proposition 2.4 *If f is a monotone function from P to P', and B is a bounded subset of P, then $B' = \{fx \mid x \in B\}$ is a bounded subset of P'.*

Proposition 2.5 *If f is a monotone function from P to P', and U is an uninteresting subset of P, then $U' = \{fx \mid x \in U\}$ is an uninteresting subset of P'.*

The easy proofs of the last two propositions are left to the reader.

2.2 Colors

Several kinds of domain are characterized by whether various kinds of subsets possess lub's, and several kinds of functions between domains are characterized by whether they respect such lub's. Moreover, the relevant kinds of subsets, which we will call "colors", are each preserved by certain operations — a fact that is critical to determining which kinds of domain and functions are preserved by constructions such as products, exponentiations, and so forth. Thus we can simplify our discussion of these constructions by making the following definition:

A *color* c is a function on the class of posets such that, for any poset P, c_P is a set of subsets of P and:

1. For any monotone function f from P to P', if $D \in c_P$, then $\{fx \mid x \in D\} \in c_{P'}$.

2. If $D' \subseteq D \subseteq P$, $D \in c_P$, and every $x \in D$ is extended by an $x' \in D'$, then $D' \in c_P$.

3. If $D \subseteq P' \subseteq P$, $D \in c_P$, and D is bounded in P' if it is bounded in P, then $D \in c'_P$.

If a color c satisfies the following strengthening of the last condition:

3′. If $D \subseteq P' \subseteq P$, $D \in c_P$, then $D \in c'_P$.

then it is called a *pure color*.

The reader may verify (with the help of the last three propositions in the previous section) that the following are colors:

$$
\begin{aligned}
\text{chain}_P &= \{C \mid C \subseteq P \text{ and } C \text{ is a chain}\} \\
\text{directed}_P &= \{D \mid D \subseteq P \text{ and } D \text{ is directed}\} \\
\text{uninteresting}_P &= \{D \mid U \subseteq P \text{ and } U \text{ is uninteresting}\} \\
\text{all}_P &= \mathcal{P}P \\
\text{no}_P &= \{\} \\
\text{empty}_P &= \{\{\}\} \\
\text{nonempty}_P &= \mathcal{P}P - \{\{\}\} \\
\text{finite}_P &= \{F \mid F \subseteq P \text{ and } F \text{ is finite}\} \\
\text{countable}_P &= \{F \mid F \subseteq P \text{ and } F \text{ is countable}\} \\
\text{bounded}_P &= \{B \mid B \subseteq P \text{ and } B \text{ is a bounded subset of } P\},
\end{aligned}
\tag{2.1}
$$

and that all except "bounded" are pure colors. (On the other hand, neither "interesting", nor "infinite", nor "containing a least element" is a color.)

From the first property of colors, it is easy to see that they are preserved under isomorphism: If ϕ is an isomorphism of posets from P to P' (so that both ϕ and its inverse are monotone), then $D \in c_P$ iff $\{\phi x \mid x \in D\} \in c_{P'}$.

By pointwise extension, we can define the subset relation on colors:

$$c \subseteq c' \text{ iff } \forall P.\ c_P \subseteq c'_P.$$

Similarly, we may also define

$$\begin{aligned}(c \cup c')_P &= c_P \cup c'_P \\ (c \cap c')_P &= c_P \cap c'_P.\end{aligned}$$

The reader may verify that, when so defined, $\cup$ and $\cap$ construct colors from colors, and pure colors from pure colors.

We will call a poset P "c-complete" iff, for all $D \in c_P$, P contains the lub of D. The term "directed-complete poset" (DCPO) coincides with conventional usage. We will also occasionally use the following conventional terms:

$$\begin{aligned}\text{complete poset (CPO)} &\stackrel{\text{def}}{=} (\text{directed} \cup \text{empty})\text{-complete poset} \\ \text{complete lattice} &\stackrel{\text{def}}{=} \text{all-complete poset.}\end{aligned}$$

Since an uninteresting subset always contains its own lub, every poset is uninteresting-complete. From Propositions 2.1 and 2.2, we find

$$\begin{aligned}&\text{chain-complete poset} \\ &\qquad = (\text{countable} \cap \text{directed})\text{-complete poset}\end{aligned}$$

$$\begin{aligned}&((\text{bounded} \cap \text{finite} \cap \text{nonempty}) \cup \text{directed})\text{-complete poset} \\ &\qquad = ((\text{bounded} \cap \text{nonempty}) \cup \text{directed})\text{-complete poset,}\end{aligned}$$

and from the second of these equalities, it is easy to infer

$$\begin{aligned}&((\text{bounded} \cap \text{finite}) \cup \text{directed})\text{-complete poset} \\ &\qquad = (\text{bounded} \cup \text{directed})\text{-complete poset}\end{aligned}$$

$$\begin{aligned}&((\text{bounded} \cap \text{finite}) \cup \text{directed} \cup \text{empty})\text{-complete poset} \\ &\qquad = (\text{bounded} \cup \text{directed} \cup \text{empty})\text{-complete poset.}\end{aligned}$$

The classes of posets described by the last two equations differ only in that the first contains the empty poset (which contains no bounded or directed subsets, but does contain an empty subset). The second is sometimes called the class of "consistent complete posets".

A function f, from a c-complete poset P to a poset P', is said to be *c-continuous* iff, for all $D \in c_P$, $f(\bigsqcup D)$ is the lub of $\{fx \mid x \in D\}$.

Proposition 2.6 *A monotone function f, from a c-complete poset P to a c-complete poset P', is c-continuous iff, for all $D \in c_P$,*

$$f(\bigsqcup D) \sqsubseteq \bigsqcup\{fx \mid x \in D\}.$$

PROOF Since $D \in c_P$ and f is monotone, $\{fx \mid x \in D\} \in c_{P'}$, so that lub on the right of the above equation exists. For any $x \in D$, the monotonicity of f implies $f(\bigsqcup D) \sqsupseteq fx$, so that $f(\bigsqcup D) \sqsupseteq \bigsqcup\{fx \mid x \in D\}$. Thus the opposite inclusion is enough to give c-continuity. END OF PROOF

Proposition 2.7 *A function f, from a poset P to a poset P', is monotone iff it is uninteresting-continuous.*

PROOF Suppose f is uninteresting-continuous. If $x \sqsubseteq y$, then $\{x, y\}$ is uninteresting, so that $f(\bigsqcup\{x, y\})$ is the lub of $\{fx, fy\}$. But $\bigsqcup\{x, y\} = y$. Thus fy is the lub of $\{fx, fy\}$, and therefore $fx \sqsubseteq fy$.

