Chapter 1

Categories and CPO's

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1.1 Relations

A *relation* is a set whose members are pairs. We will sometimes write $\rho: x \mapsto x'$ or $x \; \rho \; x'$ as synonyms for $[x, x'] \in \rho$, and describe this relationship by saying that ρ *relates* x to x'.

The empty set is a relation, often called the *empty relation*. When two relations satisfy $\rho \subseteq \rho'$, ρ is said to be a *restriction* of ρ', and ρ' is said to be an *extension* of ρ.

When every pair $[x, x'] \in \rho$ satisfies $x = x'$, ρ is said to be an *identity relation*. For each set S there is an identity relation

$$I_S \stackrel{\text{def}}{=} \{[x, x] \mid x \in S\},$$

called the *identity relation on* S. Moreover, every identity relation is I_S for a unique set S.

A variety of additional operations upon relations are significant. For relations ρ and ρ', we define

- The *domain* of ρ:

$$\operatorname{dom} \rho \stackrel{\text{def}}{=} \{x \mid (\exists x') \; [x, x'] \in \rho\},$$

- The *range* of ρ:

$$\operatorname{ran} \rho \stackrel{\text{def}}{=} \{x' \mid (\exists x) \; [x, x'] \in \rho\},$$

- The *composition* of ρ with ρ':

$$\rho \,;\, \rho' \stackrel{\text{def}}{=} \rho' \cdot \rho \stackrel{\text{def}}{=} \{[x, x''] \mid (\exists x')\; [x, x'] \in \rho \text{ and } [x', x''] \in \rho'\},$$

- The *reflection* of ρ:

$$\rho^\dagger \stackrel{\text{def}}{=} \{[x', x] \mid [x, x'] \in \rho\}.$$

Note that the two operators ; and $\cdot$ describe composition in opposite orders. The order for ; is often called *diagrammatic* order, while the order for $\cdot$ is called *functional* order.

These operations obey the following laws, in which ρ, ρ', ρ'', ρ_1, ρ_2, ρ_1', and ρ_2' denote relations, S and T denote sets, and R and R' denote sets of relations:

- Associativity of composition:

$$(\rho \,;\, \rho') \,;\, \rho'' = \rho \,;\, (\rho' \,;\, \rho''),$$

- Identity laws:

$$I_S \,;\, \rho = \rho \text{ when } \operatorname{dom} \rho \subseteq S$$

$$\rho \,;\, I_S = \rho \text{ when } \operatorname{ran} \rho \subseteq S$$

$$I_S \,;\, \rho \subseteq \rho$$

$$\rho \,;\, I_S \subseteq \rho$$

$$\rho \subseteq I_S \text{ iff } (\exists T)\; T \subseteq S \text{ and } \rho = I_T$$

$$I_S \,;\, I_T = I_{S \cap T}$$

$$\operatorname{dom} I_S = \operatorname{ran} I_S = S,$$

- Reflection laws:

$$(\rho^\dagger)^\dagger = \rho$$

$$(\rho \,;\, \rho')^\dagger = \rho'^\dagger \,;\, \rho^\dagger$$

$$I_S{}^\dagger = I_S,$$

- Laws for the empty relation:

$$\{\}\,;\rho = \{\}$$

$$\rho\,;\{\} = \{\}$$

$$I_{\{\}} = \{\}$$

$$\{\}^\dagger = \{\},$$

- Monotonicity laws:

$$\text{If } \rho_1 \subseteq \rho_2 \text{ and } \rho_1' \subseteq \rho_2' \text{ then } \rho_1\,;\rho_1' \subseteq \rho_2\,;\rho_2'$$

$$\text{If } \rho_1 \subseteq \rho_2 \text{ then } {\rho_1}^\dagger \subseteq {\rho_2}^\dagger,$$

- Additivity laws:

$$\left(\bigcup R\right);\left(\bigcup R'\right) = \bigcup\{\rho\,;\rho' \mid \rho \in R \text{ and } \rho' \in R'\}$$

$$\left(\bigcup R\right)^\dagger = \bigcup\{\rho^\dagger \mid \rho \in R\}.$$

Since composition is associative, we can write multiple compositions without parentheses, e.g. $\rho\,;\rho'\,;\rho''$.

1.2 Functions

A function is a relation in which no first component is related to more than one second component. More precisely, a relation ρ is a *function* (or *map*) iff, whenever $[x, x_1'] \in \rho$ and $[x, x_2'] \in \rho$, $x_1' = x_2'$.

Equivalently, ρ is a function iff $\rho^\dagger\,;\rho$ is an identity relation. It follows that, if f and f' are functions, S is a set, and ρ is a relation, then

$$f\,;f' \text{ is a function,}$$

$$I_S \text{ is a function,}$$

$$\{\} \text{ is a function,}$$

$$\text{If } \rho \subseteq f \text{ then } \rho \text{ is a function.}$$

On the other hand it may be the case that $f^\dagger$ is not a function; if both f and $f^\dagger$ are functions then f is called an *injection*.

If f is a function and $x \in \operatorname{dom} f$, then f relates x to a unique x' that is called the *result of applying* f to x, and is denoted by fx (often without parenthesizing x). We also say f *maps* x into x' instead of f relates x to x'. Syntactically, we will assume that function application is left-associative, so that fxy means $(fx)y$.

If f and g are functions such that $x \in \operatorname{dom}(f\,;g)$ (which will be a subset of $\operatorname{dom} f$), then $(f\,;g)x = (g \cdot f)x = g(fx)$ (which shows why $\cdot$ is called composition in functional order).

If S denotes a set and E is an expression that is defined whenever the variable x denotes a member of S, then $\lambda x \in S.\ E$ denotes the function f with domain S such that $fx = E$ for all $x \in S$. For example, $\lambda x \in \mathbf{N}.\ x^2$ denotes the function $\{[x, x^2] \mid x \in \mathbf{N}\}$ that squares any nonnegative integer, and $\lambda x \in \mathbf{Z}.\ x + y$ denotes the function $\{[x, x + y] \mid x \in \mathbf{Z}\}$ that adds y to any integer. (Note that this is a different function for each value of y.)

If f is a function, then $[\,f \mid x\colon x'\,]$ denotes the function f' such that $\operatorname{dom} f' = \operatorname{dom} f \cup \{x\}$, $f'x = x'$, and, for all $y \in \operatorname{dom} f' - \{x\}$, $f'y = fy$. Note that this definition is meaningful regardless of whether x belongs to the domain of f.

We will abbreviate $[\ldots[\,f \mid x_1\colon x'_1\,]\ldots \mid x_n\colon x'_n\,]$ by $[\,f \mid x_1\colon x'_1 \mid \ldots \mid x_n\colon x'_n\,]$, and abbreviate $[\,\{\} \mid x_1\colon x'_1 \mid \ldots \mid x_n\colon x'_n\,]$ by $[\,x_1\colon x'_1 \mid \ldots \mid x_n\colon x'_n\,]$. In the special case where the x_i are distinct, the latter notation denotes the function with domain $\{x_1, \ldots, x_n\}$ that maps each x_i into x'_i. In particular, $[\,]$ denotes the empty function and is therefore a synonym of $\{\}$.

Finally, we abbreviate $[\,1\colon x'_1 \mid \ldots \mid n\colon x'_n\,]$ by $\langle x'_1, \ldots, x'_n\rangle$, which denotes the function with domain 1 to n that maps each i into x'_i. A function with domain 1 to n is called a *sequence of length* n or an n-tuple. Note that the 0-tuple $\langle\rangle$ is another synonym for $\{\}$.

We will succumb to the temptation of calling the 2-tuple $\langle x, y\rangle$ a *pair*, even though it is a different kind of pair than $[x, y]$.

There are several important notations for sets of functions. Suppose θ is a function, called a *set map*, that maps every member of its domain into a set. Then the *Cartesian product of* θ is the set of functions

$$\prod \theta \stackrel{\text{def}}{=} \{f \mid \operatorname{dom} f = \operatorname{dom}\theta \text{ and } (\forall i \in \operatorname{dom}\theta)\ fi \in \theta i\}.$$

If A denotes a set and, whenever the variable i denotes a member of A, the expression S denotes a set, then

$$\prod_{i \in A} S \stackrel{\text{def}}{=} \prod \lambda i \in A.\ S,$$

so that

$$\prod \theta = \prod_{i \in \operatorname{dom} \theta} \theta i.$$

For example, $\prod_{i \in \mathbf{Z}} \{j \mid j > i\}$ denotes the set of functions with domain $\mathbf{Z}$ that map each integer into a larger integer.

We can now develop more specialized forms of Cartesian product in much the same way as we did for union. If a is either an integer or $-\infty$ and b is either an integer or ∞, then

$$\prod_{i=a}^{b} S \stackrel{\text{def}}{=} \prod_{i \in a \text{ to } b} S,$$

and if $S_1, \ldots, S_n$ denote sets then

$$S_1 \times \ldots \times S_n \stackrel{\text{def}}{=} \prod_{i=1}^{n} \begin{cases} S_1 & \text{if } i = 1 \\ \vdots & \\ S_n & \text{if } i = n. \end{cases}$$

Thus $S_1 \times \ldots \times S_n$ is the set of n-tuples $\langle x_1, \ldots, x_n \rangle$ such that each $x_i \in S_i$. In particular, $S_1 \times S_2$, called the *binary Cartesian product* of S_1 with S_2, is the set of pairs $\langle x_1, x_2 \rangle$ such that $x_1 \in S_1$ and $x_2 \in S_2$.

The members of the powerset $\mathcal{P}(S_1 \times \ldots \times S_n)$ are called *n-ary relations*; as a special case, the members of $\mathcal{P}(S_1 \times S_2)$ are called *binary relations*. Thus a binary relation is similar to a relation, except that its members are pairs of the form $\langle x_1, x_2 \rangle$ instead of $[x_1, x_2]$.

It is also useful to specialize the Cartesian product to constant set maps. If A and S are expressions denoting sets, and the variable i does not occur in S, then

$$S^A \stackrel{\text{def}}{=} \prod_{i \in A} S = \{f \mid \operatorname{dom} f = A \text{ and } \operatorname{ran} f \subseteq S\}.$$

As a special case, using the convention that n stands for the set 1 to n,

$$S^n \stackrel{\text{def}}{=} S^{1 \text{ to } n} = \underbrace{S \times \cdots \times S}_{n \text{ times}}$$

is the set of n-tuples whose components all belong to S.

Note that $S^0 = S^{\{\}}$ is the singleton (i.e. one-element) set $\{\langle\rangle\}$, and $S_1 \times \ldots \times S_n$ is the empty set if any S_i is empty. More generally, $\prod_{i\in\{\}} S$ is $\{\langle\rangle\}$, and if S is empty for any $i \in A$, $\prod_{i\in A} S$ is empty.

It is unnecessary to introduce functions of several arguments as novel entities. In place of a function accepting n arguments in $S_1, \ldots, S_n$ and producing results in S', we will usually use a function in ${S'}^{S_1\times\cdots\times S_n}$, i.e. a function accepting a single argument that is an n-tuple. Thus we will write $f\langle x_1, \ldots, x_n\rangle$ instead of $f(x_1, \ldots, x_n)$.

To facilitate this approach, we define the notation

$$\lambda\langle\, x_1 \in S_1,\ \ldots,\ x_n \in S_n \,\rangle.\ E$$
$$\stackrel{\text{def}}{=} \lambda t \in S_1 \times \cdots \times S_n.\ (\lambda x_1 \in S_1.\ \ldots\ \lambda x_n \in S_n.\ E)(t1)\ldots(tn).$$

For example, a binary addition function would be

$$\lambda\langle\, x \in \mathbf{Z},\ y \in \mathbf{Z} \,\rangle.\ x + y \in \mathbf{Z}^{\mathbf{Z}\times\mathbf{Z}}.$$

Note that $\lambda\langle\, x \in S \,\rangle.\ E$ is not the same as $\lambda x \in S.\ E$, and that $\lambda\langle\rangle.\ E$ is the function with domain $\{\langle\rangle\}$ that maps $\langle\rangle$ into E.

Occasionally, we will take an alternative view of multi-argument functions called *Currying* (after the logician Haskell Curry). In this approach an n-argument function accepting arguments in $S_1, \ldots, S_n$ and producing results in S' is regarded as a function with domain S_1 whose results are functions with domain S_2 ... whose results are functions with domain S_n whose results are members of S', so that the set of such functions is $(\ldots(S')^{S_n}\ldots)^{S_1}$. Such functions can be defined by writing $\lambda x_1 \in S_1.\ \ldots\ \lambda x_n \in S_n.\ E$ and applied by writing $(\ldots(f x_1)\ldots x_n)$ or, since function application is left-associative, just $f x_1 \ldots x_n$.

1.3 The Category-Theoretic View of Relations and Functions

Instead of considering the collection of all relations or functions, in this section we will focus on sets of relations or functions *from* some specified set *to* some specified set.

When $\operatorname{dom} \rho \subseteq S$ and $\operatorname{ran} \rho \subseteq S'$, or equivalently when every $[x, x'] \in \rho$ satisfies $x \in S$ and $x' \in S'$, we say that ρ is a *relation from S to S'*. We write

$S \xrightarrow[\text{REL}]{} S'$ for the set of relations from S to S'. A relation from S to S is also called a *relation on* S.

Suppose S, S', and S'' are sets, $\rho \in S \xrightarrow[\text{REL}]{} S'$, and $\rho' \in S' \xrightarrow[\text{REL}]{} S''$. Then

$$\rho\,;\rho' \in S \xrightarrow[\text{REL}]{} S'',$$

$$I_S \in S \xrightarrow[\text{REL}]{} S,$$

$$\rho^\dagger \in S' \xrightarrow[\text{REL}]{} S,$$

$$\{\} \in S \xrightarrow[\text{REL}]{} S',$$

$$\{\} \in! \{\} \xrightarrow[\text{REL}]{} S',$$

$$\{\} \in! S \xrightarrow[\text{REL}]{} \{\}.$$

(Here the symbol $\in!$ means "is the unique member of".) In addition, for $\rho \in S \xrightarrow[\text{REL}]{} S'$, the first two identity laws can be written in the simpler form

$$I_S\,;\rho = \rho,$$

$$\rho\,;I_{S'} = \rho.$$

We now consider certain subsets of $S \xrightarrow[\text{REL}]{} S'$. If $\rho \in S \xrightarrow[\text{REL}]{} S'$ relates every member of S to at least one member of S', i.e. if $\operatorname{dom}\rho = S$ or equivalently

$$(\forall x \in S)\ (\exists x' \in S')\ [x, x'] \in \rho,$$

then ρ is said to be a *total relation from S to S'*. We write $S \xrightarrow[\text{TREL}]{} S'$ for the set of total relations from S to S'.

