Power Flow in AC Networks

• Basic question: how is electricity supplied in the grid?
  – If these generators produce this much power, how much current will flow through particular transmission lines?

• A fundamentally harder problem than linear circuit analysis (focus on power makes it a non-linear problem)
• Simple example setup (showing one of three phases)
• Simpler drawing of the above in standard format

- Generator
- Load (distribution station)
- Busbar (zero impedance)
- Transmission Line
• Imagine we knew bus voltages (and impedance of load/lines)

\[ v = Zi \iff i = Yv \quad v, i \in \mathbb{R}^3, \quad Z, Y \in \mathbb{R}^{3 \times 3} \]

• Then we could compute all bus currents via linear system

\[ i_{12} = \frac{v_1 - v_2}{r_{12} + jx_{12}} \]

• Could also compute current along each line by Ohm’s law
• Power flow is similar problem conceptually, except that instead of voltages, we may just know (complex) power injections at each node

![Diagram of power flow](image)

• Problem #1: We now know product of voltage and current \( s_k = v_k i_k^* \), but how can we find each voltage/current individually?
• Solution: Using the fact that \( i = Y v \), we can write power as

\[
s = \text{diag}(v)i^* \\
= \text{diag}(v)Y^*v^*
\]

where \( A^* \) denotes elementwise conjugations (though it doesn’t matter too much here, since \( Y \) is symmetric)

• So we can determine voltage from power, but we need to solve a non-linear equation to do so
  – But we’ve already seen how to do this with Newton’s method
• Problem #2: While it’s reasonable to assume we can know (complex) power for loads, generators can’t directly control reactive power (can control voltage magnitude)

• Solution: For loads, assume real and reactive power are known (PQ load); for generators, we assume real power and voltage magnitude are known (PV generators)

• Problem #3: Can’t know the exact amount of real power we need until we’ve finished the analysis (losses due to line resistance, but not known until currents are known)

• Solution: Treat one generator as a “slack” generator that generates enough extra power to overcome line losses
Outline of the final power flow problem:

- Variables \((i = 1, \ldots, n)\):
  
  Power: \(s_i = p_i + jq_i\),
  
  Voltage: \(v_i = \hat{v}_i \angle \theta_i \equiv \hat{v}_i (\cos \theta_i + j \sin \theta_i)\)

- Problem data:

<table>
<thead>
<tr>
<th>Bus Type</th>
<th>Known Variables</th>
<th>Unknown Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slack Generator</td>
<td>(\hat{v}_i, \theta_i)</td>
<td>(p_i, q_i)</td>
</tr>
<tr>
<td>PV Generator</td>
<td>(\hat{v}_i, p_i)</td>
<td>(\theta_i, q_i)</td>
</tr>
<tr>
<td>PQ Load</td>
<td>(p_i, q_i)</td>
<td>(\hat{v}_i, \theta_i)</td>
</tr>
</tbody>
</table>

- Problem:

Find: \(s, v \in \mathbb{C}^n\), such that \(s = \text{diag}(v)Y^*v^*\)

\(2n\) equations, \(2n\) unknowns: can apply Newton’s method to solve
Aside: Computing the Admittance Matrix

• For simplified model of transmission lines, admittance matrix $Y$ is very easy to compute

• Let

$$r_{ij} = \text{Resistance of transmission line between buses } i \text{ and } j$$

($\infty$ if $i$ and $j$ not connected)

$$x_{ij} = \text{Reactance between buses } i \text{ and } j$$

• Then

$$Y_{ij} = \begin{cases} 
-\frac{1}{r_{ij} + jx_{ij}} & i \neq j \text{ (0 if } i, j \text{ not connected)} \\
\sum_{k \neq i} \frac{1}{r_{ik} + jx_{ik}} & i = j
\end{cases}$$
Mathematical Details of Power Flow

• Common to write out complex equation explicitly

\[ p_i + j q_i = v_i \sum_{k=1}^{n} Y_{ik}^* v_k^* \]

\[ = \hat{v}_i (\cos \theta_i + j \sin \theta_i) \sum_{k=1}^{n} \hat{v}_k (G_{ik} - j B_{ik}) (\cos \theta_k - j \sin \theta_k) \]

where \( Y \equiv G + j B \)