On the other hand, suppose f is monotone. If D is uninteresting, then $\bigsqcup D$ is a member of D, so that $f(\bigsqcup D)$ belongs to $\{fx \mid x \in D\}$ and must approximate the lub of this set. Then, by Proposition 2.6, f is uninteresting-continuous. END OF PROOF

Since uninteresting sets are directed, directed-continuous functions are uninteresting-continuous, and therefore monotone. In fact, if c is any of the colors defined by Equations 2.1 except "no" or "empty", then $\{x, y\} \in c_P$ whenever $x \sqsubseteq y$. Thus, for any of these colors, c-continuous functions are monotone.

Since monotone functions are uninteresting-continuous, we can strengthen Proposition 2.6 by limiting the universal quantification over $D \in c_P$ to interesting $D \in c_P$.

We will often use the following conventional terms:

$$\text{continuous function} \stackrel{\text{def}}{=} \text{directed-continuous function}$$

$$\text{strict function} \stackrel{\text{def}}{=} \text{empty-continuous function}$$

$$\text{additive function} \stackrel{\text{def}}{=} \text{bounded continuous function.}$$

By using the constructions in the proofs of Propositions 2.1 and 2.2, the reader may verify that

$$\begin{aligned}
&\text{chain-continuous function}\\
&\qquad = (\text{countable} \cap \text{directed})\text{-continuous function}\\[1ex]
&((\text{bounded} \cap \text{finite} \cap \text{nonempty}) \cup \text{directed})\text{-continuous function}\\
&\qquad = ((\text{bounded} \cap \text{nonempty}) \cup \text{directed})\text{-continuous function}\\[1ex]
&((\text{bounded} \cap \text{finite}) \cup \text{directed})\text{-continuous function}\\
&\qquad = (\text{bounded} \cup \text{directed})\text{-continuous function}\\[1ex]
&((\text{bounded} \cap \text{finite}) \cup \text{directed} \cup \text{empty})\text{-continuous function}\\
&\qquad = (\text{bounded} \cup \text{directed} \cup \text{empty})\text{-continuous function.}
\end{aligned}$$

For any color c, since c-completeness guarantees the existence of the lub's that are preserved by c-continuity, there is a subcategory of PARO (the category of posets and monotone functions) whose objects are c-complete posets and whose morphisms are c-continuous monotone functions.

Often, however, we need to consider functions that do not preserve all existing lub's. (For example, the least fixed-point theorem holds for all directed-continuous functions on a (directed $\cup$ empty)-complete poset.) Thus, we need to define subcategories of PARO indexed by both a color for objects and a color for morphisms.

When c and d are colors such that $c \subseteq d$, we write $\mathcal{K}_c^d$ for the subcategory of PARO whose objects are d-complete posets and whose morphisms are c-continuous functions. It is easy to see that:

- If $c \subseteq d \subseteq d'$, then $\mathcal{K}_c^{d'}$ is a full subcategory of $\mathcal{K}_c^d$.
- If $c \subseteq c' \subseteq d$, then $\mathcal{K}_{c'}^d$ is a subcategory of $\mathcal{K}_c^d$.

We will occasionally use more conventional names:

$$\begin{aligned}
\text{DCPO} &\stackrel{\text{def}}{=} \mathcal{K}_{\text{directed}}^{\text{directed}}\\
\text{CPO} &\stackrel{\text{def}}{=} \mathcal{K}_{\text{directed}}^{\text{directed} \cup \text{empty}}\\
\text{SCPO} &\stackrel{\text{def}}{=} \mathcal{K}_{\text{directed} \cup \text{empty}}^{\text{directed} \cup \text{empty}}.
\end{aligned}$$

2.3 Products

The (binary) *Cartesian product* of two posets P_1 and P_2 is formed from the Cartesian product of the underlying sets,

$$P_1 \times P_2 \stackrel{\text{def}}{=} \{\langle x_1, x_2\rangle \mid x_1 \in P_1 \text{ and } x_2 \in P_2\},$$

by imposing a pairwise ordering,

$$\langle x_1, x_2\rangle \sqsubseteq_{P_1 \times P_2} \langle y_1, y_2\rangle \text{ iff } x_1 \sqsubseteq_{P_1} y_1 \text{ and } x_2 \sqsubseteq_{P_2} y_2.$$

(We leave it to the read to check that $\sqsubseteq_{P_1 \times P_2}$ is a partial order when $\sqsubseteq_{P_1}$ and $\sqsubseteq_{P_2}$ are partial orders.)

Associated with this product are two *projection* functions $\pi_1 \in P_1 \times P_2 \to P_1$ and $\pi_2 \in P_1 \times P_2 \to P_2$ such that

$$\pi_1\langle x_1, x_2\rangle = x_1 \qquad \pi_2\langle x_1, x_2\rangle = x_2.$$

(We leave it to the reader to check that the pairwise ordering makes these functions monotone.)

This Cartesian product is an instance of the general category-theoretic concept of a product (applied to the category PARO): For any poset Q and any monotone functions $g_1 \in Q \to P_1$ and $g_2 \in Q \to P_2$, there is exactly one monotone function, which we will denote by $g_1 \otimes g_2$, from Q to $P_1 \times P_2$ that makes both triangles in the following diagram commute:

$$\begin{tikzcd}
& Q \arrow[dl, "g_1"'] \arrow[d, dashed, "g_1 \otimes g_2"] \arrow[dr, "g_2"] & \\
P_1 & P_1 \times P_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & P_2
\end{tikzcd}$$

To see this, suppose $z \in Q$, and let $\langle x_1, x_2\rangle$ be $(g_1 \otimes g_2)z$. Then the definition of the π's and the commutivity of the triangles implies

$$x_1 = \pi_1\langle x_1, x_2\rangle = g_1 z \qquad x_2 = \pi_2\langle x_1, x_2\rangle = g_2 z,$$

so that

$$(g_1 \otimes g_2)z = \langle g_1 z, g_2 z\rangle.$$

Since this equation must hold for all $z \in Q$, it uniquely determines the function $g_1 \otimes g_2 \in Q \to P_1 \times P_2$.

What remains is to be sure that this function actually exists as a morphism in the category PARO, i.e. that, if g_1 and g_2 are monotone, then $g_1 \otimes g_2$ is monotone. We leave this to the reader.

Now suppose we have a pair of colors $c \subseteq d$, and consider the subcategory $\mathcal{K}_c^d$ of PARO. We want to show that the Cartesian product, along with its associated morphisms, is also a product in $\mathcal{K}_c^d$.

Since composition is the same in $\mathcal{K}_c^d$ as in PARO, the diagram will continue to commute just when $(g_1 \otimes g_2)z = \langle g_1 z, g_2 z\rangle$, *providing* all of the relevant entities exist in the subcategory. Specifically, we must show that, when P_1 and P_2 are d-complete,

- $P_1 \times P_2$ is d-complete.
- π_1 and π_2 are c-continuous.
- If Q is d-complete and g_1 and g_2 are c-continuous, then $g_1 \otimes g_2$ is c-continuous.