Notice that a total relation from S to S' is not just a relation from S to S' that is a total relation; the concept of "total relation" is meaningless without the qualification "from S to S'". (Indeed any relation is a total relation from its domain to any set containing its range.) We will call such a concept *morphic*; in general, if a "blue" "thing" from S to S' is not just a "thing" from S to S' that is "blue", we will say that "blue" is a morphic concept.

"Identity" is another morphic concept, since a relation on S that is an identity relation need not be an identity relation on S.

If ρ is a relation from S to S', then it is a total relation from S to S' iff $\rho\,;\rho^\dagger \supseteq I_S$. If $\rho \in S \xrightarrow[\text{TREL}]{} S'$ and $\rho' \in S' \xrightarrow[\text{TREL}]{} S''$ then

$$\rho\,;\rho' \in S \xrightarrow[\text{TREL}]{} S'',$$

$$I_S \in S \xrightarrow[\text{TREL}]{} S,$$
$$\{\} \in! \{\} \xrightarrow[\text{TREL}]{} S'.$$

A relation from S to S' that is a function is called (rather oddly) a *partial function from S to S'*. We write $S \xrightarrow[\text{PFUN}]{} S'$ for the set of partial functions from S to S'.

If $\rho \in S \xrightarrow[\text{REL}]{} S'$, then $\rho \in S \xrightarrow[\text{PFUN}]{} S'$ iff $\rho^\dagger \,;\, \rho \subseteq I_{S'}$. If $\rho \in S \xrightarrow[\text{PFUN}]{} S'$ and $\rho' \in S' \xrightarrow[\text{PFUN}]{} S''$ then

$$\rho \,;\, \rho' \in S \xrightarrow[\text{PFUN}]{} S'',$$
$$I_S \in S \xrightarrow[\text{PFUN}]{} S,$$
$$\{\} \in S \xrightarrow[\text{PFUN}]{} S',$$
$$\{\} \in! \{\} \xrightarrow[\text{PFUN}]{} S',$$
$$\{\} \in! S \xrightarrow[\text{PFUN}]{} \{\}.$$

A partial function from S to S' whose domain is S is called a *function from S to S'*; we write $S \xrightarrow[\text{SET}]{} S'$ for the set of functions from S to S'. This is another morphic concept, since a function f that is a relation from S to S' need only satisfy

$$\operatorname{dom} f \subseteq S \text{ and } \operatorname{ran} f \subseteq S'.$$

This makes f a partial function from S to S', but to be a function from S to S' it must satisfy the stronger condition

$$\operatorname{dom} f = S \text{ and } \operatorname{ran} f \subseteq S'.$$

It follows that $S \xrightarrow[\text{SET}]{} S'$ is also the Cartesian product of S' over S:

$$S \xrightarrow[\text{SET}]{} S' = (S')^S.$$

From another viewpoint, $S \xrightarrow[\text{SET}]{} S' = (S \xrightarrow[\text{TREL}]{} S') \cap (S \xrightarrow[\text{PFUN}]{} S')$. Thus if $\rho \in S \xrightarrow[\text{REL}]{} S'$, then $\rho \in S \xrightarrow[\text{SET}]{} S'$ iff $\rho \,;\, \rho^\dagger \supseteq I_S$ and $\rho^\dagger \,;\, \rho \subseteq I_{S'}$. If $\rho \in S \xrightarrow[\text{SET}]{} S'$ and $\rho' \in S' \xrightarrow[\text{SET}]{} S''$ then

$$\rho \,;\, \rho' \in S \xrightarrow[\text{SET}]{} S'',$$
$$I_S \in S \xrightarrow[\text{SET}]{} S,$$
$$\{\} \in! \{\} \xrightarrow[\text{SET}]{} S'.$$

Much of the preceding development can be systematized by introducing the concept of a category. A *category* $\mathcal{C}$ consists of:

- A class $\mathrm{Ob}\,\mathcal{C}$ of *objects*.
- For each C, $C' \in \mathrm{Ob}\,\mathcal{C}$, a set $C \xrightarrow[\mathcal{C}]{} C'$ of *morphisms from C to C'*. The union of these sets is denoted by $\mathrm{Mor}\,\mathcal{C}$.
- A partial function from $(\mathrm{Mor}\,\mathcal{C}) \times (\mathrm{Mor}\,\mathcal{C})$ to $\mathrm{Mor}\,\mathcal{C}$, denoted by the infix operator $;_\mathcal{C}$ or, in reverse order, $\cdot_\mathcal{C}$. The application $\rho ;_\mathcal{C} \rho'$ (or $\rho' \cdot_\mathcal{C} \rho$) is called the *composition of ρ with ρ'*, and must satisfy the requirement that, whenever $\rho \in C \xrightarrow[\mathcal{C}]{} C'$ and $\rho' \in C' \xrightarrow[\mathcal{C}]{} C''$, $\rho ;_\mathcal{C} \rho'$ must be defined and must belong to $C \xrightarrow[\mathcal{C}]{} C''$.
- For each $C \in \mathrm{Ob}\,\mathcal{C}$, a morphism $I^\mathcal{C}_C \in C \xrightarrow[\mathcal{C}]{} C$, called the *identity morphism on C*.

These entities must satisfy the following laws, for all $\rho \in C \xrightarrow[\mathcal{C}]{} C'$, $\rho' \in C' \xrightarrow[\mathcal{C}]{} C''$, and $\rho'' \in C'' \xrightarrow[\mathcal{C}]{} C'''$:

- Associativity of composition:

$$(\rho ;_\mathcal{C} \rho') ;_\mathcal{C} \rho'' = \rho ;_\mathcal{C} (\rho' ;_\mathcal{C} \rho''),$$

- Identity laws:

$$I^\mathcal{C}_C ;_\mathcal{C} \rho = \rho,$$
$$\rho ;_\mathcal{C} I^\mathcal{C}_{C'} = \rho.$$

In the light of this definition, what we have done is to define four categories: REL, TREL, PFUN, and SET. In each of these categories $\mathrm{Ob}\,\mathcal{C}$ is the class of all sets. In REL and TREL, $\mathrm{Mor}\,\mathcal{C}$ is the class of relations and composition is the composition of relations, while in the remaining categories $\mathrm{Mor}\,\mathcal{C}$ is the class of functions and composition is the composition of functions (which is a restriction of the composition of relations). In all four categories the identity morphism on S is the identity relation (a.k.a. the identity function) on S.

Notice that, in each of these categories, $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}\,\mathcal{C}$ are classes that are too large to be sets. Strictly speaking, when we defined "category", we should have called composition a "large" function, i.e. one whose domain can be a class. However, the restriction of composition to $(C \xrightarrow[\mathcal{C}]{} C') \times (C' \xrightarrow[\mathcal{C}]{} C'')$ is an ordinary function from $(C \xrightarrow[\mathcal{C}]{} C') \times (C' \xrightarrow[\mathcal{C}]{} C'')$ to $(C \xrightarrow[\mathcal{C}]{} C'')$.

A category $\mathcal{C}_1$ is said to be a *subcategory* of a category $\mathcal{C}_2$ iff:

- $\mathrm{Ob}\,\mathcal{C}_1 \subseteq \mathrm{Ob}\,\mathcal{C}_2$,
- For all $C,\, C' \in \mathrm{Ob}\,\mathcal{C}_1$, $C \xrightarrow[\mathcal{C}_1]{} C' \subseteq C \xrightarrow[\mathcal{C}_2]{} C'$,
- For $\rho \in C \xrightarrow[\mathcal{C}_1]{} C'$ and $\rho' \in C' \xrightarrow[\mathcal{C}_1]{} C''$, $\rho \,;_{\mathcal{C}_1} \rho' = \rho \,;_{\mathcal{C}_2} \rho'$,
- For $C \in \mathrm{Ob}\,\mathcal{C}_1$, $I_C^{\mathcal{C}_1} = I_C^{\mathcal{C}_2}$.

For example:

- TREL and PFUN are subcategories of REL.
- SET is a subcategory of TREL and of PFUN.

Each of the above subcategories is formed by retaining the same collection of objects and reducing the morphism sets in a way that retains identity morphisms and closure under composition. One can also obtain subcategories by reducing the collection of objects but retaining all morphisms between retained objects; in this case one says one has a *full* subcategory. More precisely, a subcategory $\mathcal{C}_1$ of $\mathcal{C}_2$ is called a *full subcategory of* $\mathcal{C}_2$ iff, for all $C,\, C' \in \mathrm{Ob}\,\mathcal{C}_1$, $C \xrightarrow[\mathcal{C}_1]{} C' = C \xrightarrow[\mathcal{C}_2]{} C'$.

Each subcollection of $\mathrm{Ob}\,\mathcal{C}_2$ is the collection of objects of a unique full subcategory of $\mathrm{Ob}\,\mathcal{C}_2$. For example, by reducing the collection of objects to the class of finite sets, we can define full subcategories $\mathrm{REL}_{\mathsf{fin}}$, $\mathrm{SET}_{\mathsf{fin}}$, etc. of each of the previously defined categories.

Since SET is the most ubiquitous category, it is convenient to write $S \to S'$ for $S \xrightarrow[\mathrm{SET}]{} S'$, I_S for I_S^{SET} and similarly for other notations that will be introduced later. More generally, we will continue to write ; and $\cdot$ without subscripts to denote both the composition of relations and its restriction to functions. However, we will also omit other category names from various notations when the relevant category is obvious from context.

In any category $\mathcal{C}$, an *isomorphism* ρ *from* C *to* C' is a morphism $\rho \in C \xrightarrow[\mathcal{C}]{} C'$ that possesses both left and right inverses, i.e. morphisms $\psi_l,\, \psi_r \in C' \xrightarrow[\mathcal{C}]{} C$ such that

$$\rho \,;\, \psi_r = I_C \text{ and } \psi_l \,;\, \rho = I_{C'}.$$

(This is another morphic concept.) These equations imply $\psi_l = \psi_l;\rho;\psi_r = \psi_r$, so that every right inverse equals every left inverse. Thus an isomorphism ρ from C to C' possesses a unique inverse, which is two-sided, and is denoted by ρ^{-1}. Moreover, ρ^{-1} is an isomorphism from C' to C, such that $(\rho^{-1})^{-1} = \rho$.

In the four categories we have defined, it can be shown that ρ is an isomorphism from S to S' iff it is a *bijection from S to S'*, which is a function from S to S' whose reflection is a function from S' to S.

In any category, when there is an isomorphism from C to C', the objects C and C' are said to be *isomorphic*. But saying that C and C' are isomorphic does not single out a particular isomorphism from C to C' when there are more than one; usually, one wishes to emphasize a particular isomorphism called the "canonical" isomorphism.

In any category $\mathcal{C}$, an object $\bot$ is said to be *initial* iff, for all $C \in \mathrm{Ob}\,\mathcal{C}$, there is exactly one morphism in $\bot \underset{\mathcal{C}}{\rightarrow} C$. An object $\top$ is said to be *terminal* iff, for all $C \in \mathrm{Ob}\,\mathcal{C}$, there is exactly one morphism in $C \underset{\mathcal{C}}{\rightarrow} \top$.

Suppose $\bot$ and $\bot'$ are both initial objects in $\mathcal{C}$. Then there is a morphism $\rho \in \bot \underset{\mathcal{C}}{\rightarrow} \bot'$ and another morphism $\rho' \in \bot' \underset{\mathcal{C}}{\rightarrow} \bot$; so that $\rho;\rho' \in \bot \underset{\mathcal{C}}{\rightarrow} \bot$. But $I_\bot$ also belongs to $\bot \underset{\mathcal{C}}{\rightarrow} \bot$ and, since $\bot$ is initial, $\bot \underset{\mathcal{C}}{\rightarrow} \bot$ contains exactly one member. Thus $\rho\,;\rho' = I_\bot$, and by a symmetric argument, $\rho'\,;\rho = I_{\bot'}$, so that $\bot$ and $\bot'$ are isomorphic.

By a similar (actually dual) argument, terminal objects are also isomorphic to one another.

In REL, TREL, PFUN, and SET, the empty set is the unique initial element. In REL and PFUN the empty set is also the unique terminal element, while in TREL and SET the terminal elements are the singleton (one-element) sets.

The reader should not gain the impression that every useful category has sets as objects and relations as morphisms, or even that every useful category is large. To provide some drastically different examples, we digress to consider certain kinds of relations *on* sets.

A relation ρ is said to be *transitive* iff $\rho\,;\rho \subseteq \rho$, *symmetric* iff $\rho^\dagger = \rho$, and *antisymmetric* iff $\rho \cap \rho^\dagger$ is an identity function. Thus a relation ρ on S is antisymmetric iff $\rho \cap \rho^\dagger \subseteq I_S$. Finally, a relation ρ on S is *reflexive on S* (a morphic concept) iff $\rho \supseteq I_S$.

A relation that is reflexive on S and transitive is a *preorder on S*; if it is also antisymmetric it is a *partial order on S*, while if it is also symmetric it is an *equivalence relation on S*.

A set S equipped with a preorder (partial order) is called a *preordered* (*partially ordered*) set. Partially ordered sets are also called *posets*.

Now suppose P equipped with $\preceq$ is a preordered set. Then we can define a category $\mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{C} = P$ and $\mathrm{Mor}\,\mathcal{C}$ is a singleton set, say $\{\langle\rangle\}$, with

$$x \underset{\mathcal{C}}{\to} x' = \begin{cases} \{\langle\rangle\} & \text{if } x \preceq x' \\ \{\} & \text{otherwise.} \end{cases}$$

Obviously, $\langle\rangle\,;\langle\rangle$ must be $\langle\rangle$, $I_x^{\mathcal{C}}$ must be $\langle\rangle$, and the associativity and identity laws will be trivially true. But the transitivity of $\preceq$ is needed to insure that if $\langle\rangle \in x \to x'$ and $\langle\rangle \in x' \to x''$ then $\langle\rangle \,;_{\mathcal{C}}\, \langle\rangle \in x \to x''$, and reflexivity is needed for $I_x^{\mathcal{C}} \in x \to x$.

We will often identify a preordered set with the category which is constructed in this way. In such a category, two objects x and x' will be isomorphic iff $x \preceq x'$ and $x' \preceq x$. Thus if $\preceq$ is a partial order, isomorphic objects will be equal (such a category is called *skeletal*). An object $\perp$ will be initial iff $(\forall x \in P)\ \perp \preceq x$, i.e. if $\perp$ is a least element of P. Similarly, $\top$ will be terminal iff it is a greatest element of P.

Any set S can be equipped with the identity relation on S (which is both a partial order and an equivalence relation on S); it is then said to be *discretely ordered*. The corresponding category will possess only identity morphisms; such a category is called a *discrete* category.