• Multiplying terms out leads to canonical power flow equations

\[ p_i = \hat{v}_i \sum_{k=1}^{n} \hat{v}_k (G_{ik} \cos (\theta_i - \theta_k) + B_{ik} \sin (\theta_i - \theta_k)) \]

\[ q_i = \hat{v}_i \sum_{k=1}^{n} \hat{v}_k (G_{ik} \sin (\theta_i - \theta_k) - B_{ik} \cos (\theta_i - \theta_k)) \]
Let $z \in \mathbb{R}^{2n}$ be a vector of all the unknown variables.

$$z = \begin{bmatrix}
p_0 & \}
slack generator
q_0 & \}
pV generators
q_1 & \}
\theta_1 & \}
\vdots & \}
nPV + 1 & \}
q_{nPV + 1} & \}
\theta_{nPV + 1} & \}
PQ Loads
\hat{v}_{nPV + 2} & \}
\theta_{nPV + 2} & \}
\vdots & \}
\hat{v}_n & \}
\theta_n & \}
\end{bmatrix}$$
Let $g : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a function that contains all the power flow equations

$$g(z) = 
\begin{bmatrix}
  p_1 - \hat{v}_1 \sum_{k=1}^{n} \hat{v}_k (G_{1k} \cos(\theta_1 - \theta_k) + B_{1k} \sin(\theta_1 - \theta_k)) \\
  \vdots \\
  q_1 - \hat{v}_1 \sum_{k=1}^{n} \hat{v}_k (G_{1k} \sin(\theta_1 - \theta_k) - B_{1k} \cos(\theta_1 - \theta_k)) \\
  \vdots
\end{bmatrix}
$$

Want to find $z$ such that $g(z) = 0$

Newton’s method, repeat:

$$z \leftarrow z - (D_z g(z))^{-1} g(z)$$

- Computing the Jacobian terms can get a bit cumbersome
• A couple Jacobian terms

\[ \frac{\partial g_{p,i}}{\partial \theta_i} = \hat{v}_i \sum_{k \neq i} \hat{v}_k (G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k)) \]

\[ \frac{\partial g_{q,i}}{\partial \theta_i} = -2\hat{v}_i B_{ii} - \sum_{k \neq i} (G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k)) \]

• Need 8 of these total, can compute Jacobian element by element

• The typical formulation of Newton’s method for power flow you’ll find in most textbooks
• Jacobians are easier to compute if we consider all variables

\[
\begin{bmatrix}
\theta \\
\hat{v} \\
p \\
q
\end{bmatrix}
\]

and

\[
g(z) = \begin{bmatrix}
\text{Re}\{\text{diag}(v)(Yv)^* - s\} \\
\text{Im}\{\text{diag}(v)(Yv)^* - s\}
\end{bmatrix}.
\]

Then

\[
J = D_z g(z) = \begin{bmatrix}
\text{Re}\{J_1\} & \text{Re}\{J_2\} & -I & 0 \\
\text{Im}\{J_1\} & \text{Im}\{J_2\} & 0 & -I
\end{bmatrix}
\]

where \((A^* \text{ denotes elementwise conjugatation})\)

\[
J_1 = j \text{ diag}(v)(\text{diag}(Yv) - Y \text{ diag}(v))^* \\
J_2 = \text{ diag}(v)(Y \text{ diag}(e^{j\theta}))^* + \text{ diag}(e^{j\theta}) \text{ diag}(Yv)^*
\]

\((J_1, J_2 \text{ have the same sparsity pattern as } Y)\)
• Let \( \mathcal{I} \) be the indices of unknown variables in \( z \) (\(|\mathcal{I}| = 2n\))

• Then Newton’s method is just

\[
z_{\mathcal{I}} \leftarrow z_{\mathcal{I}} - J_{\mathcal{I} \mathcal{I}}^{-1} g
\]

• Some notes:
  – Newton’s method is not guaranteed to converge in all cases (even if solution exists)
  
  – But, works very well for many “typical” power flow problems
• Typically, not much relevance to absolute voltage magnitudes; what matters is *relative* magnitudes

• Therefore, common to report voltages in terms of per-unit (p.u.) units, scaled by nominal voltage of the system

• Also, power, admittance, etc are scaled (typical parameter of power flow problems is “base MVA” that says how to scale power)
• A simple three-bus example:

\[ v = 1.0 \angle 0^\circ \]

\[ |v| = 1.0 \]
\[ p = 0.5 \]

Slack generator

\[ 0.04 + j0.2 \]

PV generator

\[ 0.05 + j0.3 \]

\[ s = -1.6 - j0.3 \]