Suppose $D \in d_{P_1 \times P_2}$. By the first property of colors, $\{\pi_1 z \mid z \in D\} \in c_{P_1}$ and $\{\pi_2 z \mid z \in D\} \in c_{P_2}$. Thus these sets have lub's in P_1 and P_2, and we may pair these lub's to obtain

$$p \stackrel{\text{def}}{=} \langle \bigsqcup_{P_1} \{\pi_1 z \mid z \in D\}, \bigsqcup_{P_2} \{\pi_2 z \mid z \in D\}\rangle \in P_1 \times P_2.$$

For any $z \in D$, $\pi_1 p$ is the lub of a set containing $\pi_1 z$, so that $\pi_1 p \sqsupseteq \pi_1 z$. Similarly, $\pi_2 p \sqsupseteq \pi_2 z$, so by the pairwise ordering, $p \sqsupseteq z$, and since this holds for all $z \in D$, p is an upper bound of D.

On the other hand, suppose p' is an upper bound of D. For any $z \in D$, $p' \sqsupseteq z$ gives $\pi_1 p' \sqsupseteq \pi_1 z$, so that $\pi_1 p' \sqsupseteq \bigsqcup\{\pi_1 z \mid x \in D\}$. Similarly, $\pi_2 p' \sqsupseteq \bigsqcup\{\pi_2 z \mid x \in D\}$, so that $p' \sqsupseteq p$. Thus p is the lub of D, i.e. for any $D \in d_{P_1 \times P_2}$,

$$\bigsqcup_{P_1 \times P_2} D = \langle \bigsqcup_{P_1} \{\pi_1 z \mid z \in D\}, \bigsqcup_{P_2} \{\pi_2 z \mid z \in D\}\rangle.$$

Now suppose $D \in c_{P_1 \times P_2}$. Since $c \subseteq d$, we may use the above equation to obtain

$$\pi_1(\bigsqcup_{P_1 \times P_2} D) = \bigsqcup_{P_1} \{\pi_1 z \mid z \in D\}.$$

Thus π_1, and similarly π_2, is c-continuous. (In fact, these projection functions are also d-continuous.)

Finally, suppose Q is d-continuous, $g_1 \in Q \to P_1$ and $g_2 \in Q \to P_2$ are c-continuous, and $D \in c_Q$. Then

$$\begin{aligned}
(g_1 \otimes g_2)(\textstyle\bigsqcup_Q D) &= \langle g_1(\textstyle\bigsqcup_Q D), g_2(\textstyle\bigsqcup_Q D)\rangle \\
&= \langle \textstyle\bigsqcup_{P_1}\{g_1 w \mid w \in D\}, \textstyle\bigsqcup_{P_2}\{g_2 w \mid w \in D\}\rangle \\
&= \langle \textstyle\bigsqcup_{P_1}\{\pi_1((g_1 \otimes g_2)w) \mid w \in D\}, \\
&\qquad\quad \textstyle\bigsqcup_{P_2}\{\pi_2((g_1 \otimes g_2)w) \mid w \in D\}\rangle \\
&= \langle \textstyle\bigsqcup_{P_1}\{\pi_1 z \mid z \in \{(g_1 \otimes g_2)w \mid w \in D\}\}, \\
&\qquad\quad \textstyle\bigsqcup_{P_2}\{\pi_2 z \mid z \in \{(g_1 \otimes g_2)w \mid w \in D\}\}\rangle \\
&= \textstyle\bigsqcup_{P_1 \times P_2} \{(g_1 \otimes g_2)w \mid w \in D\}.
\end{aligned}$$

From the basic categorical property of the product, one can see that, if $f_1 \in P_1' \to P_1$ and $f_2 \in P_2' \to P_2$ are morphisms in $\mathcal{K}_c^d$, then $(f_1 \cdot \pi_1') \otimes (f_2 \cdot \pi_2')$ is the unique morphism making

$$\begin{tikzcd}
P_1' \arrow[d, "f_1"'] & P_1' \times P_2' \arrow[l, "\pi_1'"'] \arrow[r, "\pi_2'"] \arrow[d, dashed, "(f_1 \cdot \pi_1') \otimes (f_2 \cdot \pi_2')"] & P_2' \arrow[d, "f_2"] \\
P_1 & P_1 \times P_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & P_2
\end{tikzcd}$$

commute. In fact, as the reader may verify, one can extend $\times$ from a binary constructor on objects to a bifunctor from $\mathcal{K}_c^d \times \mathcal{K}_c^d$ to $\mathcal{K}_c^d$ by defining its action on morphisms to be

$$f_1 \times f_2 = (f_1 \cdot \pi_1') \otimes (f_2 \cdot \pi_2').$$

When $c \subseteq c'$ and $d \subseteq d'$, $\mathcal{K}_{c'}^{d'}$ is a subcategory of $\mathcal{K}_c^d$. (In the extreme case, $\mathcal{K}_{\mathrm{no}}^{\mathrm{no}}$ is PARO.) Whenever P_1 and P_2 are objects of $\mathcal{K}_{c'}^{d'}$, $P_1 \times P_2$ is their product in both $\mathcal{K}_{c'}^{d'}$ and $\mathcal{K}_c^d$, with the same associated projection functions. Thus the inclusion functor from $\mathcal{K}_{c'}^{d'}$ to $\mathcal{K}_c^d$ is product-preserving.

2.4 Pullbacks

Suppose P_0, P_1, and P_2 are posets and $f_1 \in P_1 \to P_0$ and $f_2 \in P_2 \to P_0$ are monotone functions. Then the pullback of these entities is the set

$$P \stackrel{\text{def}}{=} \{\langle x_1, x_2\rangle \mid x_1 \in P_1 \text{ and } x_2 \in P_2 \text{ and } f_1 x_1 = f_2 x_2\}.$$

This set (which is also the pullback in the category SET) is a subset of the Cartesian product $P_1 \times P_2$, and is given the same pairwise ordering as the product:

$$\langle x_1, x_2\rangle \sqsubseteq_{P_1 \times P_2} \langle y_1, y_2\rangle \text{ iff } x_1 \sqsubseteq_{P_1} y_1 \text{ and } x_2 \sqsubseteq_{P_2} y_2.$$

There are also two associated projection functions, $\pi_1 \in P \to P_1$ and $\pi_2 \in P \to P_2$, which are restrictions of those for the product:

$$\pi_1\langle x_1, x_2\rangle = x_1 \qquad \pi_2\langle x_1, x_2\rangle = x_2.$$

(The argument that the ordering of P is a partial order, and that the projection functions are monotone, is the same as for the product.)

The poset P satisfies the general category-theoretic concept of a pullback: In the diagram

$$\begin{tikzcd}
& Q \arrow[d, dashed, "\eta"] \arrow[ddl, "g_1"'] \arrow[ddr, "g_2"] & \\
& P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
P_1 \arrow[dr, "f_1"'] & & P_2 \arrow[dl, "f_2"] \\
& P_0 &
\end{tikzcd}$$

the inner diamond commutes, and if the outer quadrilateral commutes, then there is exactly one $\eta \in Q \to P$ such that the two triangles commute.