1.4 Functors

A *functor* F from a category $\mathcal{C}$ to a category $\mathcal{C}'$ consists of:

- A function $F_{\mathrm{ob}} \in \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}'$, called the *object part* of F,

- For C, $C' \in \mathrm{Ob}\,\mathcal{C}$, functions $F_{CC'} \in (C \underset{\mathcal{C}}{\to} C') \to (F_{\mathrm{ob}}C \underset{\mathcal{C}'}{\to} F_{\mathrm{ob}}C')$, called the *morphism parts* of F,

that satisfy the *functor laws*:

- For $\rho \in C \underset{\mathcal{C}}{\to} C'$ and $\rho' \in C' \underset{\mathcal{C}}{\to} C''$,

$$F_{CC''}(\rho \,;_{\mathcal{C}}\, \rho') = (F_{CC'}\rho) \,;_{\mathcal{C}'}\, (F_{C'C''}\rho'),$$

- For $C \in \mathrm{Ob}\,\mathcal{C}$,

$$F_{CC}(I_C^{\mathcal{C}}) = I_{F_{\mathrm{ob}}C}^{\mathcal{C}'}.$$

In using functors we will often omit the subscript "ob" and the subscripts on morphism parts when they are evident from context.

An example is the functor F from SET to SET such that

$$F_{\mathrm{ob}} S = S \times S \quad \text{and} \quad F_{SS'}\rho = \lambda\langle\, x \in S,\ y \in S\,\rangle.\ \langle \rho x, \rho y\rangle.$$

Since functors "go between" categories in much the same way that functions go between sets, one would expect that categories and functors should form a category. However, it would be paradoxical for a category to be an object of itself. In fact, categories and functors form a "large" category, which is an entity satisfying all of the definition of a category except the constraints on its size. Specifically, the large category CAT is such that:

- $\mathrm{Ob}\,\mathrm{CAT}$ is the collection of all categories.
- $\mathcal{C} \xrightarrow[\mathrm{CAT}]{} \mathcal{C}'$ is the collection of all functors from $\mathcal{C}$ to $\mathcal{C}'$.
- If $F \in \mathcal{C} \xrightarrow[\mathrm{CAT}]{} \mathcal{C}'$ and $F' \in \mathcal{C}' \xrightarrow[\mathrm{CAT}]{} \mathcal{C}''$ then
$$(F\,;_{\mathrm{CAT}}\,F')_{\mathrm{ob}} = F_{\mathrm{ob}}\,;\,F'_{\mathrm{ob}},$$
$$(F\,;_{\mathrm{CAT}}\,F')_{CC'} = F_{CC'}\,;\,F'_{F_{\mathrm{ob}}C, F_{\mathrm{ob}}C'}.$$
- If $\mathcal{C}$ is a category then
$$(I_{\mathcal{C}}^{\mathrm{CAT}})_{\mathrm{ob}} = I_{\mathrm{Ob}\,\mathcal{C}},$$
$$(I_{\mathcal{C}}^{\mathrm{CAT}})_{CC'} = I_{C \xrightarrow[\mathcal{C}]{} C'}.$$

The reader may verify that composition is associative and that the identity laws hold.

1.5 Preorders, Monotone Functions, and Categories

Let P be a set equipped with a preorder $\preceq$ and P' be a set equipped with a preorder $\preceq'$. A function $f \in P \to P'$ is said to be a *monotone function from P to P'* iff $(\forall x_1, x_2 \in P)\ x_1 \preceq x_2$ implies $f x_1 \preceq' f x_2$.

If F is a monotone function from P to P' and f' is a monotone function from P' to P'', then $f\,;\,f'$ is a monotone function from P to P''. Moreover, I_P is a monotone function from P to P. Thus there is a category, which we will call PREO, such that

- Ob(PREO) is the class of preordered sets,
- $P \xrightarrow[\mathrm{PREO}]{} P'$ is the set of monotone functions from P to P',
- $f\,;_{\mathrm{PREO}}\,f' = f\,;f'$,
- $I_P^{\mathrm{PREO}} = I_P$.

There is also a full subcategory of PREO, obtained by reducing the collection of objects to the class of partially ordered sets, which we will call PARO.

We have already seen a quite different connection between preorders and categories, where a preordered set P is itself viewed as a category, with $\mathrm{Ob}\,P = P$, $\mathrm{Mor}\,P = \{\langle\rangle\}$, and

$$x \xrightarrow[P]{} x' = \begin{cases} \{\langle\rangle\} & \text{if } x \preceq x' \\ \{\} & \text{otherwise.} \end{cases}$$

Suppose P and P' are preordered sets viewed as categories, and F is a functor from P to P'. Since the morphism part $F_{xx'}$ has domain $x \xrightarrow[P]{} x'$, it must be

$$F_{xx'} = \begin{cases} \lambda\langle\rangle.\ \langle\rangle & \text{if } x \preceq x' \\ \{\} & \text{otherwise.} \end{cases}$$

But in order for this function to belong to the required set

$$F_{xx'} \in (x \xrightarrow[P]{} x') \to (Fx \xrightarrow[P']{} Fx'),$$

the set $Fx \xrightarrow[P']{} Fx'$ must be nonempty whenever $x \xrightarrow[P]{} x'$ is nonempty, i.e. $Fx \preceq' Fx'$ must hold whenever $x \preceq x'$. Thus the object part of F must be a monotone function from P to P'.

On the other hand, any monotone function from P to P' can be taken as the object part of a functor, whose morphism parts are defined as above (in which case it is obvious that the functor laws are satisfied). Thus there is a one-to-one correspondence between monotone functions and functors from P to P', so that we can view a monotone function as the unique functor of which it is the object part.

In fact, when we identify preordered sets with categories we will also identify monotone functions with functors, so that PREO is a full subcategory of CAT.

1.6 Products

Suppose $\mathcal{C}$ is a category, J is a set (often called the *index set*), and $\bar{C} \in (\mathrm{Ob}\,\mathcal{C})^J$. Then a *product over J of $\bar{C}$ in $\mathcal{C}$* consists of:

- An object $C \in \mathrm{Ob}\,\mathcal{C}$,
- A function $\bar{\rho} \in \prod_{i \in J}(C \underset{\mathcal{C}}{\to} \bar{C}i)$, mapping each $i \in J$ into a morphism from C to $\bar{C}i$,

such that

- For all $D \in \mathrm{Ob}\,\mathcal{C}$ and $\bar{\alpha} \in \prod_{i \in J}(D \underset{\mathcal{C}}{\to} \bar{C}i)$, there is a unique $\beta \in D \underset{\mathcal{C}}{\to} C$ such that the diagram

$$\begin{tikzcd}
D \arrow[d, "\beta"'] \arrow[dr, "\bar{\alpha} i"] & \\
C \arrow[r, "\bar{\rho} i"] & \bar{C} i
\end{tikzcd}$$

  commutes in $\mathcal{C}$ for all $i \in J$.

One sometimes speaks of C by itself as a product over J of $\bar{C}$, calling each morphism $\bar{\rho}i$ the *projection morphism from C to $\bar{C}i$.*

Suppose $C, \bar{\rho}$ and $C', \bar{\rho}'$ are both products over J of $\bar{C}$. Since $C, \bar{\rho}$ is a product, there is a (unique) $\beta \in C' \underset{\mathcal{C}}{\to} C$ such that the lower triangle in

$$\begin{tikzcd}
C \arrow[d, "\beta'"'] \arrow[ddr, "\bar{\rho} i"] & \\
C' \arrow[d, "\beta"'] \arrow[dr, "\bar{\rho}' i"] & \\
C \arrow[r, "\bar{\rho} i"] & \bar{C} i
\end{tikzcd}$$

commutes for all $i \in J$, and since $C', \bar{\rho}'$ is a product, there is a (unique) $\beta' \in C \underset{\mathcal{C}}{\to} C'$ such that the upper triangle commutes for all $i \in J$. Putting these triangles together, we find that the outer triangle in

$$\begin{tikzcd}
C \arrow[d, shift right, "\beta' ; \beta"'] \arrow[d, shift left, "I^{\mathcal{C}}_{C}"] \arrow[dr, "\bar{\rho}i"] & \\
C \arrow[r, "\bar{\rho}i"] & \bar{C}i
\end{tikzcd}$$

commutes for all $i \in J$. But the inner triangle obviously commutes, and since $C, \bar{\rho}$ is a product, the morphism from C to C that makes these two triangles commute must be unique. Thus $\beta' ; \beta = I^{\mathcal{C}}_{C}$.

By a similar argument, putting the diagrams together in the reverse order, $\beta ; \beta' = I^{\mathcal{C}}_{C'}$. Thus, β is an isomorphism from C' to C such that $\bar{\rho}'i = \beta ; \bar{\rho}i$ and $\bar{\rho}i = \beta^{-1} ; \bar{\rho}'i$ for all $i \in J$. In brief, any two products over J of $\bar{C}$ are isomorphic.

On the other hand, suppose $C, \bar{\rho}$ is a product over J of $\bar{C}$, and $\phi \in C' \underset{\mathcal{C}}{\rightarrow} C$ is an isomorphism. For any $D \in \mathrm{Ob}\, \mathcal{C}$, $\bar{\alpha} \in \prod_{i \in J}(D \underset{\mathcal{C}}{\rightarrow} \bar{C}i)$, and $\beta' \in D \underset{\mathcal{C}}{\rightarrow} C'$,

$$\begin{tikzcd}
D \arrow[d, "\beta'"'] \arrow[ddrr, "\bar{\alpha}i"] & & \\
C' \arrow[dr, "\phi"'] & & \\
& C \arrow[r, "\bar{\rho}i"'] & \bar{C}i
\end{tikzcd}$$

will commute for all $i \in J$ iff $\beta' ; \phi = \beta$, where β is the unique morphism that makes

$$\begin{tikzcd}
D \arrow[d, "\beta"'] \arrow[dr, "\bar{\alpha}i"] & \\
C \arrow[r, "\bar{\rho}i"] & \bar{C}i
\end{tikzcd}$$

commute for all $i \in J$. Then the assumption that ϕ is an isomorphism determines β' uniquely (to be $\beta ; \phi^{-1}$). Thus $C', \lambda i \in J.\ \phi ; \bar{\rho} i$ is also a product over J of $\bar{C}$.

Finally, consider the special case of a product over the empty set. Then $\bar{C}$ is the empty function, there are no projection morphisms, and there are no commutative diagrams to be satisfied by β. Thus a product over the empty set in $\mathcal{C}$ is simply an object $C \in \mathrm{Ob}\,\mathcal{C}$ such that, for all $D \in \mathrm{Ob}\,\mathcal{C}$, there is a unique morphism $\beta \in D \xrightarrow[\mathcal{C}]{} C$. In other words, a product over the empty set is a terminal element.

1.7 Distinguished Products

In the category SET, the Cartesian product is a particular or "distinguished" product. Specifically, when J is a set and $\bar{C} \in (\mathrm{Ob}\,\mathrm{SET})^J$, we define $\prod \bar{C}$ to be the set

$$\prod \bar{C} \stackrel{\text{def}}{=} \{\bar{x} \mid \operatorname{dom} \bar{x} = J \text{ and } (\forall i \in J)\ \bar{x} i \in \bar{C} i\}$$

and $\mathsf{P}\bar{C}$ to be the function

$$\mathsf{P}\bar{C} \in \prod_{i \in J} \left(\prod \bar{C} \to \bar{C} i\right)$$

such that

$$\mathsf{P}\bar{C} \stackrel{\text{def}}{=} \lambda i \in J.\ \lambda \bar{x} \in \prod \bar{C}.\ \bar{x} i.$$

Then $\prod \bar{C}$, $\mathsf{P}\bar{C}$ is a product over J of $\bar{C}$.

To see this, suppose D is a set and $\bar{\alpha} \in \prod_{i \in J}(D \to \bar{C} i)$. Then a function $\beta \in D \to \prod \bar{C}$ will make the diagram

$$\begin{tikzcd}
D \arrow[d, "\beta"'] \arrow[dr, "\bar{\alpha} i"] & \\
\prod \bar{C} \arrow[r, "\mathsf{P}\bar{C} i"] & \bar{C} i
\end{tikzcd}$$

commute for all $i \in J$ iff

$$(\forall i \in J)(\forall y \in D)\ \mathsf{P}\bar{C} i(\beta y) = \bar{\alpha} i y,$$

or equivalently

$$(\forall i \in J)(\forall y \in D)\ \beta y i = \bar{\alpha} i y,$$

which uniquely determines β to be the function

$$\beta = \lambda y \in D.\ \lambda i \in J.\ \bar{\alpha} i y.$$

This unique morphism is important enought to merit a special notation. We define, for $\bar{\alpha} \in \prod_{i \in J}(D \to \bar{C}i)$,

$$\prod \bar{\alpha} \stackrel{\text{def}}{=} \lambda y \in D.\ \lambda i \in J.\ \bar{\alpha} i y.$$

(Strictly speaking, we should write $\prod^{D}_{\bar{C}}$, but we will elide D and $\bar{C}$, which should be obvious from context.)

For many other categories, we will distinguish particular products over all sets, or at least all finite sets, and denote such products by a notation similar to that for the Cartesian product. Specifically, a *category with distinguished (finite) products* is a category $\mathcal{C}$ equipped with functions

$$\prod\nolimits^{\mathcal{C}}_{J} \in (\mathrm{Ob}\,\mathcal{C})^{J} \to \mathrm{Ob}\,\mathcal{C},$$

$$\mathsf{P}^{\mathcal{C}}_{J} \in \prod_{\bar{C} \in (\mathrm{Ob}\,\mathcal{C})^{J}} \Big(\prod_{i \in J}\big(\prod\nolimits^{\mathcal{C}}_{J} \bar{C} \underset{\mathcal{C}}{\to} \bar{C}i\big)\Big)$$

for each (finite) set J, such that $\prod^{\mathcal{C}}_{J} \bar{C}$, $\mathsf{P}^{\mathcal{C}}_{J}\bar{C}$ is a product over J of $\bar{C}$ for all $\bar{C} \in (\mathrm{Ob}\,\mathcal{C})^{J}$. When $D \in \mathrm{Ob}\,\mathcal{C}$ and $\bar{\alpha} \in \prod_{i \in J}(D \underset{\mathcal{C}}{\to} \bar{C}i)$, we write $\prod^{\mathcal{C}} \bar{\alpha}$ for the unique morphism from D to $\prod^{\mathcal{C}}_{J} \bar{C}$ such that

$$\begin{tikzcd}
D \arrow[d, "\prod^{\mathcal{C}} \bar{\alpha}"'] \arrow[dr, "\bar{\alpha} i"] & \\
\prod^{\mathcal{C}}_{J} \bar{C} \arrow[r, "\mathsf{P}^{\mathcal{C}}_{J}\bar{C}i"] & \bar{C}i
\end{tikzcd}$$

commutes in $\mathcal{C}$ for all $i \in J$.