PQ load
- Power flow solution:

\[ v = 1.0 \angle 0^\circ \]
\[ s = 1.17 + j0.36 \]

\[ v = 1.0 \angle -2.56^\circ \]
\[ s = 0.5 + j0.34 \]

\[ v = 0.90 \angle -12.95^\circ \]
\[ s = -1.6 - j0.3 \]
- Power flow solution with resulting currents:

\[
\begin{align*}
\gamma &= 1.0 \angle 0^\circ \\
\delta &= 1.17 + j0.36 \\
\iota_{12} &= 0.10 \angle 25.3^\circ \\
\delta &= 1.0 \angle -2.56^\circ \\
\gamma &= 0.5 + j0.34 \\
\iota_{13} &= 1.16 \angle -20.5^\circ \\
\iota_{23} &= 0.65 \angle -17.5^\circ \\
\gamma &= 0.90 \angle -12.95^\circ \\
\delta &= -1.6 - j0.3 \\
\end{align*}
\]
Progress of Newton’s method on 3 bus example
Progress of Newton’s method on IEEE 300 bus example
• Some notes:
  – Many extensions to Newton’s method are possible, help with convergence or speed things up (less of an issue with current computation)

  – E.g. Fast Decoupled Power Flow (FDPF): ignore off-diagonal blocks in Jacobian (typically smaller in value)

  – Solution to power flow also not guaranteed to exist (can be physically unrealizable)
DC Power Flow

• A linear approximation to power flow

• Not power flow computation for DC network (this is still a non-linear problem)

• Simplifying assumptions:
  – All line resistances are zero (reactances are non-zero)
  – All voltage magnitudes are equal (to 1 p.u.)
  – Voltage angles are small, so that

\[
\sin(\theta_i - \theta_k) \approx \theta_i - \theta_k, \quad \cos(\theta_i - \theta_k) \approx 1
\]
• Under these assumptions

\[ p = -B\theta \quad \text{(imaginary part of admittance matrix } Y = G + jB) \]
\[ q = 0 \quad \text{(all reactive power is equal to zero)} \]

(You will derive these in problem set)

• Given solution to \( \theta \)'s, can again compute current flowing through each line

\[ i_{12} = \frac{v_1 - v_2}{jx_{12}} = \frac{\theta_1 - \theta_2}{x_{12}} = B_{12}(\theta_1 - \theta_2) \]

• We are back to a set of \textit{linear} equations, easier for solving and optimization
Optimal Power Flow

- In power flow problem, real powers at all generators and loads (except slack) are given as input.

- In some situations (e.g. power markets, demand response, etc), we might want to optimize the power at each generator (or loads, if we have demand response capabilities).

- The general term for power flow where we are optimizing certain variables (which are typically fixed in power flow) is optimal power flow (OPF).
• General formulation

\[
\begin{align*}
\text{minimize}_{z} & \quad f(z) \\
\text{subject to} & \quad h_e(z) = 0 \\
& \quad h_i(z) \leq 0 \\
& \quad s = \text{diag}(v)(Yv)^* \\
\end{align*}
\]

(some objective function)

(equality constraints)

(inequality constraints)

(power flow constraints)

where \( z \) is defined as before

\[
z = \begin{bmatrix}
\theta \\
\hat{v} \\
p \\
q
\end{bmatrix}
\]

• Note that we must pre-specify fewer variables than in the power flow problem, or there won’t be anything to optimize over
• In principle, just a (non-convex) optimization problem, could solve with YALMIP
  – In practice, YALMIP has a lot of difficulties with the non-convex equality constraint $s = \text{diag}(v)(Yv)^*$. \\

• If $f$, $h_i$ and $h_e$ are such that the OPF problem would be convex without the power flow constraint, can solve with iterative optimization

• Basic idea: linearize the power flow constraint at each iteration, and solve resulting convex problem
Iterative Optimization Procedure

• Begin with some initial guess $z_0$

• Repeat:
  – Compute Jacobian
    \[ J \leftarrow D_z g(z_0) \]
  – Solve linearized optimization problem
    \[
    \begin{align*}
    \text{minimize} & \quad f(z) \\
    \text{subject to} & \quad h_e(z) = 0 \\
    & \quad h_i(z) \leq 0 \\
    & \quad J(z - z_0) = -g(z_0)
    \end{align*}
    \]
  – Set $z_0 \leftarrow z^*$ (solution to optimization problem)
• Three-bus example with power costs

\[ f(z) = p_1^2 + 2p_1 + 2p_2^2 + p_2 \]
• OPF Solution

\[ v = 1.0 \angle 0^\circ \]
\[ s = 0.95 + j0.42 \]

\[ v = 1.0 \angle 0.84^\circ \]
\[ s = 0.72 + j0.27 \]

\[ v = 0.90 \angle -11.55^\circ \]
\[ s = -1.6 - j0.3 \]
Convergence of solution in iterative optimization procedure
Power Markets

• How do we actually coordinate and price generation with consumption?