To see this, suppose the outer quadrilateral commutes. The argument for the uniqueness of η is the same as for the product:

Suppose $z \in Q$, and let $\langle x_1, x_2 \rangle$ be ηz. Then the definition of the π's and the commutivity of the triangles implies

$$x_1 = \pi_1 \langle x_1, x_2 \rangle = g_1 z \qquad x_2 = \pi_2 \langle x_1, x_2 \rangle = g_2 z,$$

so that

$$\eta z = \langle g_1 z, g_2 z \rangle,$$

which uniquely determines η.

As with the product, it is also easy to see that the monotonicity of g_1 and g_2 implies the monotonicity of η. But now we must check an additional condition: that the range of η falls within P, i.e. that, for all $z \in Q$, if $\langle x_1, x_2 \rangle = \eta z$ then $f_1 x_1 = f_2 x_2$. Since $\eta z = \langle g_1 z, g_2 z \rangle$, this condition is just $f_1(g_1 z) = f_2(g_2 z)$, which is insured by the commutativity of the outer quadrilateral.

Now suppose we have a pair of colors $c \subseteq d$, and consider whether the pullback exists in the category $\mathcal{K}_c^d$. Assuming that P_0, P_1, P_2, f_1, and f_2 all live in this category, we must show, as with the product, that

- P is d-complete.
- π_1 and π_2 are c-continuous.
- If g_1 and g_2 are c-continuous, then $g_1 \otimes g_2$ is c-continuous.

The argument is similar to that for the product, with one significant addition: When $D \in d_P$, in order to show that the pair

$$p \stackrel{\text{def}}{=} \langle \textstyle\bigsqcup_{P_1} \{\pi_1 z \mid z \in D\}, \bigsqcup_{P_2} \{\pi_2 z \mid z \in D\} \rangle$$

is the lub of D in P, we must show that this pair actually belongs to P, i.e. that

$$f_1(\textstyle\bigsqcup_{P_1} \{\pi_1 z \mid z \in D\}) = f_2(\bigsqcup_{P_2} \{\pi_2 z \mid z \in D\}).$$

To do this, we must assume that f_1 and f_2 are d-continuous, not just c-continuous. Then we can rewrite the above condition as

$$\textstyle\bigsqcup_{P_0} \{f_1(\pi_1 z) \mid z \in D\} = \bigsqcup_{P_0} \{f_2(\pi_2 z) \mid z \in D\},$$

which holds because the $z \in D$ belong to P, so that $f_1(\pi_1 z) = f_2(\pi_2 z)$.

Thus we only know that the pullback of P_0, P_1, P_2, f_1, and f_2 exists in the category $\mathcal{K}_c^d$ in the special case where f_1 and f_2 are d-continuous. This is a vacuous restriction when $c = d$, but not otherwise. For example, in $\mathrm{CPO} = \mathcal{K}_{\mathrm{directed}}^{\mathrm{directed} \cup \mathrm{empty}}$, if $f_1\bot \neq f_2\bot$ (which can only occur if at least one of these functions is not strict), then (possibly) P has no least element.

2.5 Exponentiation

Suppose $c \subseteq d$, Q is a c-complete poset, and R is a d-complete poset. We write $Q \xrightarrow{c} R$ for the set of c-continuous monotone functions from Q to R, and $Q \overset{c}{\Rightarrow} R$ for the poset obtained from $Q \xrightarrow{c} R$ by imposing a pointwise ordering:

$$f \sqsubseteq_{Q \overset{c}{\Rightarrow} R} g \text{ iff } \forall x \in Q.\ fx \sqsubseteq_R gx$$

(which is obviously a partial order).

In fact, the poset $Q \overset{c}{\Rightarrow} R$ is d-complete. For suppose $F \in d_{Q \overset{c}{\Rightarrow} R}$. The pointwise ordering insures that applying a function to a fixed argument is a monotone operation, so that, by the first property of colors, $\{fx \mid f \in F\} \in d_R$. Thus we can define g be the (obviously monotone) function from Q to R such that

$$gx = \bigsqcup_R \{fx \mid f \in F\}.$$

It is easy to see that g is the lub of F, provided it actually belongs to $Q \overset{c}{\Rightarrow} R$, i.e. provided it is c-continuous. To see this, let $D \in c_Q$. Then, since the member of F are c-continuous,

$$\begin{aligned}
g(\textstyle\bigsqcup_Q D) &= \textstyle\bigsqcup_R \{f(\bigsqcup_Q D) \mid f \in F\} \\
&= \textstyle\bigsqcup_R \{\bigsqcup_R \{fx \mid x \in D\} \mid f \in F\} \\
&= \textstyle\bigsqcup_R \{fx \mid x \in D \text{ and } f \in F\} \\
&= \textstyle\bigsqcup_R \{\bigsqcup_R \{fx \mid f \in F\} \mid x \in D\} \\
&= \textstyle\bigsqcup_R \{gx \mid x \in D\}.
\end{aligned}$$

Thus

$$\bigsqcup_{Q \overset{c}{\Rightarrow} R} F = \lambda x \in Q.\ \bigsqcup_R \{fx \mid f \in F\}.$$

We can extend $\overset{c}{\Rightarrow}$ from a binary operation on objects to a bifunctor from $(\mathcal{K}_c^c)^{\mathrm{op}} \times \mathcal{K}_c^d$ to $\mathcal{K}_c^d$ as follows: If Q and Q' are c-complete posets, R and R'

are d-complete posets, $g \in Q' \xrightarrow{c} Q$, and $h \in R \xrightarrow{c} R'$, then

$$g \stackrel{c}{\Rightarrow} h \in (Q \stackrel{c}{\Rightarrow} R) \xrightarrow{c} (Q' \stackrel{c}{\Rightarrow} R')$$

is the function such that

$$(g \stackrel{c}{\Rightarrow} h)f = h \cdot f \cdot g.$$

To see that this function is c-continuous, suppose $F \in c_{Q \stackrel{c}{\Rightarrow} R}$ and $x \in Q'$. Then

$$\begin{aligned}
(g \stackrel{c}{\Rightarrow} h)(\textstyle\bigsqcup_{Q \stackrel{c}{\Rightarrow} R} F)x &= h((\textstyle\bigsqcup_{Q \stackrel{c}{\Rightarrow} R} F)(gx)) \\
&= h(\textstyle\bigsqcup_{R}\{f(gx) \mid f \in F\}) \\
&= \textstyle\bigsqcup_{R'}\{h(f(gx)) \mid f \in F\} \\
&= \textstyle\bigsqcup_{R'}\{(g \stackrel{c}{\Rightarrow} h)f\,x \mid f \in F\} \\
&= (\textstyle\bigsqcup_{Q' \stackrel{c}{\Rightarrow} R'}\{(g \stackrel{c}{\Rightarrow} h)f \mid f \in F\})x.
\end{aligned}$$