Just as with the Cartesian product, it is useful to develop more specialized notations. In applications such as $\prod^{\mathcal{C}}_J \bar{C}$ and $\mathsf{P}^{\mathcal{C}}_J \bar{C}$, we will usually omit the subscript J, since it must denote the domain of $\bar{C}$. Then we define

$$\prod_{i \in J}^{\mathcal{C}} C \stackrel{\text{def}}{=} \prod^{\mathcal{C}} \lambda i \in J.\ C,$$

$$\textcircled{$\prod$}_{i \in J}^{\mathcal{C}} \alpha \stackrel{\text{def}}{=} \textcircled{$\prod$}^{\mathcal{C}} \lambda i \in J.\ \alpha,$$

$$\prod_{i=a}^{b}{}^{\mathcal{C}} C \stackrel{\text{def}}{=} \prod_{i \in a \text{ to } b}^{\mathcal{C}} C,$$

$$\textcircled{$\prod$}_{i=a}^{b}{}^{\mathcal{C}} \alpha \stackrel{\text{def}}{=} \textcircled{$\prod$}_{i \in a \text{ to } b}^{\mathcal{C}} \alpha,$$

$$C^J_{\mathcal{C}} \stackrel{\text{def}}{=} \prod_{i \in J}^{\mathcal{C}} C \text{ where } i \text{ does not occur in } C,$$

$$\mathsf{P}[C^J_{\mathcal{C}}] \stackrel{\text{def}}{=} \mathsf{P}^{\mathcal{C}}(\lambda i \in J.\ C) \text{ where } i \text{ does not occur in } C,$$

$$C_1 \times_{\mathcal{C}} \cdots \times_{\mathcal{C}} C_n \stackrel{\text{def}}{=} \prod_{i=1}^{n}{}^{\mathcal{C}} \begin{cases} C_1 & \text{if } i = 1 \\ \vdots & \\ C_n & \text{if } i = n, \end{cases}$$

$$\mathsf{p}_i[C_1 \times_{\mathcal{C}} \cdots \times_{\mathcal{C}} C_n] \stackrel{\text{def}}{=} \mathsf{P}^{\mathcal{C}}\left(\lambda i \in 1 \text{ to } n.\ \left\{ \begin{array}{ll} C_1 & \text{if } i = 1 \\ \vdots & \\ C_n & \text{if } i = n \end{array} \right\}\right) i,$$

$$\alpha_1 \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} \alpha_n \stackrel{\text{def}}{=} \textcircled{$\prod$}_{i=1}^{n}{}^{\mathcal{C}} \begin{cases} \alpha_1 & \text{if } i = 1 \\ \vdots & \\ \alpha_n & \text{if } i = n. \end{cases}$$

Thus, in the simple case of a *binary* product, $\alpha_1 \otimes_{\mathcal{C}} \alpha_2$ is the unique morphism from D to $C_1 \times_{\mathcal{C}} C_2$ such that both triangles in

$$\begin{tikzcd}
 & D \arrow[dl, "\alpha_1"'] \arrow[d, "\alpha_1 \otimes \alpha_2"'] \arrow[dr, "\alpha_2"] & \\
C_1 & C_1 \times C_2 \arrow[l, "\mathsf{p}_1{[C_1 \times C_2]}"'] \arrow[r, "\mathsf{p}_2{[C_1 \times C_2]}"] & C_2
\end{tikzcd}$$

commute.

Note that

$$C^n_{\mathcal{C}} = C^{1\,\mathrm{to}\,n}_{\mathcal{C}} = \underbrace{C \times_{\mathcal{C}} \cdots \times_{\mathcal{C}} C}_{n\text{ times}},$$

and

$$\mathsf{P}[C^n_{\mathcal{C}}]i = \mathsf{P}[C^{1\,\mathrm{to}\,n}_{\mathcal{C}}]i = \mathsf{p}_i[\underbrace{C \times_{\mathcal{C}} \cdots \times_{\mathcal{C}} C}_{n\text{ times}}].$$

The following proposition establishes two basic properties of the $\textcircled{\prod}$ operation that hold for any distinguished product:

Proposition 1.1 *In any category $\mathcal{C}$ with a distinguished product over J, if $\bar{C} \in (\mathrm{Ob}\,\mathcal{C})^J$ then*

$$\textcircled{\prod}_{i\in J}^{\mathcal{C}} \mathsf{P}^{\mathcal{C}}\bar{C}i = I^{\mathcal{C}}_{\prod^{\mathcal{C}}\bar{C}},$$

and if $\bar{\alpha} \in \prod_{i\in J}(D \xrightarrow[\mathcal{C}]{} \bar{C}i)$ and $\beta \in D' \xrightarrow[\mathcal{C}]{} D$ then

$$\beta\,;\Big(\textcircled{\prod}_{i\in J}^{\mathcal{C}} \bar{\alpha}i\Big) = \textcircled{\prod}_{i\in J}^{\mathcal{C}}(\beta\,;\bar{\alpha}i).$$

PROOF The first part of the proposition is an immediate consequence of the commutativity of

$$\begin{tikzcd}
\prod^{\mathcal{C}}\bar{C} \arrow[d, "I^{\mathcal{C}}_{\prod^{\mathcal{C}}\bar{C}}"'] \arrow[dr, "\mathsf{P}^{\mathcal{C}}\bar{C}i"] & \\
\prod^{\mathcal{C}}\bar{C} \arrow[r, "\mathsf{P}^{\mathcal{C}}\bar{C}i"] & \bar{C}i
\end{tikzcd}$$

for all $i \in J$. The second part holds since the diagram defining $\textcircled{\prod}$,

$$\begin{tikzcd}
D \arrow[d, "\textcircled{\prod}_{i\in J}^{\mathcal{C}}\bar{\alpha}i"'] \arrow[dr, "\bar{\alpha}i"] & \\
\prod^{\mathcal{C}}\bar{C} \arrow[r, "\mathsf{P}^{\mathcal{C}}\bar{C}i"] & \bar{C}i
\end{tikzcd}$$

implies that

$$\begin{tikzcd}
D' \arrow[d, "\beta\,;\,\left(\textcircled{\scriptsize$\prod$}^{\mathcal{C}}_{i\in J}\bar{\alpha}i\right)"'] \arrow[dr, "\beta\,;\,\bar{\alpha}i"] & \\
\prod^{\mathcal{C}}\bar{C} \arrow[r, "\mathsf{P}^{\mathcal{C}}\bar{C}i"'] & \bar{C}i
\end{tikzcd}$$

commutes for all $i \in J$. END OF PROOF

Notice that, in the case of a binary product, this proposition reduces to

$$\mathsf{p}_1[C_1 \times C_2] \otimes \mathsf{p}_2[C_1 \times C_2] = I^{\mathcal{C}}_{C_1 \times C_2}$$

and

$$\beta\,;(\alpha_1 \otimes \alpha_2) = (\beta\,;\alpha_1) \otimes (\beta\,;\alpha_2).$$

1.8 Distinguished Products In Other Categories

In PREO (and PARO) we distinguish a product that is similar to the Cartesian product of sets, except that the set

$$\prod\nolimits_J^{\mathrm{PREO}} \bar{C} \stackrel{\mathrm{def}}{=} \{\bar{x} \mid \operatorname{dom}\bar{x} = J \text{ and } (\forall i \in J)\ \bar{x}i \in \bar{C}i\}$$

is equipped with a "pointwise" ordering:

$$\bar{x} \preceq_{\prod^{\mathrm{PREO}}\bar{C}} \bar{x}' \text{ iff } (\forall i \in J)\ \bar{x}i \preceq_{\bar{C}i} \bar{x}'i.$$

The proof that this construction gives a product is similar to that for the Cartesian product, except that one must also show:

- If $\preceq_{\bar{C}i}$ is a preorder (partial order) for all $i \in J$, then $\preceq_{\prod^{\mathrm{PREO}}\bar{C}}$ is a preorder (partial order).
- The projection functions $\mathsf{P}\bar{C}i$ are monotone.
- If $\bar{\alpha}i$ is monotone for all $i \in J$ then $\textcircled{\scriptsize$\prod$}\,\bar{\alpha}$ is monotone.

The details are left to the reader.

It is also possible to distinguish a product in CAT. The general construction works for arbitrary index sets, but is notationally complicated; thus we will limit this exposition to the simple binary case where $J = \{1,2\}$. For categories $\mathcal{C}_1$ and $\mathcal{C}_2$, $\mathcal{C}_1 \times_{\mathrm{CAT}} \mathcal{C}_2$ is the category such that

- $\mathrm{Ob}(\mathcal{C}_1 \times_{\mathrm{CAT}} \mathcal{C}_2) = \mathrm{Ob}\,\mathcal{C}_1 \times \mathrm{Ob}\,\mathcal{C}_2$,
- For $\langle C_1, C_2\rangle$, $\langle C_1', C_2'\rangle \in \mathrm{Ob}(\mathcal{C}_1 \times_{\mathrm{CAT}} \mathcal{C}_2)$,
$$\langle C_1, C_2\rangle \xrightarrow[\mathcal{C}_1\times\mathcal{C}_2]{} \langle C_1', C_2'\rangle = (C_1 \xrightarrow[\mathcal{C}_1]{} C_1') \times (C_2 \xrightarrow[\mathcal{C}_2]{} C_2'),$$
- For $\langle \rho_1, \rho_2\rangle \in \langle C_1, C_2\rangle \xrightarrow[\mathcal{C}_1\times\mathcal{C}_2]{} \langle C_1', C_2'\rangle$ and $\langle \rho_1', \rho_2'\rangle \in \langle C_1', C_2'\rangle \xrightarrow[\mathcal{C}_1\times\mathcal{C}_2]{} \langle C_1'', C_2''\rangle$,
$$\langle \rho_1, \rho_2\rangle \,;_{\mathcal{C}_1\times\mathcal{C}_2} \langle \rho_1', \rho_2'\rangle = \langle \rho_1 \,;_{\mathcal{C}_1} \rho_1', \rho_2 \,;_{\mathcal{C}_2} \rho_2'\rangle,$$
- For $\langle C_1, C_2\rangle \in \mathrm{Ob}(\mathcal{C}_1 \times_{\mathrm{CAT}} \mathcal{C}_2)$,
$$I^{\mathcal{C}_1\times\mathcal{C}_2}_{\langle C_1, C_2\rangle} = \langle I^{\mathcal{C}_1}_{C_1}, I^{\mathcal{C}_2}_{C_2}\rangle.$$

(The reader may verify that composition is associative and the identity laws hold.) Then $\mathsf{p}_1[\mathcal{C}_1 \times \mathcal{C}_2]$ and $\mathsf{p}_2[\mathcal{C}_1 \times \mathcal{C}_2]$ are the functors from $\mathcal{C}_1 \times_{\mathrm{CAT}} \mathcal{C}_2$ to $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively such that, for $\langle C_1, C_2\rangle \in \mathrm{Ob}(\mathcal{C}_1 \times_{\mathrm{CAT}} \mathcal{C}_2)$,
$$\mathsf{p}_i[\mathcal{C}_1 \times \mathcal{C}_2]\langle C_1, C_2\rangle = C_i,$$
and for $\langle \rho_1, \rho_2\rangle \in \langle C_1, C_2\rangle \xrightarrow[\mathcal{C}_1\times\mathcal{C}_2]{} \langle C_1', C_2'\rangle$,
$$\mathsf{p}_i[\mathcal{C}_1 \times \mathcal{C}_2]\langle \rho_1, \rho_2\rangle = \rho_i.$$
(The reader may verify the functor laws.)

To see that this construction gives a product, suppose that $\mathcal{D}$ is a category and that α_1 and α_2 are functors from $\mathcal{D}$ to $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. Then a functor β from $\mathcal{D}$ to $\mathcal{C}_1 \times_{\mathrm{CAT}} \mathcal{C}_2$ will make

$$\begin{tikzcd}
& \mathcal{D} \arrow[dl, "\alpha_1"'] \arrow[d, "\beta"'] \arrow[dr, "\alpha_2"] & \\
\mathcal{C}_1 & \mathcal{C}_1 \times \mathcal{C}_2 \arrow[l, "{\mathsf{p}_1[\mathcal{C}_1 \times \mathcal{C}_2]}"'] \arrow[r, "{\mathsf{p}_2[\mathcal{C}_1 \times \mathcal{C}_2]}"] & \mathcal{C}_2
\end{tikzcd}$$

commute iff two conditions hold:

- For all $i \in \{1,2\}$ and $D \in \mathrm{Ob}\,\mathcal{D}$,

  $$\mathsf{p}_i[\mathcal{C}_1 \times \mathcal{C}_2](\beta D) = \alpha_i D,$$

  or equivalently

  $$(\beta D)_i = \alpha_i D,$$

  which uniquely determines the object part of β to be

  $$\beta_{\mathrm{ob}} = \lambda D \in \mathrm{Ob}\,\mathcal{D}.\ \langle \alpha_1 D, \alpha_2 D\rangle.$$

- For all $i \in \{1,2\}$, $D,\ D' \in \mathrm{Ob}\,\mathcal{D}$, and $\rho \in D \xrightarrow[\mathcal{D}]{} D'$,

  $$\mathsf{p}_i[\mathcal{C}_1 \times \mathcal{C}_2](\beta \rho) = \alpha_i \rho,$$

  or equivalently

  $$(\beta\rho)_i = \alpha_i \rho,$$

  which uniquely determines the morphism parts of β to be

  $$\beta_{DD'} = \lambda \rho \in D \xrightarrow[\mathcal{D}]{} D'.\ \langle \alpha_1 \rho, \alpha_2 \rho\rangle.$$

(We leave it to the reader to verify that β satisfies the functor laws since the α_i's satisfy these laws.) Thus $\mathcal{C}_1 \times_{\mathrm{CAT}} \mathcal{C}_2$, $\mathsf{p}_i[\mathcal{C}_1 \times \mathcal{C}_2]$ is a product in CAT, and $\alpha_1 \otimes_{\mathrm{CAT}} \alpha_2$ is the functor such that

$$(\alpha_1 \otimes_{\mathrm{CAT}} \alpha_2)_{\mathrm{ob}} = \lambda D \in \mathrm{Ob}\,\mathcal{D}.\ \langle \alpha_1 D, \alpha_2 D\rangle,$$

$$(\alpha_1 \otimes_{\mathrm{CAT}} \alpha_2)_{DD'} = \lambda \rho \in D \xrightarrow[\mathcal{D}]{} D'.\ \langle \alpha_1 \rho, \alpha_2 \rho\rangle.$$

When $\mathcal{C}_1$ and $\mathcal{C}_2$ are preorders viewed as categories, one can form two product categories, $\mathcal{C}_1 \times_{\mathrm{CAT}} \mathcal{C}_2$ and $\mathcal{C}_1 \times_{\mathrm{PREO}} \mathcal{C}_2$. The only difference between these isomorphic categories is that the unique morphism in $\mathcal{C}_1 \times_{\mathrm{CAT}} \mathcal{C}_2$ is $\langle\langle\rangle, \langle\rangle\rangle$, instead of the unique morphism $\langle\rangle$ in $\mathcal{C}_1 \times_{\mathrm{PREO}} \mathcal{C}_2$. We will often ignore this difference and regard the two product categories as identical.