Since this holds for all $x \in Q'$, we have the continuity equation

$$(g \stackrel{c}{\Rightarrow} h)(\textstyle\bigsqcup_{Q \stackrel{c}{\Rightarrow} R} F) = \textstyle\bigsqcup_{Q' \stackrel{c}{\Rightarrow} R'}\{(g \stackrel{c}{\Rightarrow} h)f \mid f \in F\}.$$

Since $\mathcal{K}_c^d$ is a subcategory of $\mathcal{K}_c^c$, $\stackrel{c}{\Rightarrow}$ can be restricted to a bifunctor from $(\mathcal{K}_c^d)^{\mathrm{op}} \times \mathcal{K}_c^d$ to $\mathcal{K}_c^d$. In the special case where $c \subseteq$ directed, this restriction (which we will denote simply by $c \Rightarrow d$) is the *exponentiation* operation for the category $\mathcal{K}_c^d$.

Associated with this exponentiation operation is the family of *application* functions $\mathsf{ap}_{QR} \in (Q \Rightarrow R) \times Q \xrightarrow{c} R$ such that

$$\mathsf{ap}_{QR}\langle f, y\rangle = y.$$

To see that ap_{QR} is continuous, suppose $D \in c_{(Q \Rightarrow R)\times Q} \subseteq \text{directed}_{(Q \Rightarrow R)\times Q}$. Then

$$\begin{aligned}
\mathsf{ap}_{QR}(\textstyle\bigsqcup D) &= \mathsf{ap}_{QR}\langle \textstyle\bigsqcup_{Q \Rightarrow R}\{\pi_1 z \mid z \in D\}, \textstyle\bigsqcup_{Q}\{\pi_2 z \mid z \in D\}\rangle \\
&= (\textstyle\bigsqcup_{Q \Rightarrow R}\{\pi_1 z \mid z \in D\})(\textstyle\bigsqcup_{Q}\{\pi_2 z \mid z \in D\}) \\
&= \textstyle\bigsqcup_{R}\{\pi_1 z(\textstyle\bigsqcup_{Q}\{\pi_2 z' \mid z' \in D\}) \mid z \in D\} \\
&= \textstyle\bigsqcup_{R}\{\textstyle\bigsqcup_{R}\{\pi_1 z(\pi_2 z') \mid z' \in D\} \mid z \in D\} \\
&= \textstyle\bigsqcup_{R}\{\pi_1 z(\pi_2 z') \mid z \in D \text{ and } z' \in D\} \\
&= \textstyle\bigsqcup_{R}\{\pi_1 z(\pi_2 z) \mid z \in D\} \\
&= \textstyle\bigsqcup_{R}\{\mathsf{ap}_{QR}\, z \mid x \in D\}.
\end{aligned}$$

Here it is the penultimate step where $c \subseteq$ directed is critical. Clearly,

$$\{\pi_1 z(\pi_2 z) \mid z \in D\}$$

is a subset of

$$\{\pi_1 z(\pi_2 z') \mid z \in D \text{ and } z' \in D\},$$

so that an upper bound of the latter set is an upper bound of the former. On the other hand, consider any member $\pi_1 z(\pi_2 z')$ of the latter set; the directed set D will contain an upper bound z'' of z and z', so that $\pi_1 z''(\pi_2 z'')$ will be a member of the former set that extends $\pi_1 z(\pi_2 z')$. Thus any upper bound of the former set will be an upper bound of the latter. Finally, since the two sets have the same set of upper bounds, they must have the same lub.

In calling $\Rightarrow$ an exponentiation operator for $\mathcal{K}_c^d$ (when $c \subseteq$ directed), we imply that $\mathcal{K}_c^d$ has the category-theoretic property of being *Cartesian closed*: For all d-complete posets P, Q, and R, and c-continuous monotone functions $f \in P \times Q \to R$, there is a unique c-continuous monotone function $\mathsf{ab}\, f \in P \to (Q \Rightarrow R)$ such that

$$\begin{tikzcd}
P \times Q \arrow[r, "\mathsf{ab}\, f \times I_Q"] \arrow[dr, "f"'] & (Q \Rightarrow R) \times Q \arrow[d, "\mathsf{ap}_{QR}"] \\
& R
\end{tikzcd}$$

commutes.

For all $x \in P$ and $y \in Q$, the definitions of ap_{QR} and of the action of $\times$ on morphisms give

$$\mathsf{ap}_{QR}((\mathsf{ab}\, f \times I_Q)\langle x, y\rangle) = (\mathsf{ab}\, f) x\, y.$$

Thus, for the diagram to commute, we must have

$$(\mathsf{ab}\, f) x\, y = f\langle x, y\rangle,$$

which completely determines $\mathsf{ab}\, f \in P \to (Q \Rightarrow R)$.

To complete the argument that our category is Cartesian closed, we must show that $\mathsf{ab}\, f$ has the necessary continuity properties:

1. For all $x \in P$, to show $(\mathsf{ab}\, f)x \in Q \overset{c}{\Longrightarrow} R$, we must show that $(\mathsf{ab}\, f)x$ is c-continuous.

2. To show that $\mathsf{ab}\, f \in P \overset{c}{\to} (Q \overset{c}{\Longrightarrow} R)$, we must also show that $\mathsf{ab}\, f$ is c-continuous.

To show (1), suppose $x \in P$ and $D \in c_Q$. Then

$$\begin{aligned}
(\mathsf{ab}\, f)x(\textstyle\bigsqcup_Q D) &= f\langle x, \textstyle\bigsqcup_Q D\rangle \\
&= f\langle \textstyle\bigsqcup_P \{x \mid y \in D\}, \bigsqcup_P \{y \mid y \in D\}\rangle \\
&= f(\textstyle\bigsqcup_{P \times Q} \{\langle x, y\rangle \mid y \in D\}) \\
&= \textstyle\bigsqcup_R \{f\langle x, y\rangle \mid y \in D\} \\
&= \textstyle\bigsqcup_R \{(\mathsf{ab}\, f)x\, y \mid y \in D\}.
\end{aligned}$$

Here the step between the first and second lines is justified because D is directed, and therefore nonempty; thus $\{x \mid y \in D\} = \{x\}$, so that $\bigsqcup_P \{x \mid y \in D\} = x$.

Property (2) is left to the reader; again the nonemptyness of D will be needed.

2.6 Disjoint Union

The disjoint union $P_1 + P_2$ of two posets P_1 and P_2 is the ordinary union of two disjoint posets that are isomorphic to P_1 and P_2 respectively. If we construct these isomorphic sets by pairing with the integers 1 and 2, we have

$$P_1 + P_2 \overset{\text{def}}{=} \{\langle 1, x\rangle \mid x \in P_1\} \cup \{\langle 2, x\rangle \mid x \in P_2\},$$

with the partial ordering

$$\langle i, x\rangle \sqsubseteq_{P_1 + P_2} \langle j, y\rangle \text{ iff } i = j \text{ and } x \sqsubseteq_{P_i} y.$$

Associated with this construction are the monotone *injections* $\iota_1 \in P_1 \to P_1 + P_2$ and $\iota_2 \in P_2 \to P_1 + P_2$ such that

$$\iota_1 x = \langle 1, x\rangle \qquad \iota_2 x = \langle 2, x\rangle.$$

This disjoint union is an instance of the general category-theoretic concept of a coproduct (in the category PARO): For any poset Q and any monotone functions $g_1 \in P_1 \to Q$ and $g_2 \in P_2 \to Q$, there is exactly one monotone function, which we will denote by $g_1 \oplus g_2$, from $P_1 + P_2$ to Q that makes both triangles in the following diagram commute:

$$\begin{tikzcd}
& Q & \\
P_1 \arrow[ur, "g_1"] \arrow[r, "\iota_1"] & P_1 + P_2 \arrow[u, dashed, "g_1 \oplus g_2"'] & P_2 \arrow[ul, "g_2"'] \arrow[l, "\iota_2"']
\end{tikzcd}$$

To see this, note that the commutivity of the left triangle requires

$$\forall x \in P_1.\ (g_1 \oplus g_2)\langle 1, x\rangle = g_1 x,$$

while that of the right triangle requires

$$\forall x \in P_2.\ (g_1 \oplus g_2)\langle 2, x\rangle = g_2 x.$$

Between them, these equations completely determine $g_1 \oplus g_2$. (The reader may verify that the ordering of $P_1 + P_2$ makes this function monotone.)

Now suppose we have two colors $c \subseteq d$, and consider whether $P_1 + P_2$ is a coproduct in the subcategory $\mathcal{K}_c^d$ of PARO. The situation here is very different than it was for the product. Clearly, $P_1 + P_2$ has no least element (unless P_1 or P_2 is empty), so that it is not empty-complete. Moreover, it is not d-complete if $d_{P_1+P_2}$ contains any set that includes members of both the images of P_1 and P_2, since such sets have no bounds at all in $P_1 + P_2$.

The arguments we are about to pursue become very cumbersome if one has to keep track of the pairings used to insure disjointness. To avoid this, we will simplify our discussion by assuming that P_1 and P_2 are themselves disjoint and taking $P_1 + P_2$ to be the ordinary union $P_1 \cup P_2$, with the ordering

$$x \sqsubseteq_{P_1+P_2} y \text{ iff } (x, y \in P_1 \text{ and } x \sqsubseteq_{P_1} y) \text{ or } (x, y \in P_2 \text{ and } x \sqsubseteq_{P_2} y).$$

(Note that this makes P_1 and P_2 subposets of $P_1 + P_2$, i.e. subsets with the same ordering.)

There is no loss of generality here. If P_1 or P_2 are not disjoint, one can replace them by disjoint isomorphic sets; our results will remain true since isomorphisms preserve upper bounds, least upper bounds, and colors.

In this simplified setting, the injections are

$$\iota_1 x = x \qquad \iota_2 x = x,$$

and

$$(g_1 \oplus g_2)x = \begin{cases} g_1 x \text{ when } x \in P_1 \\ g_2 x \text{ when } x \in P_2. \end{cases}$$

If D is a subset of $P_1 + P_2$ that contains members of both P_1 and P_2, then D itself, as well as some pair of members of D, will have no upper bound. Thus D will be neither directed nor bounded. This implies that, if $D \in ((\text{directed} \cup \text{bounded}) \cap \text{nonempty})_{P_1+P_2}$, then D will be a nonempty subset of either P_1 or P_2.

Thus, we will require

$$c \subseteq d \subseteq (\text{directed} \cup \text{bounded}) \cap \text{nonempty}.$$

(Note that the color on the right is the same as directed $\cup$ (bounded $\cap$ nonempty), since directed sets are never empty.) Under this assumption, $P_1 + P_2$ will be a coproduct in $\mathcal{K}_c^d$. To see this we will show that, when P_1 and P_2 are d-complete,

- $P_1 + P_2$ is d-complete.
- ι_1 and ι_2 are c-continuous.
- If Q is d-complete, and g_1 and g_2 are c-continuous, then $g_1 \oplus g_2$ is c-continuous.

In fact, it is simplest to begin with the continuity of the injections. Suppose $D \in d_{P_1}$. Then $\bigsqcup_{P_1} D$ exists, and $\iota_1(\bigsqcup_{P_1} D) = \bigsqcup_{P_1} D$. Moreover, since $d \subseteq$ nonempty, D is nonempty. Thus no member of P_2 is a bound on D, so that D has the same lub in $P_1 + P_2$ as in P_1. Thus $\iota_1(\bigsqcup_{P_1} D) = \bigsqcup_{P_1+P_2} D$, i.e. ι_1 is d-continuous (and therefore c-continuous). (The argument for ι_2 is obviously similar.)

To see the completeness of $P_1 + P_2$, suppose $D \in d_{P_1+P_2}$. Our restrictions on d insure that D will be a subset of either P_1 or P_2; without loss of generality we can assume P_1. Then, since D is nonempty and no member of P_2 extends any member of P_1, D has no upper bound in P_2. Thus, if D is bounded in $P_1 + P_2$, it is bounded in P_1, so that, by the third property of colors, $D \in d_{P_1}$.

Then since P_1 is d-complete, $\bigsqcup_{P_1} D$ exists, and since ι_1 is d-continuous, $\bigsqcup_{P_1} D = \iota_1(\bigsqcup_{P_1} D) = \bigsqcup_{P_1+P_2} D$.

Finally, assume that Q is d-complete, and g_1 and g_2 are c-continuous, and let $D \in c_{P_1+P_2}$. By the argument of the last two paragraphs (with c replacing d), D is a nonempty subset of either P_1 or P_2, and $\bigsqcup_{P_1+P_2} D$ is either $\bigsqcup_{P_1} D$ or $\bigsqcup_{P_2} D$. In the first case, the continuity of g_1 gives

$$\begin{aligned}(g_1 \oplus g_2)(\textstyle\bigsqcup_{P_1+P_2} D) &= (g_1 \oplus g_2)(\textstyle\bigsqcup_{P_1} D)\\ &= g_1(\textstyle\bigsqcup_{P_1} D)\\ &= \textstyle\bigsqcup_Q \{g_1 x \mid x \in D\}\\ &= \textstyle\bigsqcup_Q \{(g_1 + g_2)x \mid x \in D\}.\end{aligned}$$

The second case is similar.