When preorders are viewed as categories, these categories themselves may possess products. In this case, all morphisms are equal to the unique morphism $\langle\rangle$. Thus all diagrams commute, so that the definition of a product reduces to an assertion that certain morphism sets are nonempty. Specifically, a product over J of $\bar{x}$ in a preorder P is an object $x \in P$ such that

$\langle\rangle \in x \xrightarrow[P]{} \bar{x}i$ for all $i \in J$ and, whenever y satisfies $\langle\rangle \in y \xrightarrow[P]{} \bar{x}i$ for all $i \in J$, $\langle\rangle \in y \xrightarrow[P]{} x$. In other words, $x \preceq \bar{x}i$ for all $i \in J$ and, whenever $y \preceq \bar{x}i$ for all $i \in J$, $y \preceq x$. Thus, x is a greatest lower bound of the set $\{\bar{x}i \mid i \in J\}$.

For an arbitrary preorder, there is no guarantee that products either exist or are unique, and there is no reason to distinguish a product. For a partial order, however, since products are isomorphic and isomorphic objects are equal, there is at most one product over J of $\bar{x}$, so that we must define $\prod_J^P \bar{x}$ to be *the* greatest lower bound of $\{\bar{x}i \mid i \in J\}$. A partial order in which all products exist is a partial order in which greatest lower bounds of all subsets exist, which is called a *complete lattice*.

1.9 Product Functors

Let $\mathcal{C}$ be some category with a distinguished binary product $\times_{\mathcal{C}}$, which will be a function from $\mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{C}$ to $\mathrm{Ob}\,\mathcal{C}$. From the development in the previous section of a binary product in CAT, we have $\mathrm{Ob}(\mathcal{C} \times_{\mathrm{CAT}} \mathcal{C}) = \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{C}$, so that $\times_{\mathcal{C}}$ is a function from $\mathrm{Ob}(\mathcal{C} \times_{\mathrm{CAT}} \mathcal{C})$ to $\mathrm{Ob}\,\mathcal{C}$. This suggests that $\times_{\mathcal{C}}$ might be the object part of a functor from $\mathcal{C} \times_{\mathrm{CAT}} \mathcal{C}$ to $\mathcal{C}$, i.e. a bifunctor from $\mathcal{C}$ and $\mathcal{C}$ to $\mathcal{C}$. (In general a functor from $\mathcal{C}_1 \times \mathcal{C}_2$ to $\mathcal{C}_3$ is called a bifunctor from $\mathcal{C}_1$ and $\mathcal{C}_2$ to $\mathcal{C}_3$.)

To see that this is the case we will define the morphism parts of this functor. (As in the previous section, this construction works for products over arbitrary index sets, but we limit our exposition to the binary case for notational simplicity.) For $\langle C_1, C_2\rangle, \langle C_1', C_2'\rangle \in \mathrm{Ob}(\mathcal{C} \times_{\mathrm{CAT}} \mathcal{C})$, we define

$$(\times_{\mathcal{C}})_{\langle C_1,C_2\rangle,\langle C_1',C_2'\rangle} \in (\langle C_1, C_2\rangle \xrightarrow[\mathcal{C}\times\mathcal{C}]{} \langle C_1', C_2'\rangle) \to (C_1 \times_{\mathcal{C}} C_2 \xrightarrow[\mathcal{C}]{} C_1' \times_{\mathcal{C}} C_2')$$

$$= (C_1 \xrightarrow[\mathcal{C}]{} C_1') \times (C_2 \xrightarrow[\mathcal{C}]{} C_2') \to (C_1 \times_{\mathcal{C}} C_2 \xrightarrow[\mathcal{C}]{} C_1' \times_{\mathcal{C}} C_2')$$

to be the functor such that

$$(\times_{\mathcal{C}})_{\langle C_1,C_2\rangle,\langle C_1',C_2'\rangle} \stackrel{\mathrm{def}}{=} \lambda\langle\, \rho_1 \in C_1 \xrightarrow[\mathcal{C}]{} C_1',\ \rho_2 \in C_2 \xrightarrow[\mathcal{C}]{} C_2'\,\rangle.$$
$$(\mathsf{p}_1[C_1 \times C_2]\,;\rho_1) \otimes_{\mathcal{C}} (\mathsf{p}_2[C_1 \times C_2]\,;\rho_2)$$

or, less formally,

$$\rho_1 \times_{\mathcal{C}} \rho_2 \stackrel{\mathrm{def}}{=} (\mathsf{p}_1[C_1 \times C_2]\,;\rho_1) \otimes_{\mathcal{C}} (\mathsf{p}_2[C_1 \times C_2]\,;\rho_2).$$

The fact that this definition obeys the composition law for functors follows from the second part of Proposition 1.1 and the definition of $\otimes$: If $\rho_1 \in C_1 \xrightarrow[\mathcal{C}]{} C_1'$, $\rho_2 \in C_2 \xrightarrow[\mathcal{C}]{} C_2'$, $\rho_1' \in C_1' \xrightarrow[\mathcal{C}]{} C_1''$, and $\rho_2' \in C_2' \xrightarrow[\mathcal{C}]{} C_2''$, then

$$(\rho_1 \times \rho_2) \mathbin{;} (\rho_1' \times \rho_2')$$

$$= \big((\mathsf{p}_1[C_1 \times C_2] \mathbin{;} \rho_1) \otimes (\mathsf{p}_2[C_1 \times C_2] \mathbin{;} \rho_2)\big) \mathbin{;} \big((\mathsf{p}_1[C_1' \times C_2'] \mathbin{;} \rho_1') \otimes (\mathsf{p}_2[C_1' \times C_2'] \mathbin{;} \rho_2')\big)$$

$$= \Big(\big(((\mathsf{p}_1[C_1 \times C_2] \mathbin{;} \rho_1) \otimes (\mathsf{p}_2[C_1 \times C_2] \mathbin{;} \rho_2)) \mathbin{;} \mathsf{p}_1[C_1' \times C_2'] \mathbin{;} \rho_1'\big)$$

$$\otimes \big(((\mathsf{p}_1[C_1 \times C_2] \mathbin{;} \rho_1) \otimes (\mathsf{p}_2[C_1 \times C_2] \mathbin{;} \rho_2)) \mathbin{;} \mathsf{p}_2[C_1' \times C_2'] \mathbin{;} \rho_2'\big)\Big)$$

$$= (\mathsf{p}_1[C_1 \times C_2] \mathbin{;} \rho_1 \mathbin{;} \rho_1') \otimes (\mathsf{p}_2[C_1 \times C_2] \mathbin{;} \rho_2 \mathbin{;} \rho_2')$$

$$= (\rho_1 \mathbin{;} \rho_1') \times (\rho_2 \mathbin{;} \rho_2').$$

The identity law for functors follows from the first part of Proposition 1.1:

$$I^{\mathcal{C}}_{C_1} \times I^{\mathcal{C}}_{C_2} = (\mathsf{p}_1[C_1 \times C_2] \mathbin{;} I^{\mathcal{C}}_{C_1}) \otimes (\mathsf{p}_2[C_1 \times C_2] \mathbin{;} I^{\mathcal{C}}_{C_2})$$

$$= \mathsf{p}_1[C_1 \times C_2] \otimes \mathsf{p}_2[C_1 \times C_2] = I^{\mathcal{C}}_{C_1 \times C_2}.$$

An additional consequence of the second part of Proposition 1.1 is:

Proposition 1.2 *In any category $\mathcal{C}$ with a distinguished binary product, if C_1, C_2, C_1', C_2', and D are objects of $\mathcal{C}$, and $\alpha_1 \in D \xrightarrow[\mathcal{C}]{} C_1$, $\alpha_2 \in D \xrightarrow[\mathcal{C}]{} C_2$, $\rho_1 \in C_1 \xrightarrow[\mathcal{C}]{} C_1'$, and $\rho_2 \in C_2 \xrightarrow[\mathcal{C}]{} C_2'$, then*

$$(\alpha_1 \otimes \alpha_2) \mathbin{;} (\rho_1 \times \rho_2) = (\alpha_1 \mathbin{;} \rho_1) \otimes (\alpha_2 \mathbin{;} \rho_2).$$

PROOF From the second part of Proposition 1.1 and the definition of $\otimes$:

$$(\alpha_1 \otimes \alpha_2) \mathbin{;} (\rho_1 \times \rho_2) = (\alpha_1 \otimes \alpha_2) \mathbin{;} \big((\mathsf{p}_1[C_1 \times C_2] \mathbin{;} \rho_1) \otimes (\mathsf{p}_2[C_1 \times C_2] \mathbin{;} \rho_2)\big)$$

$$= \big((\alpha_1 \otimes \alpha_2) \mathbin{;} \mathsf{p}_1[C_1 \times C_2] \mathbin{;} \rho_1\big) \otimes \big((\alpha_1 \otimes \alpha_2) \mathbin{;} \mathsf{p}_2[C_1 \times C_2] \mathbin{;} \rho_2\big) = (\alpha_1 \mathbin{;} \rho_1) \otimes (\alpha_2 \mathbin{;} \rho_2).$$

END OF PROOF

1.10 Natural Transformations

Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories, and F and F' are functors from $\mathcal{C}$ to $\mathcal{D}$. Then a *natural transformation from F to F'* is a function

$$\eta \in \prod_{C \in \mathrm{Ob}\,\mathcal{C}} (FC \xrightarrow[\mathcal{D}]{} F'C)$$

that satisfies the following *naturality condition*: For all C, $C' \in \mathrm{Ob}\,\mathcal{C}$ and $\rho \in C \xrightarrow[\mathcal{C}]{} C'$, the diagram

$$\begin{tikzcd}
FC \arrow[r, "\eta C"] \arrow[d, "F\rho"] & F'C \arrow[d, "F'\rho"] \\
FC' \arrow[r, "\eta C'"] & F'C'
\end{tikzcd}$$

commutes in $\mathcal{D}$. We write $F \xrightarrow[\mathcal{C} \Rightarrow \mathcal{D}]{} F'$ for the class of natural transformations from F to F'.

As suggested by this notation, natural transformations are morphisms in a category. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then $\mathcal{C} \Rightarrow \mathcal{D}$ is the category such that

- $\mathrm{Ob}(\mathcal{C} \Rightarrow \mathcal{D})$ is the class of functors from $\mathcal{C}$ to $\mathcal{D}$,

- $F \xrightarrow[\mathcal{C} \Rightarrow \mathcal{D}]{} F'$ is the class of natural transformations from F to F',

- If $\eta \in F \xrightarrow[\mathcal{C} \Rightarrow \mathcal{D}]{} F'$ and $\eta' \in F' \xrightarrow[\mathcal{C} \Rightarrow \mathcal{D}]{} F''$ then

$$\eta \,;_{\mathcal{C} \Rightarrow \mathcal{D}}\, \eta' \;=\; \lambda C \in \mathrm{Ob}\,\mathcal{C}.\ \eta C \,;_{\mathcal{D}}\, \eta' C,$$

- If $F \in \mathrm{Ob}(\mathcal{C} \Rightarrow \mathcal{D})$ then

$$I_F^{\mathcal{C} \Rightarrow \mathcal{D}} \;=\; \lambda C \in \mathrm{Ob}\,\mathcal{C}.\ I_{FC}^{\mathcal{D}}.$$

The reader may verify that $\eta;_{\mathcal{C} \Rightarrow \mathcal{D}}\eta'$ and $I_F^{\mathcal{C} \Rightarrow \mathcal{D}}$ satisfy the naturality condition, and that composition is associative and the identity laws hold.

There are also two ways to compose functors with natural transformations, both called *Godement multiplication*: If $F \in \mathcal{C} \xrightarrow[\mathrm{CAT}]{} \mathcal{D}$, G, $G' \in \mathcal{D} \xrightarrow[\mathrm{CAT}]{} \mathcal{E}$, and $\eta \in G \xrightarrow[\mathcal{D}\Rightarrow\mathcal{E}]{} G'$, then

$$F \mathbin{\boxed{;}} \eta \in F \,;\, G \xrightarrow[\mathcal{C}\Rightarrow\mathcal{E}]{} F \,;\, G'$$

is the natural transformation such that

$$(F \mathbin{\boxed{;}} \eta)C \;=\; \eta(FC).$$

If F, $F' \in \mathcal{C} \xrightarrow[\mathrm{CAT}]{} \mathcal{D}$, $\eta \in F \xrightarrow[\mathcal{C}\Rightarrow\mathcal{D}]{} F'$, and $G \in \mathcal{D} \xrightarrow[\mathrm{CAT}]{} \mathcal{E}$, then

$$\eta \mathbin{\boxed{;}} G \in F \,;\, G \xrightarrow[\mathcal{C}\Rightarrow\mathcal{E}]{} F' \,;\, G$$

is the natural transformation such that

$$(\eta \mathbin{\boxed{;}} G)C \;=\; G(\eta C).$$

The reader may verify the naturality condition for both cases, as well as the following *Godement rules*: If

$$\mathcal{A} \xrightarrow{D} \mathcal{B} \xrightarrow{E} \mathcal{C} \xrightarrow{F,\,F',\,F''} \mathcal{D} \xrightarrow{G} \mathcal{E} \xrightarrow{H} \mathcal{F}$$

is a diagram of categories and functors, $\eta \in F \xrightarrow[\mathcal{C}\Rightarrow\mathcal{D}]{} F'$, and $\eta' \in F' \xrightarrow[\mathcal{C}\Rightarrow\mathcal{D}]{} F''$, then

$$(D \,;\, E) \mathbin{\boxed{;}} \eta = D \mathbin{\boxed{;}} (E \mathbin{\boxed{;}} \eta),$$

$$(E \mathbin{\boxed{;}} \eta) \mathbin{\boxed{;}} G = E \mathbin{\boxed{;}} (\eta \mathbin{\boxed{;}} G),$$

$$(\eta \mathbin{\boxed{;}} G) \mathbin{\boxed{;}} H = \eta \mathbin{\boxed{;}} (G \,;\, H),$$

$$E \mathbin{\boxed{;}} (\eta \,;_{\mathcal{C}\Rightarrow\mathcal{D}}\, \eta') = (E \mathbin{\boxed{;}} \eta) \,;_{\mathcal{B}\Rightarrow\mathcal{D}}\, (E \mathbin{\boxed{;}} \eta'),$$

$$(\eta \,;_{\mathcal{C}\Rightarrow\mathcal{D}}\, \eta') \mathbin{\boxed{;}} G = (\eta \mathbin{\boxed{;}} G) \,;_{\mathcal{C}\Rightarrow\mathcal{E}}\, (\eta' \mathbin{\boxed{;}} G).$$