From the basic categorical property of the coproduct, one can see that, if $f_1 \in P_1 \to P_1'$ and $f_2 \in P_2 \to P_2'$ are morphisms in $\mathcal{K}_c^d$, then $(\iota_1' \cdot f_1) \oplus (\iota_2' \cdot f_2)$ is the unique morphism making

$$\begin{tikzcd}
P_1' \arrow[r, "\iota_1'"] & P_1' + P_2' & P_2' \arrow[l, "\iota_2'"'] \\
P_1 \arrow[u, "f_1"] \arrow[r, "\iota_1"] & P_1 + P_2 \arrow[u, dashed, "(\iota_1' \cdot f_1) \oplus (\iota_2' \cdot f_2)"'] & P_2 \arrow[u, "f_2"'] \arrow[l, "\iota_2"']
\end{tikzcd}$$

commute. Just as with the product, one can use the unique mediating morphism here to extend $+$ from a binary constructor on objects to a bifunctor from $\mathcal{K}_c^d \times \mathcal{K}_c^d$ to $\mathcal{K}_c^d$, by defining its action on morphisms to be

$$f_1 + f_2 = (\iota_1' \cdot f_1) \oplus (\iota_2' \cdot f_2).$$

When $c \subseteq c'$ and $d \subseteq d' \subseteq$ (directed $\cup$ bounded) $\cap$ nonempty (as well as $c \subseteq d$ and $c' \subseteq d'$), the disjoint union is a coproduct in both $\mathcal{K}_c^d$ and its subcategory $\mathcal{K}_{c'}^{d'}$, and the inclusion functor is coproduct-preserving.

2.7 Lifting

For a poset P, the *lifting* $P_\perp$ is the poset obtained from P by adding a new least element $\perp$. The general definition of this concept is complicated by the possibility that $\perp$ may already be a member of P; to be sure that the new least element is distinct, we replace P by an isomorphic set that cannot contain $\perp$. Thus $P_\perp$ is the set

$$P_\perp \stackrel{\text{def}}{=} \{\perp\} \cup \{\langle x \rangle \mid x \in P\},$$

with the partial ordering

$$x \sqsubseteq_{P_\perp} y \text{ iff } x = \perp \text{ or } \exists x', y' \in P.\ x = \langle x' \rangle \text{ and } y = \langle y' \rangle \text{ and } x' \sqsubseteq_P y'.$$

Much as in the previous section, however, we will simplify our discussion by assuming that P itself does not contain $\perp$, and that $P_\perp = \{\perp\} \cup P$, with the ordering $x \sqsubseteq_{P_\perp} y$ iff $x = \perp$ or $x \sqsubseteq_P y$. (No generality is lost, since isomorphisms preserve upper bounds, least upper bounds, and colors.)

Associated with this construction is the *injection* function $\iota \in P \to P_\perp$ such that $\iota x = x$ (which is obviously monotone).

The lifting operation has the following property: If Q is a poset with a least element and g is a monotone function from P to Q, then there is exactly one strict monotone function $g_{\perp\!\!\perp}$ from $P_\perp$ to Q that makes

$$\begin{tikzcd}
P \arrow[r, "\iota"] \arrow[dr, "g"'] & P_\perp \arrow[d, dashed, "g_{\perp\!\!\perp}"] \\
& Q
\end{tikzcd}$$

commute, namely,

$$g_{\perp\!\!\perp} x = \textbf{if } x = \perp \textbf{ then } \perp \textbf{ else } gx.$$

Now suppose c and d are colors such that $c \subseteq d \subseteq$ nonempty, and P is d-complete. Then $P_\perp$ is ($d \cup$empty)-complete and ι is d-continuous. Moreover, if Q is $(c \cup \text{empty})$-complete and g is c-continuous, then $g_{\perp\!\!\perp}$ is $(c \cup \text{empty})$-continuous.

It is convenient to show the continuity of ι first. Suppose $D \in d_P$. Then $\bigsqcup_P D$ exists, and $\iota(\bigsqcup_P D) = \bigsqcup_P D$. Moreover, since $d \subseteq$ nonempty, D is nonempty. Thus $\perp$ is not a bound of D, so that D has the same lub in $P_\perp$ as in P.

To see the completeness of $P_\perp$, suppose $D \in (d \cup \text{empty})_{P_\perp}$. If $D \subseteq \{\perp\}$, then the lub of D in $P_\perp$ is obviously $\perp$. Otherwise $D - \{\perp\}$ is nonempty and contains a member of P. Then:

1. Since $\perp$ is extended by any member of the nonempty set $D - \{\perp\}$, and every other member of D belongs to $D - \{\perp\}$, the second property of colors gives $D - \{\perp\} \in (d \cup \text{empty})_{P_\perp}$.

2. Since $D - \{\perp\}$ is nonempty, $D - \{\perp\} \in d_{P_\perp}$.

3. Since $D - \{\perp\}$ contains a member of P, so that $\perp$ is not a bound of $D - \{\perp\}$, the third property of colors gives $D - \{\perp\} \in d_P$.

4. Since P is d-complete, $\bigsqcup_P(D - \{\perp\})$ exists.

5. Since ι is d-continuous, $\bigsqcup_P(D - \{\perp\}) = \iota(\bigsqcup_P(D - \{\perp\})) = \bigsqcup_{P_\perp}(D - \{\perp\})$.

6. Since adding a least element to a set never changes its lub, $\bigsqcup_{P_\perp}(D - \{\perp\}) = \bigsqcup_{P_\perp} D$.

Finally, suppose Q is $(c \cup \text{empty})$-complete, and g is c-continuous. To see that $g_{\perp\!\!\perp}$ is $(c \cup \text{empty})$-continuous, suppose $D \in (c \cup \text{empty})_{P_\perp}$. If $D \subseteq \{\perp\}$, then $g_{\perp\!\!\perp}(\bigsqcup_{P_\perp} D) = g_{\perp\!\!\perp}\perp = \perp$, while $\{g_{\perp\!\!\perp} x \mid x \in D\}$ is either empty or $\{\perp\}$, so that $\bigsqcup_Q\{g_{\perp\!\!\perp} x \mid x \in D\} = \perp$.

Otherwise $D - \{\perp\}$ is nonempty and contains a member of P. Then

$$
\begin{aligned}
g_{\perp\!\!\perp}(\textstyle\bigsqcup_{P_\perp} D) &= g_{\perp\!\!\perp}(\textstyle\bigsqcup_{P_\perp}(D - \{\perp\})) \\
&= g_{\perp\!\!\perp}(\textstyle\bigsqcup_P(D - \{\perp\})) \\
&= g(\textstyle\bigsqcup_P(D - \{\perp\})) \\
&= \textstyle\bigsqcup_Q\{gx \mid x \in D - \{\perp\}\} \\
&= \textstyle\bigsqcup_Q\{g_{\perp\!\!\perp} x \mid x \in D - \{\perp\}\} \\
&= \textstyle\bigsqcup_Q\{g_{\perp\!\!\perp} x \mid x \in D\}.
\end{aligned}
$$

Here the first step follows from (6) above, the second step from (5), and the fourth from the c-continuity of g and the fact that $D - \{\perp\} \in c_P$, which in turn follows from (1-3) above (with d replaced by c).

From a categorical viewpoint, lifting can be described as an adjunction. Suppose $c \subseteq d \subseteq$ nonempty, and let U be the identity-injection functor from $\mathcal{K}_{c \,\cup\, \mathsf{empty}}^{d \,\cup\, \mathsf{empty}}$ to $\mathcal{K}_c^d$. Then the operation of lifting is the object part of a functor $(-)_\perp$ from $\mathcal{K}_c^d$ to $\mathcal{K}_{c \,\cup\, \mathsf{empty}}^{d \,\cup\, \mathsf{empty}}$ that is left adjoint to U.

In detail, for each object P of $\mathcal{K}_c^d$ there is a morphism $\iota \in P \xrightarrow[\mathcal{K}_c^d]{} U(P_\perp)$ such that, for each object Q of $\mathcal{K}_{c \,\cup\, \mathsf{empty}}^{d \,\cup\, \mathsf{empty}}$ and morphism $g \in P \xrightarrow[\mathcal{K}_c^d]{} UQ$, there is exactly one morphism $g_{\perp\!\!\perp} \in P_\perp \xrightarrow[\mathcal{K}_{c \,\cup\, \mathsf{empty}}^{d \,\cup\, \mathsf{empty}}]{} Q$ that makes the following diagram commutes in $\mathcal{K}_c^d$:

$$\begin{tikzcd}
P \arrow[r, "\iota"] \arrow[dr, "g"'] & U(P_\perp) \arrow[d, dashed, "U(g_{\perp\!\!\perp})"] \\
& UQ
\end{tikzcd}$$

As a special case, suppose P' is also an object of $\mathcal{K}_c^d$, with an associated injection ι', and f is a morphism in $P \xrightarrow[\mathcal{K}_c^d]{} P'$. Then $(\iota' \cdot f)_{\perp\!\!\perp}$ is the unique morphism making

$$\begin{tikzcd}
P \arrow[r, "\iota"] \arrow[d, "f"'] & U(P_\perp) \arrow[d, dashed, "U((\iota' \cdot f)_{\perp\!\!\perp})"] \\
P' \arrow[r, "\iota'"] & U(P'_\perp)
\end{tikzcd}$$

commute. In fact, the action of $(-)_\perp$ on morphisms is given by the mediating morphism in this diagram:

$$f_\perp = (\iota' \cdot f)_{\perp\!\!\perp}.$$

2.8 Topping

For a poset P, the *topping* $P_\top$ is the poset obtained from P by adding a new greatest element $\top$. As with lifting, to be sure that the new element is distinct, we replace P by an isomorphic set that cannot contain $\top$. Thus $P_\top$ is the set

$$P_\top \stackrel{\text{def}}{=} \{\top\} \cup \{\langle x \rangle \mid x \in P\},$$

with the partial ordering

$$x \sqsubseteq_{P_\top} y \text{ iff } y = \top \text{ or } \exists x', y' \in P.\ x = \langle x' \rangle \text{ and } y = \langle y' \rangle \text{ and } x' \sqsubseteq_P y'.$$

Again, however, we will simplify our discussion by assuming that P itself does not the new element, and that $P_\top = \{\top\} \cup P$, with the ordering $x \sqsubseteq_{P_\top} y$ iff $y = \top$ or $x \sqsubseteq_P y$.

Associated with this construction is the *injection* function $\iota \in P \to P_\top$ such that $\iota x = x$ (which is obviously monotone).

The topping operation has the following property: If Q is a poset with a greatest element and g is a monotone function from P to Q, then there is exactly one monotone function $g_{\top\top}$ from $P_\top$ to Q that maps $\top$ into $\top$ and makes

$$\begin{tikzcd}
P \arrow[r, "\iota"] \arrow[dr, "g"'] & P_\top \arrow[d, dashed, "g_{\top\top}"] \\
& Q
\end{tikzcd}$$

commute, namely,

$$g_{\top\top}\, x = \textbf{if } x = \top \textbf{ then } \top_Q \textbf{ else } g x.$$

So far, our development has paralleled that for lifting, since the concepts of poset and monotone function are self-dual. But completeness and continuity give a special role to lub's that breaks this symmetry, and the development is somewhat different. In particular, there is a role for pure colors.

Now suppose c is a pure color and d is a color such that $c \subseteq d$, and P is d-complete. Then $P_\top$ is d-complete and ι is d-continuous. Moreover, if Q

is c-complete and has a greatest element, and g is c-continuous, then $g_{\top\!\top}$ is c-continuous and maps $\top$ into $\top_Q$.

Again, it is convenient to show the continuity of ι first. Suppose $D \in d_P$. Then $\bigsqcup_P D$ exists, and $\iota(\bigsqcup_P D) = \bigsqcup_P D$. Moreover, although $\top$ is an upper bound of D, it is an extension of $\bigsqcup_P D$, which will therefore continue to be the lub of D in $P_\top$. Thus $\iota(\bigsqcup_P D) = \bigsqcup_{P_\top} D$.

To see the completeness of $P_\top$, suppose $D \in d_{P_\top}$. If $\top$ is the only upper bound of D, then it is the lub of D; otherwise $D \subseteq P$ and if D is bounded in $P_\top$ it is bounded in P. Thus, by the third property of colors, $D \in d_P$. Then, since P is d-complete, $\bigsqcup_P D$ exists, and since ι is d-continuous, $\bigsqcup_P D = \iota(\bigsqcup_P D) = \bigsqcup_{P_\top} D$.

Finally, suppose Q is c-complete and has a greatest element, and g is c-continuous. Let $D \in c_{P_\top}$. If $\top \in D$ then $g_{\top\!\top}(\bigsqcup D) = g_{\top\!\top}\top = \top_Q$. But $\{g_{\top\!\top}x \mid x \in D\}$ contains $\top_Q$, so that $\bigsqcup_Q\{g_{\top\!\top}x \mid x \in D\} = \top_Q$.

On the other hand, if $\top \notin D$, then $D \subseteq P$, and by the special property of pure colors, $D \in c_P \subseteq d_P$. Then $\bigsqcup_P D$ exists, and $\bigsqcup_P D = \iota(\bigsqcup_P D) = \bigsqcup_{P_\top} D$. Thus

$$
\begin{aligned}
g_{\top\!\top}(\textstyle\bigsqcup_{P_\top} D) &= g_{\top\!\top}(\textstyle\bigsqcup_P D)\\
&= g(\textstyle\bigsqcup_P D)\\
&= \textstyle\bigsqcup_Q\{gx \mid x \in D\}\\
&= \textstyle\bigsqcup_Q\{g_{\top\!\top}x \mid x \in D\}.
\end{aligned}
$$