1.11 Duality

For any category $\mathcal{C}$, the *dual* of $\mathcal{C}$, written $\mathcal{C}^{\mathrm{op}}$, is the category such that

- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}\,\mathcal{C}$,

- For C, $C' \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$,

$$C \xrightarrow[\mathcal{C}^{\mathrm{op}}]{} C' = C' \xrightarrow[\mathcal{C}]{} C,$$

- For C, C', $C'' \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, $\rho \in C \xrightarrow[\mathcal{C}^{\mathrm{op}}]{} C'$, and $\rho' \in C' \xrightarrow[\mathcal{C}^{\mathrm{op}}]{} C''$,

$$\rho \;;_{\mathcal{C}^{\mathrm{op}}} \rho' = \rho' \;;_{\mathcal{C}} \rho,$$

- For $C \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$,

$$I_C^{\mathcal{C}^{\mathrm{op}}} = I_C^{\mathcal{C}}.$$

For any functor F from $\mathcal{C}$ to $\mathcal{C}'$, we write F^{op} for the functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}'^{\mathrm{op}}$ such that

$$(F^{\mathrm{op}})_{\mathrm{ob}} = F_{\mathrm{ob}},$$
$$F^{\mathrm{op}}{}_{CC'} = F_{C'C}.$$

It is easily seen that $(—)^{\mathrm{op}}$, which maps $\mathcal{C}$ into $\mathcal{C}^{\mathrm{op}}$ and F into F^{op}, is a functor from CAT to CAT, and that $(—)^{\mathrm{op}} \,;\, (—)^{\mathrm{op}}$ is the identity functor from CAT to CAT, i.e.

$$(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C} \text{ and } (F^{\mathrm{op}})^{\mathrm{op}} = F.$$

We have seen that a preordered set $(P, \preceq)$ (i.e. the set P equipped with the preorder $\preceq$) is regarded as a category as follows:

$$\mathrm{Ob}(P, \preceq) = P,$$

$$x \xrightarrow[(P,\preceq)]{} y = \begin{cases} \{\langle\rangle\} & \text{if } x \preceq y \\ \{\} & \text{otherwise,} \end{cases}$$

$$\langle\rangle \;;_{(P,\preceq)} \langle\rangle = \langle\rangle,$$

$$I_x^{(P,\preceq)} = \langle\rangle.$$

Thus,

$$\mathrm{Ob}((P, \preceq)^{\mathrm{op}}) = P,$$

$$x \xrightarrow[(P,\preceq)^{\mathrm{op}}]{} y = \begin{cases} \{\langle\rangle\} & \text{if } x \preceq^{\dagger} y \\ \{\} & \text{otherwise,} \end{cases}$$

$$\langle\rangle \;;_{(P,\preceq)^{\mathrm{op}}} \langle\rangle = \langle\rangle,$$

$$I_x^{(P,\preceq)^{\mathrm{op}}} = \langle\rangle.$$

so that $(P,\preceq)^{\mathrm{op}}$ is the preordered set $(P,\preceq^\dagger)$, regarded as a category.

Similarly, a monotone function $f \in (P,\preceq) \xrightarrow[\mathrm{PREO}]{} (P',\preceq')$ is regarded as the functor such that

$$f_{\mathrm{ob}} = f,$$

$$f_{x,y} = \begin{cases} \lambda\langle\rangle.\ \langle\rangle & \text{if } x \preceq y \\ \{\} & \text{otherwise.} \end{cases}$$

Thus,

$$(f^{\mathrm{op}})_{\mathrm{ob}} = f,$$

$$(f^{\mathrm{op}})_{x,y} = \begin{cases} \lambda\langle\rangle.\ \langle\rangle & \text{if } x \preceq^\dagger y \\ \{\} & \text{otherwise.} \end{cases}$$

But f is also a monotone function from $(P,\preceq^\dagger)$ to $(P',\preceq'^\dagger)$ which, when regarded as a functor, satisfies the above equations.

Thus the restriction of $(\text{—})^{\mathrm{op}}$ to PREO is a functor such that

$$(P,\preceq)^{\mathrm{op}} = (P,\preceq^\dagger) \text{ and } f^{\mathrm{op}} = f.$$

Similarly, since the reflection of a partial order is a partial order, $(\text{—})^{\mathrm{op}}$ can be restricted to a functor from PARO to PARO.

1.12 DCPO's and CPO's

In this section, we consider several special kinds of partially ordered sets. We will denote the partial ordering by $\sqsubseteq$ and its reflection by $\sqsupseteq$; the relationship $x \sqsubseteq y$ is often read as "x approximates y", "y extends x", "x is below y", or "y is above x".

Suppose P is a partially ordered set, $X \subseteq P$ and $y \in P$. If $x \sqsubseteq y$ for all $x \in X$ then we write $X \sqsubseteq^* y$ and say that y is an *upper bound* of X; if any such y exists we say that X is *upper-bounded* or just *bounded.* If $X \sqsubseteq^* y$ and, for all $z \in P$ such that $X \sqsubseteq^* z$, $y \sqsubseteq z$ then we say that y is a *least upper bound* of X.

Similarly, if $y \sqsubseteq x$ for all $x \in X$ then we write $y \sqsubseteq^* X$ and say that y is a *lower bound* of X; if any such y exists we say that X is *lower bounded.* If $y \sqsubseteq^* X$ and, for all $z \in P$ such that $z \sqsubseteq^* X$, $z \sqsubseteq y$ then we say that y is a *greatest lower bound* of X.

We have seen that in a partially ordered set (as opposed to a preordered set), a subset X will have at most one greatest lower bound; similarly it will have at most one least upper bound. When they exist, we write

$$\bigsqcup X \text{ for the least upper bound of } X,$$

$$x \sqcup y \text{ for the least upper bound of } \{x, y\},$$

$$\bigsqcap X \text{ for the greatest lower bound of } X,$$

$$x \sqcap y \text{ for the greatest lower bound of } \{x, y\}.$$

Being the least upper bound of the empty set is equivalent to being the *least* element of P, i.e. the unique y such that $y \sqsubseteq z$ for all $z \in P$. Similarly, being the greatest lower bound of the empty set is equivalent to being the *greatest* element of P, i.e. the unique y such that $z \sqsubseteq y$ for all $z \in P$. When they exist, we write

$$\perp \text{ for the least element,}$$

$$\top \text{ for the greatest element.}$$

Proposition 1.3 *If $\mathcal{X} \subseteq \mathcal{P}P$, and every member of $\mathcal{X}$ possesses a least upper bound, and either $\bigcup \mathcal{X}$ or $\{\bigsqcup X \mid X \in \mathcal{X}\}$ possesses a least upper bound, then both $\bigcup \mathcal{X}$ and $\{\bigsqcup X \mid X \in \mathcal{X}\}$ possess the same least upper bound, i.e.*

$$\bigsqcup(\bigcup \mathcal{X}) = \bigsqcup\{\bigsqcup X \mid X \in \mathcal{X}\}.$$

PROOF Show that $\bigcup \mathcal{X}$ and $\{\bigsqcup X \mid X \in \mathcal{X}\}$ have the same set of upper bounds. END OF PROOF

A subset X of a partially ordered set P is said to be a *directed subset of* P iff, for all finite $F \subseteq X$, there exists a $z \in X$ that is an upper bound of F. Equivalently, X is a directed subset of P iff X is nonempty and, for all x, $y \in X$, there is a $z \in X$ that is an upper bound of $\{x, y\}$. (Proof by induction on the size of the finite set.)

A subset of a partially ordered set P is said to be *uninteresting* iff it contains an upper bound of itself; otherwise it is said to be *interesting*. When a set is uninteresting, its contained upper bound must be least, so that uninteresting sets all contain least upper bounds. An uninteresting set is always directed, and a finite directed set is always uninteresting. But if P is infinite then it may possess infinite directed subsets that are interesting.

A *chain* is an increasing sequence of members of a partially ordered set, i.e.

$$x_1 \sqsubseteq x_2 \sqsubseteq x_3 \sqsubseteq \cdots .$$

The set $\{x_1, x_2, x_3, \ldots\}$ of members of a chain is directed, and is interesting iff it is infinite.

Suppose P is the set of all finite subsets of some set U, with $x \sqsubseteq y$ iff x is a subset of y. Then an example of a directed set that is not a chain is provided by the set of all finite members of P, which is interesting iff it is infinite (iff U is infinite iff P is infinite).

A partially ordered set is called a *directed-complete partial order* (DCPO) iff every directed subset has a least upper bound. (Note that this condition is trivially true for uninteresting directed sets; thus every finite partially ordered set is a DCPO.) A DCPO is called a *complete partial order* (CPO) iff it possesses a least element. A DCPO is said to be *bounded complete* if every bounded subset possesses a least upper bound.

If a nonempty DCPO P is bounded complete, then the empty set is bounded (by any element) and possesses a least upper bound, which will be the least element of P; thus P is a CPO. (However, the empty set is a bounded complete DCPO that is not a CPO.)

Any set, when equipped with a discrete ordering, becomes a DCPO, since its only directed subsets are singletons.

In the definition of DCPO and CPO, some authors only require the existence of least upper bounds of chains, rather than all directed sets. We will refer to these slightly variant concepts as ω-DCPO and ω-CPO.

Some authors call CPO's "domains", but other authors (or even the same author in different publications) use this term to denote a CPO with special properties (such as bounded completeness and algebraicity). DCPO's are sometimes called "predomains".

1.13 Continuous and Strict Functions

A function f from a DCPO P to a DCPO P' is said to be *continuous* iff, for all directed $X \subseteq P$, $f(\bigsqcup X)$ is the least upper bound of $\{fx \mid x \in X\}$.

If $x \sqsubseteq y$ then $\{x, y\}$ is directed. From this it is easily seen that a continuous function is monotone. On the other hand, a monotone function will be continuous only if it satisfies the following:

Proposition 1.4 *Suppose f is a monotone function from a DCPO P to a DCPO P'. Then f will be continuous iff, for all interesting directed $X \subseteq P$,*

$$f(\bigsqcup X) \sqsubseteq \bigsqcup'\{fx \mid x \in X\}.$$

PROOF If f is monotone and X is directed, then it is easily seen that $\{fx \mid x \in X\}$ is directed, and thus possesses a least upper bound. Thus f will be continuous iff

$$f(\bigsqcup X) = \bigsqcup'\{fx \mid x \in X\}.$$

However, $\bigsqcup X$ is an upper bound of X, so that, since f is monotone, $f(\bigsqcup X)$ is an upper bound of $\{fx \mid x \in X\}$, and

$$\bigsqcup'\{fx \mid x \in X\} \sqsubseteq f(\bigsqcup X).$$

Moreover, the opposite order holds when X is uninteresting, for then $\bigsqcup X \in X$, so that $f(\bigsqcup X) \in \{fx \mid x \in X\}$. END OF PROOF

When P is finite, or is a discretely ordered set, it possesses no interesting directed subsets, so that every monotone function from P to P' is continuous. Moreover, when P is discretely ordered, all functions from P to P' are monotone, and therefore continuous.

A function f from a CPO P to a CPO P' is said to be *strict* iff it preserves least elements, i.e. $f(\perp) = \perp'$.

It is easily seen that the composition of continuous functions is a continuous function, and that the identity function on a DCPO is continuous. Moreover, the composition of strict functions is strict, and the identity function on a CPO is strict. Thus we can define three categories:

- DCPO, whose objects are DCPO's and whose morphisms are continuous functions,

- CPO, whose objects are CPO's and whose morphisms are continuous functions,

- SCPO, whose objects are CPO's and whose morphisms are strict continuous functions.

In each of these categories, composition is functional composition and the identity morphisms are identity functions.

Notice that SCPO is a subcategory of CPO, which is a full subcategory of DCPO (which is a subcategory of PARO, which is a full subcategory of PREO). Moreover, if sets are regarded as discretely ordered, then SET is a full subcategory of DCPO.

1.14 Products in DCPO, CPO, and SCPO

We have already defined distinguished products in the category PARO as follows: For an index set J and $\bar{P} \in (\mathrm{Ob}\,\mathrm{PARO})^J$, $\prod \bar{P}$ is the set

$$\prod \bar{P} = \{\bar{x} \mid \mathrm{dom}\,\bar{x} = J \text{ and } (\forall i \in J)\ \bar{x}i \in \bar{P}i\},$$

with the partial ordering

$$\bar{x} \sqsubseteq_{\prod \bar{P}} \bar{x}' \text{ iff } (\forall i \in J)\ \bar{x}i \sqsubseteq_{\bar{P}i}\ \bar{x}'i.$$

Then for $i \in J$, $\mathsf{P}\bar{P}i$ is the monotone function from $\prod \bar{P}$ to $\bar{P}i$ such that

$$(\mathsf{P}\bar{P}i)\bar{x} = \bar{x}i,$$

and when P is a partially ordered set and, for $i \in J$, $\bar{\alpha}i$ is a monotone function from P to $\bar{P}i$, $\textcircled{\scriptsize$\prod$}\,\bar{\alpha}$ is the monotone function from P to $\prod \bar{P}$ such that

$$\left(\textcircled{\scriptsize$\prod$}\,\bar{\alpha}\right)y = \lambda i \in J.\ \bar{\alpha}iy.$$

The same construction also gives a distinguished product in DCPO since (as may be verified by the reader), when the $\bar{P}i$ are DCPO's,

- $\prod \bar{P}$ is a DCPO, in which, for all directed $X \subseteq \prod \bar{P}$,

$$\bigsqcup_{\prod \bar{P}} X = \lambda i \in J.\ \bigsqcup_{\bar{P}i}\{\bar{x}i \mid \bar{x} \in X\},$$

- Each $\mathsf{P}\bar{P}i$ is a continuous function,

- If each $\bar{\alpha}i$ is continuous then $\textcircled{\scriptsize$\prod$}\,\bar{\alpha}$ is continuous.

Moreover, the same constructions give a product in CPO and SCPO since, when the $\bar{P}i$ have least elements,

- $\prod \bar{P}$ has the least element

$$\perp_{\prod \bar{P}} = \lambda i \in J.\ \perp_{\bar{P}i},$$

- Each $\mathsf{P}\bar{P}i$ is strict,

- If each $\bar{\alpha}i$ is strict then $\textcircled{\scriptsize$\prod$}\,\bar{\alpha}$ is strict.

1.15 The Least Fixed-Point Theorem

A fundamental property of CPO's and continuous functions, which underlies the semantics of recursion, is given by the following proposition:

Proposition 1.5 *If P is a CPO and f is a continuous function from P to P then*

$$x = \bigsqcup_{n=0}^{\infty} f^n \perp$$

is the least fixed-point of f, i.e. $fx = x$ and, whenever $fy = y$, $x \sqsubseteq y$.

PROOF Clearly $\perp \sqsubseteq f\perp$, and if $f^n\perp \sqsubseteq f^{n+1}\perp$ then $f^{n+1}\perp = f(f^n\perp) \sqsubseteq f(f^{n+1}\perp) = f^{n+2}\perp$, since f is monotone. Thus, by induction on n,

$$\perp \sqsubseteq f\perp \sqsubseteq f^2\perp \sqsubseteq \cdots$$

is an increasing sequence, so that x is well-defined.

Then, since f is continuous and x is the least upper bound of an increasing sequence,

$$fx = f\Big(\bigsqcup_{n=0}^{\infty} f^n\perp\Big) = \bigsqcup_{n=0}^{\infty} f^{n+1}\perp = \bigsqcup_{n=1}^{\infty} f^n\perp = \bigsqcup_{n=0}^{\infty} f^n\perp = x,$$

where the penultimate step is justified since one can always add $\perp$ to a set without changing its least upper bound.

Finally, suppose $fy = y$. Then $\perp \sqsubseteq y$ and if $f^n\perp \sqsubseteq y$ then $f^{n+1}\perp = f(f^n\perp) \sqsubseteq fy = y$. Thus, by induction on n, y is an upper bound on all the $f^n\perp$, so that $x = \bigsqcup_{n=0}^{\infty} f^n\perp \sqsubseteq y$. END OF PROOF

Notice that, in the final paragraph of the above proof, a sufficient hypothesis is $fy \sqsubseteq y$. Thus x is the least solution of $fx \sqsubseteq x$ (sometimes called the least *pre-fixed-point* of f), as well as of $fx = x$.

1.16 Cartesian Closed Categories

Let $\mathcal{C}$ be a category with distinguished products, and B, $C \in \mathrm{Ob}\,\mathcal{C}$. Then an *exponentiation* (*weak exponentiation*) *of C by B* consists of:

- An object $E \in \mathrm{Ob}\,\mathcal{C}$,

- A morphism $a \in E \times B \underset{\mathcal{C}}{\to} C$,

such that, for all $A \in \mathrm{Ob}\,\mathcal{C}$ and $\rho \in A \times B \underset{\mathcal{C}}{\to} C$, there is exactly one (at least one) $\rho' \in A \underset{\mathcal{C}}{\to} E$ such that

$$\begin{tikzcd} A \times B \arrow[r, "\rho' \times I_B"] \arrow[dr, "\rho"'] & E \times B \arrow[d, "a"] \\ & C \end{tikzcd}$$

commutes in $\mathcal{C}$.

Now suppose E_1, a_1 and E_2, a_2 are both exponentiations of C by B. Since E_1, a_1 is an exponentiation, there is an $a_2' \in E_2 \underset{\mathcal{C}}{\to} E_1$ such that

$$\begin{tikzcd} E_2 \times B \arrow[r, "a_2' \times I_B"] \arrow[dr, "a_2"'] & E_1 \times B \arrow[d, "a_1"] \\ & C \end{tikzcd}$$

commutes, and since E_2, a_2 is an exponentiation, there is an $a_1' \in E_1 \underset{\mathcal{C}}{\to} E_2$ such that

$$\begin{tikzcd} E_1 \times B \arrow[r, "a_1' \times I_B"] \arrow[dr, "a_1"'] & E_2 \times B \arrow[d, "a_2"] \\ & C \end{tikzcd}$$

commutes. Then, since $(a_1' \times I_B)\,;(a_2' \times I_B) = a_1'\,;a_2' \times I_B\,;I_B = a_1'\,;a_2' \times I_B$,

$$\begin{tikzcd}
E_1 \times B \arrow[rr, "a_1'\,;\,a_2' \times I_B"] \arrow[drr, "a_1"'] & & E_1 \times B \arrow[d, "a_1"] \\
& & C
\end{tikzcd}$$

commutes. But this diagram also commutes if $a_1';a_2'$ is replaced by I_{E_1}. Thus, since E_1, a_1 is an exponentiation, $a_1'\,;a_2' = I_{E_1}$.

By a similar argument, $a_2'\,;a_1' = I_{E_2}$. Thus a_1' is an isomorphism from E_1 to E_2, with inverse $a_1'^{-1} = a_2'$, such that $a_2 = (a_1'^{-1} \times I_B)\,;a_1$ and $a_1 = (a_1' \times I_B)\,;a_2$. In brief, any two exponentiations of C by B (though not any two weak exponentiations) are isomorphic.

A (*weak*) *Cartesian closed category* is a category $\mathcal{C}$ with distinguished finite products, including a distinguished terminal element, and distinguished (weak) exponentiations of all pairs of objects. We write $B \xRightarrow[\mathcal{C}]{} C$, $\mathsf{ap}^{\mathcal{C}}_{BC} \in (B \xRightarrow[\mathcal{C}]{} C) \times B \xrightarrow[\mathcal{C}]{} C$ for the distinguished exponentiation of C by B. When $\rho \in A \times B \to C$, we write $\mathsf{ab}\,\rho$ for the unique morphism (or, in the weak case, some distinguished morphism) in $A \to (B \xRightarrow[\mathcal{C}]{} C)$ such that

$$\begin{tikzcd}
A \times B \arrow[rr, "\mathsf{ab}\,\rho \times I_B"] \arrow[drr, "\rho"'] & & (B \Rightarrow C) \times B \arrow[d, "\mathsf{ap}_{BC}"] \\
& & C
\end{tikzcd}$$

commutes.

In the category SET, we distinguish

$$B \Rightarrow C \overset{\text{def}}{=} B \to C,$$

$$\mathsf{ap}_{BC} \overset{\text{def}}{=} \lambda\langle\, f \in B \to C,\ y \in B\,\rangle.\ fy \in (B \Rightarrow C) \times B \to C.$$

Then the above diagram commutes iff, for all $x \in A$ and $y \in B$,

$$\rho\langle x, y\rangle = \mathsf{ap}_{BC}\langle \mathsf{ab}\,\rho x, y\rangle = \mathsf{ab}\,\rho x y,$$

which uniquely determines $\mathsf{ab}\,\rho$ to be

$$\mathsf{ab}\,\rho = \lambda x \in A.\ \lambda y \in B.\ \rho\langle x, y\rangle.$$

Thus SET is Cartesian closed.

The same construction also works in PREO and PARO, providing the set $B \Rightarrow C$ is ordered pointwise,

$$f \sqsubseteq_{B\Rightarrow C} f' \text{ iff } (\forall y \in B)\ fy \sqsubseteq_C f'y,$$

since, as the reader may verify:

- If $\sqsubseteq_C$ is a preorder (partial order) then $\sqsubseteq_{B\Rightarrow C}$ is a preorder (partial order),
- ap_{BC} is monotone,
- If ρ is monotone then, for all $x \in A$, $\mathsf{ab}\,\rho x$ is monotone,
- If ρ is monotone then $\mathsf{ab}\,\rho$ is monotone.

Moreover, the same construction also works in DCPO, where, for all directed $F \subseteq B \Rightarrow C$,

$$\textstyle\bigsqcup_{B\Rightarrow C} F = \lambda x \in B.\ \bigsqcup_C \{fx \mid f \in F\}.$$

To see this, we must first establish that the above function is well-defined and continuous. The ordering on $B \Rightarrow C$ makes the mapping that carries f into fx (for fixed x) monotone, so that $\{fx \mid f \in F\}$ is a directed subset of C. Thus this set has a least upper bound in the DCPO C, so that the above function is well-defined. To see that it is continuous, suppose X is a directed subset of B. Then, since each $f \in F$ is continuous,

$$\begin{aligned} &(\textstyle\bigsqcup_{B\Rightarrow C} F)(\bigsqcup_B X) \\ &= \textstyle\bigsqcup_C \{f(\bigsqcup_B X) \mid f \in F\} \\ &= \textstyle\bigsqcup_C \Big\{\bigsqcup_C \{fx \mid x \in X\} \;\Big|\; f \in F\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \textstyle\bigsqcup_C \bigcup\Big\{\{fx \mid x \in X\} \;\Big|\; f \in F\Big\} \\
&= \textstyle\bigsqcup_C \bigcup\Big\{\{fx \mid f \in F\} \;\Big|\; x \in X\Big\} \\
&= \textstyle\bigsqcup_C \Big\{\bigsqcup_C \{fx \mid f \in F\} \;\Big|\; x \in X\Big\} \\
&= \textstyle\bigsqcup_C \{(\bigsqcup_{B \Rightarrow C} F)x \mid x \in X\}.
\end{aligned}
$$

This establishes that $\bigsqcup_{B \Rightarrow C} F$ actually belongs to $B \Rightarrow C$; it is then straightforward to check that it is the least upper bound of F.

Next, we show that ap_{BC} is a continuous function. Suppose P is a directed subset of $(B \Rightarrow C) \times B$, and let

$$P_1 = \{f \mid (\exists y_0 \in B)\ \langle f, y_0 \rangle \in P\},$$

$$P_2 = \{y \mid (\exists f_0 \in B \Rightarrow C)\ \langle f_0, y \rangle \in P\},$$

which are easily seen to be directed subsets of $B \Rightarrow C$ and B respectively. Then

$$
\begin{aligned}
&\mathsf{ap}_{BC}(\textstyle\bigsqcup_{(B \Rightarrow C) \times B} P) \\
&= \mathsf{ap}_{BC}\langle \textstyle\bigsqcup_{B \Rightarrow C} P_1, \bigsqcup_B P_2 \rangle \\
&= (\textstyle\bigsqcup_{B \Rightarrow C} P_1)(\bigsqcup_B P_2) \\
&= \textstyle\bigsqcup_C \{f(\bigsqcup_B P_2) \mid f \in P_1\} \\
&= \textstyle\bigsqcup_C \Big\{\bigsqcup_C \{fy \mid y \in P_2\} \;\Big|\; f \in P_1\Big\} \\
&= \textstyle\bigsqcup_C \{fy \mid f \in P_1 \text{ and } y \in P_2\},
\end{aligned}
$$

where the penultimate step follows from the continuity of the $f \in P_1$. Now if $\langle f, y \rangle \in P$ then $f \in P_1$ and $y \in P_2$, so that fy belongs to the above set. On the other hand, if fy belongs to the above set then $f \in P_1$ and $y \in P_2$, so that there are $y_0 \in B$ and $f_0 \in B \Rightarrow C$ such that $\langle f, y_0 \rangle$, $\langle f_0, y \rangle \in P$. Then since P is directed, there is an $\langle f', y' \rangle \in P$ such that $\langle f, y_0 \rangle \sqsubseteq \langle f', y' \rangle$ and $\langle f_0, y \rangle \sqsubseteq \langle f', y' \rangle$, so that $f \sqsubseteq f'$, $y \sqsubseteq y'$, and thus $fy \sqsubseteq f'y'$. Therefore,

$$
\begin{aligned}
&\textstyle\bigsqcup_C \{fy \mid f \in P_1 \text{ and } y \in P_2\} \\
&= \textstyle\bigsqcup_C \{fy \mid \langle f, y \rangle \in P\} \\
&= \textstyle\bigsqcup_C \{\mathsf{ap}_{BC}\langle f, y \rangle \mid \langle f, y \rangle \in P\}.
\end{aligned}
$$

Next, we show that, for any $x \in A$, $\mathsf{ab}\,\rho x$ is a continuous function. Suppose Y is a directed subset of B. Then $\{\langle x, y\rangle \mid y \in Y\}$ is a directed subset of $A \times B$, and

$$\textstyle\bigsqcup_{A\times B}\{\langle x, y\rangle \mid y \in Y\} \;=\; \langle \bigsqcup_A\{x \mid y \in Y\}, \bigsqcup_B\{y \mid y \in Y\}\rangle$$

$$= \langle x, \textstyle\bigsqcup_B Y\rangle,$$

where the last step follows since Y, being directed, is nonempty. Then, since ρ is continuous,

$$\mathsf{ab}\,\rho x(\textstyle\bigsqcup_B Y) = \rho\langle x, \bigsqcup_B Y\rangle$$

$$= \rho\Big(\textstyle\bigsqcup_{A\times B}\{\langle x, y\rangle \mid y \in Y\}\Big)$$

$$= \textstyle\bigsqcup_C\{\rho\langle x, y\rangle \mid y \in Y\}$$

$$= \textstyle\bigsqcup_C\{\mathsf{ab}\,\rho x y \mid y \in Y\}.$$

By a symmetric argument, if X is a directed subset of A and $y \in B$,

$$\mathsf{ab}\,\rho(\textstyle\bigsqcup_A X)y = \bigsqcup_C\{\mathsf{ab}\,\rho x y \mid x \in X\},$$

so that

$$\mathsf{ab}\,\rho(\textstyle\bigsqcup_A X)y = \Big(\bigsqcup_{B\Rightarrow C}\{\mathsf{ab}\,\rho x \mid x \in X\}\Big)y.$$

Since this equality holds for all $y \in B$, we have

$$\mathsf{ab}\,\rho(\textstyle\bigsqcup_A X) = \bigsqcup_{B\Rightarrow C}\{\mathsf{ab}\,\rho x \mid x \in X\}$$

for all directed $X \subseteq A$. Thus $\mathsf{ab}\,\rho$ is a continuous function.

Finally, we note that our construction also works in CPO, since our ordering of $B \Rightarrow C$ insures that if C has the least element $\bot_C$ then $B \Rightarrow C$ has the least element

$$\bot_{B\Rightarrow C} = \lambda y \in B.\ \bot_C.$$

However, the construction does not work for the category SCPO. The difficulty is that the strictness of ρ is not enough to insure the strictness of $\mathsf{ab}\,\rho x$ or of $\mathsf{ab}\,\rho$. In fact, it can be shown that SCPO does not possess exponentiations of all pairs of objects and thus is not a Cartesian closed category.

1.17 Limits

The concept of a product over an index set of a function can be generalized to that of a limit over an index category of a functor. Suppose J is a category (often called the *index* category), $\mathcal{H}$ is a category, and $\bar{H}$ is a functor from J to $\mathcal{H}$. Then

$$H \in \mathcal{H}, \quad \bar{\alpha} \in \prod_{i \in \mathrm{Ob}\, J} (H \underset{\mathcal{H}}{\rightarrow} \bar{H} i)$$

is a *cone* of $\bar{H}$ iff, for all i_1, $i_2 \in \mathrm{Ob}\, J$ and $\iota \in i_1 \underset{J}{\rightarrow} i_2$,

$$\begin{tikzcd}
H \arrow[r, "\bar{\alpha} i_1"] \arrow[dr, "\bar{\alpha} i_2"'] & \bar{H} i_1 \arrow[d, "\bar{H} \iota"] \\
& \bar{H} i_2
\end{tikzcd}$$

commutes in $\mathcal{H}$. A cone $H_0, \bar{\alpha}_0$ is a *limit* of $\bar{H}$ (over J in $\mathcal{H}$) iff, for all cones $H, \bar{\alpha}$ of $\bar{H}$, there is a unique $\beta \in H \underset{\mathcal{H}}{\rightarrow} H_0$ such that

$$\begin{tikzcd}
H \arrow[d, "\beta"'] \arrow[dr, "\bar{\alpha} i"] & \\
H_0 \arrow[r, "\bar{\alpha}_0 i"] & \bar{H} i
\end{tikzcd}$$

commutes in $\mathcal{H}$ for all $i \in \mathrm{Ob}\, J$.

In the special case where J is a set, viewed as a discrete category, and $\bar{H}$ is a function from J to $\mathrm{Ob}\, \mathcal{H}$, viewed as a functor, a limit of $\bar{H}$ is a product of $\bar{H}$.

Suppose $H_0, \bar{\alpha}_0$ and $H_1, \bar{\alpha}_1$ are both limits of $\bar{H}$. Then, since $H_0, \bar{\alpha}_0$ is a limit and $H_1, \bar{\alpha}_1$ is a cone, there is a $\beta \in H_1 \xrightarrow[\mathcal{H}]{} H_0$ such that

$$\begin{tikzcd} H_1 \arrow[d, "\beta"'] \arrow[dr, "\bar{\alpha}_1 i"] & \\ H_0 \arrow[r, "\bar{\alpha}_0 i"] & \bar{H} i \end{tikzcd}$$

commutes for all $i \in \mathrm{Ob}\, J$, and since $H_1, \bar{\alpha}_1$ is a limit and $H_0, \bar{\alpha}_0$ is a cone, there is a $\beta' \in H_0 \xrightarrow[\mathcal{H}]{} H_1$ such that

$$\begin{tikzcd} H_0 \arrow[d, "\beta'"'] \arrow[dr, "\bar{\alpha}_0 i"] & \\ H_1 \arrow[r, "\bar{\alpha}_1 i"] & \bar{H} i \end{tikzcd}$$

commutes for all $i \in \mathrm{Ob}\, J$. Combining these diagrams, we find that

$$\begin{tikzcd} H_0 \arrow[d, "\beta' ; \beta"'] \arrow[dr, "\bar{\alpha}_0 i"] & \\ H_0 \arrow[r, "\bar{\alpha}_0 i"] & \bar{H} i \end{tikzcd}$$

commutes for all $i \in \mathrm{Ob}\, J$. But this diagram also commutes if $\beta';\beta$ is replaced by I_{H_0}. Thus, since $H_0, \bar{\alpha}_0$ is a limit, $\beta' ; \beta = I_{H_0}$.

By a symmetric argument, $\beta ; \beta' = I_{H_1}$. Thus β is an isomorphism from H_1 to H_0, with inverse $\beta^{-1} = \beta'$, such that $\bar{\alpha}_1 i = \beta ; \bar{\alpha}_0 i$ and $\bar{\alpha}_0 i = \beta^{-1} ; \bar{\alpha}_1 i$ for all $i \in \mathrm{Ob}\, J$. In brief, any two limits of the same functor are isomorphic.

For several categories, we will define distinguished limits, at least for certain index categories. We will write $\Lambda \bar{H}.$, $\mathsf{C}\bar{H}$ for the distinguished limit of $\bar{H} \in J \xrightarrow[\mathrm{CAT}]{} \mathcal{H}$. For each cone $H, \bar{\alpha}$ of $\bar{H}$, we will write $\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}$ for the unique morphism in $H \xrightarrow[\mathcal{H}]{} \Lambda\bar{H}.$ such that

$$\begin{tikzcd}
H \arrow[d, "\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}"'] \arrow[dr, "\bar{\alpha}i"] & \\
\Lambda\bar{H}. \arrow[r, "\mathsf{C}\bar{H}i"] & \bar{H}i
\end{tikzcd}$$

commutes in $\mathcal{H}$ for all $i \in \mathrm{Ob}\, J$.

Suppose SCAT is a subcategory of CAT such that $\mathcal{H}$ has distinguished limits over all $J \in \mathrm{Ob}\,\mathrm{SCAT}$. (In some applications, SCAT will be the category of all *small* categories, i.e. all categories whose collection of objects is a set. In other applications, SCAT will be the category PREO, viewed as a subcategory of CAT.) Then we define $\hat{\mathcal{H}}$ to be the category such that

$$\mathrm{Ob}\,\hat{\mathcal{H}} = \{\langle J, \bar{H}\rangle \mid J \in \mathrm{Ob}\,\mathrm{SCAT} \text{ and } \bar{H} \in J \xrightarrow[\mathrm{CAT}]{} \mathcal{H}\},$$

$$\langle J, \bar{H}\rangle \xrightarrow[\hat{\mathcal{H}}]{} \langle J', \bar{H}'\rangle = \{\theta \mid \theta \in J \xrightarrow[\mathrm{SCAT}]{} J' \text{ and } \bar{H} = \theta\,;\bar{H}'\},$$

with composition and identities inherited from SCAT:

$$\theta\,;_{\hat{\mathcal{H}}}\,\theta' = \theta\,;_{\mathrm{SCAT}}\,\theta',$$

$$I^{\hat{\mathcal{H}}}_{\langle J,\bar{H}\rangle} = I^{\mathrm{SCAT}}_{J}.$$

(The reader should verify that $\hat{\mathcal{H}}$ contains the above identities, and is closed under the above composition.)

Next, we define a functor L from $\hat{\mathcal{H}}^{\mathrm{op}}$ to $\mathcal{H}$, such that

$$L\langle J, \bar{H}\rangle = \Lambda\bar{H}.\,.$$

To define the morphism parts of L, suppose $\theta \in \langle J, \bar{H}\rangle \xrightarrow[\hat{\mathcal{H}}]{} \langle J', \bar{H}'\rangle$, so that

$\theta \in J \xrightarrow[\text{SCAT}]{} J'$ and $\bar{H} = \theta\,; \bar{H}'$. Since $\Lambda\bar{H}.'$, $\mathsf{C}\bar{H}'$ is a cone of $\bar{H}'$, the diagram

$$\begin{tikzcd}
\Lambda\bar{H}.' \arrow[rr, "\mathsf{C}\bar{H}'i_1'"] \arrow[drr, "\mathsf{C}\bar{H}'i_2'"'] & & \bar{H}'i_1' \arrow[d, "\bar{H}'\iota'"] \\
& & \bar{H}'i_2'
\end{tikzcd}$$

commutes for every morphism $\iota' \in i_1' \xrightarrow[J']{} i_2'$ of J'. Then

$$\begin{tikzcd}
\Lambda\bar{H}.' \arrow[rr, "\mathsf{C}\bar{H}'(\theta i_1)"] \arrow[drr, "\mathsf{C}\bar{H}'(\theta i_2)"'] & & \bar{H}'(\theta i_1) = \bar{H}i_1 \arrow[d, "\bar{H}(\theta\iota) = \bar{H}\iota"] \\
& & \bar{H}'(\theta i_2) = \bar{H}i_2
\end{tikzcd}$$

commutes for every morphism $\iota \in i_1 \xrightarrow[J]{} i_2$ of J, so that

$$\Lambda\bar{H}.',\ \lambda i \in \operatorname{Ob} J.\ \mathsf{C}\bar{H}'(\theta i)$$

is a cone of $\bar{H}$. Thus we may define

$$L\theta = \text{Lm}_{\Lambda\bar{H}.',\bar{H}}\ \lambda i \in \operatorname{Ob} J.\ \mathsf{C}\bar{H}'(\theta i),$$

so that $L\theta$ is the unique morphism in $\Lambda\bar{H}.' \xrightarrow[\mathcal{H}]{} \Lambda\bar{H}.$ such that

$$\begin{tikzcd}
\Lambda\bar{H}.' \arrow[d, "L\theta"'] \arrow[dr, "\mathsf{C}\bar{H}'(\theta i)"] & \\
\Lambda\bar{H}. \arrow[r, "\mathsf{C}\bar{H}i"] & \bar{H}i
\end{tikzcd}$$

commutes for all $i \in J$. (The reader may verify the functor laws.)

1.18 Limits in SET

When J is a small category and $\bar{H}$ is a functor from J to SET, we define

$$\Lambda\bar{H}. = \{\bar{x} \mid \bar{x} \in \prod_{i \in \mathrm{Ob}\, J} \bar{H}i \text{ and } (\forall i_1, i_2 \in \mathrm{Ob}\, J)(\forall \iota \in i_1 \xrightarrow[J]{} i_2)\ \bar{H}\iota(\bar{x}i_1) = \bar{x}i_2\},$$

and, for each $i \in \mathrm{Ob}\, J$, we define $\mathsf{C}\bar{H}i \in \Lambda\bar{H}. \to \bar{H}i$ to be the function such that

$$\mathsf{C}\bar{H}i\bar{x} = \bar{x}i.$$

To see that $\Lambda\bar{H}.\,, \mathsf{C}\bar{H}$ is a cone, note that, for each $\iota \in i_1 \xrightarrow[J]{} i_2$,

$$\begin{tikzcd}
\Lambda\bar{H}. \arrow[r, "\mathsf{C}\bar{H}i_1"] \arrow[dr, "\mathsf{C}\bar{H}i_2"'] & \bar{H}i_1 \arrow[d, "\bar{H}\iota"] \\
& \bar{H}i_2
\end{tikzcd}$$

commutes by virtue of the restriction placed on the $\bar{x}$ in $\Lambda\bar{H}.\,$.

To see that this construction gives a limit, suppose $H, \bar{\alpha}$ is any cone of $\bar{H}$, so that $\bar{\alpha}i \in H \to \bar{H}i$ for each $i \in \mathrm{Ob}\, J$, and $\bar{H}\iota(\alpha i_1 z) = \alpha i_2 z$ for each $\iota \in i_1 \xrightarrow[J]{} i_2$ and $z \in H$. Then

$$\begin{tikzcd}
H \arrow[d, "\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}"'] \arrow[dr, "\bar{\alpha}i"] & \\
\Lambda\bar{H}. \arrow[r, "\mathsf{C}\bar{H}i"] & \bar{H}i
\end{tikzcd}$$

commutes for all $i \in \mathrm{Ob}\, J$ iff

$$(\forall i \in \mathrm{Ob}\, J)(\forall z \in H)\ \mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}zi = \bar{\alpha}iz,$$

which uniquely determines $\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}$ to be

$$\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha} = \lambda z \in H.\ \lambda i \in \mathrm{Ob}\, J.\ \bar{\alpha}iz.$$

We must also be sure that, for each $z \in H$, $\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}z$ actually belongs to $\Lambda\bar{H}.$, i.e.

$$(\forall z \in H)(\forall i \in \mathrm{Ob}\,J)\ \mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}zi \in \bar{H}i,$$

$$(\forall z \in H)(\forall i_1, i_2 \in \mathrm{Ob}\,J)(\forall \iota \in i_1 \xrightarrow[J]{} i_2)\ \bar{H}\iota(\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}zi_1) = \mathrm{Lm}_{H\bar{H}'}\,\bar{\alpha}zi_2.$$

However, since $\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}zi = \bar{\alpha}iz$, these conditions are consequences of the assumption that $H, \bar{\alpha}$ is a cone.

1.19 Limits in Other Categories

Our construction for SET also gives limits for PREO (and PARO) if $\Lambda\bar{H}.$ is given a pointwise ordering:

$$\bar{x} \preceq_{\Lambda\bar{H}.} \bar{x}' \text{ iff } (\forall i \in \mathrm{Ob}\,J)\ \bar{x}i \preceq_{\bar{H}i} \bar{x}'i.$$

We already know that this definition gives a preorder (partial order) for the product $\prod_{i \in \mathrm{Ob}\,J} \bar{H}i$ if each $\preceq_{\bar{H}i}$ is a preorder (partial order); it obviously does the same for any subset of the product, such as $\Lambda\bar{H}.$. The reader may verify that this ordering makes the $\mathsf{C}\bar{H}i$ monotone, and makes the $\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}$ monotone when the $\bar{\alpha}i$ are monotone.

The construction also works for DCPO. When the $\bar{H}i$ are DCPO's, $\Lambda\bar{H}.$ is a DCPO, in which, for all directed $X \subseteq \Lambda\bar{H}.$,

$$\textstyle\bigsqcup_{\Lambda\bar{H}.} X = \lambda i \in \mathrm{Ob}\,J.\ \bigsqcup_{\bar{H}i}\{\bar{x}i \mid \bar{x} \in X\}.$$

To see this, note that the above equation gives least upper bounds for directed subsets of the product $P = \prod_{i \in \mathrm{Ob}\,J} \bar{H}i$. Thus we need only show that the condition $\bar{H}\iota(\bar{x}i_1) = \bar{x}i_2$ is satisfied by $\bar{x} = \bigsqcup_P X$ when it is satisfied by all $\bar{x} \in X$. But since $\{\bar{x}i_1 \mid \bar{x} \in X\}$ is directed and $\bar{H}\iota$ is continuous,

$$\textstyle\bar{H}\iota((\bigsqcup_P X)i_1) = \bar{H}\iota(\bigsqcup_{\bar{H}i_1}\{\bar{x}i_1 \mid \bar{x} \in X\})$$

$$\textstyle= \bigsqcup_{\bar{H}i_2}\{\bar{H}\iota(\bar{x}i_1) \mid \bar{x} \in X\} = \bigsqcup_{\bar{H}i_2}\{\bar{x}i_2 \mid \bar{x} \in X\} = (\bigsqcup_P X)i_2.$$

The reader may verify that the $\mathsf{C}\bar{H}i$ are continuous, and that $\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}$ is continuous when the $\bar{\alpha}i$ are continuous.

The construction also works for SCPO. When the $\bar{H}i$ are CPO's, $\Lambda\bar{H}.$ is a CPO, with

$$\perp_{\Lambda\bar{H}.} = \lambda i \in \mathrm{Ob}\,J.\ \perp_{\bar{H}i}.$$

Here the crucial fact is that $H\iota(\perp_{\Lambda\bar{H}.}\, i_1) = \perp_{\Lambda\bar{H}.}\, i_2$, since $\bar{H}\iota$ is strict; once this fact is established it is obvious that the $\mathsf{C}\bar{H}i$ are strict, and that $\mathrm{Lm}_{H\bar{H}}\,\bar{\alpha}$ is strict when the $\bar{\alpha}i$ are strict.

However, our construction does not work for CPO. The difficulty is that $\lambda i \in \mathrm{Ob}\, J.\ \perp_{\bar{H}i}$ will not belong to $\Lambda\bar{H}.$ if any $\bar{H}\iota$ is not strict. Nevertheless, if $\bar{H}$ is a functor from J to SCPO, then $\Lambda\bar{H}.\,, \mathsf{C}\bar{H}$ is a limit of $\bar{H}$ in CPO as well as SCPO. In other words, the identity functor on SCPO, viewed as a functor from SCPO to CPO, preserves limits.