• Previously, done by single monopoly, could observe all loads and optimize generation accordingly.

• With competition (even just in generation), need a way to coordinate market instantaneously
  – Unlike traditional markets, can’t wait for market to converge through bidding system
• Theory of the firm

• Given market price $\pi$, determine how much to make by optimization problem

$$\max_y \left\{ \pi y - C(y) \right\} \implies \pi = \frac{dC}{dy}$$
• \( \frac{dC}{dy} \) is known as marginal price, cost to make one more item
  
  – By above relationship, a producer will produce items until the marginal price equals the market price

• In traditional market, market price is set by interaction of supply and demand
• Many idiosyncracies to power markets
  – Need to respond instantaneously to demand (no time for market to “settle” via a manual bidding system)
  – Can’t (yet) store significant portions of energy
  – Demand is typically viewed as fixed, producers (generators) must in total produce enough generation to match
  – Distribution of product (i.e., transmission of power) must obey the laws of physics
Operation of centralized power market

1a. Loads report demand

1b. Generators report costs

2. ISO returns power to generate (by solving OPF)

3. Power flow provides power
Optimal Power Flow for Power Markets

• For generality, we’ll usually consider case of power generators and loads at the different nodes

\[ s_i = s_i^G - s_i^L \]

• We’ll also explicitly separate out generator and load powers (if no generator at a node, then we’ll just add a constraint that \( s_i^G = 0 \))
• A typical OPF formulation for power markets

$$\begin{align*}
\text{minimize} \quad & C(p^G) \equiv \sum_{i=1}^{n} C_i(p^G_i) \\
\text{subject to} \quad & s^G - s^L = \text{diag}(v)(Yv)^* \\
& -v_iY_{ij}^*(v_i - v_j)^* \leq S_{ij}^{\max}, \quad \forall i \neq j \\
& s^G \leq s^{G\max}
\end{align*}$$

where constraints are 1) power flow constraints, 2) transmission constraints, and 3) generation constraints

• Can view this as a “black box” that takes as input node loads, and outputs power generation, power flow, prices

$$s^L \rightarrow \text{OPF}(s^L) \rightarrow z^*, C(z^*)$$
• Locational marginal prices (LMPs)
  – One of the most important elements of power markets is that the marginal price of power can vary at different nodes

  – Leads to locational marginal pricing, or nodal pricing

  \[ \text{LMP}_i = \text{cost per unit of a little more (real) power at node } i \]

  \[ = \frac{\partial C(z^*)}{\partial p^L_i} \]

  – Can compute these numerically (or, optimization methods often generate these types of derivatives automatically via dual variables)
Example power market costs and (real) loads

\[ C_2 = pG_1^2 + 3pG_1 \]

\[ C_2 = 2pg_2^2 + 2pg_2 \]

\[ pL_1 = 1.0 \]

\[ pL_2 = 0.1 \]

\[ pL_3 = 1.5 \]
Resulting generation and LMPs

pG1 = 1.71 MW

LMP1 = 6.41 $/MW

LMP2 = 5.79 $/MW

pG1 = 0.94 MW

LMP3 = 6.95 $/MW
• In above example, LMP was highest at the node without a generator (intuitive, since it had to move power from elsewhere, incur losses)

• LMPs can also be less than marginal cost at any generator, or even be negative (much less intuitive)
  – Basic issue is that power flow equations + transmission constraints can conspire so that, for example, to obtain 1MW more at node 3 requires we decrease generator 2 by 1MW and increase generator 1 by 2MW

  – Given certain prices, this can lower the overall cost of power by increasing demand at node 3
• Many more elements to power markets
  – Run markets both on real-time and one day ahead

  – How do we handle reserve? Viewed as a “service” to grid and often bought separately by ISO

  – What about reactive power? Losses are much higher, so it’s not economical to supply it over longer distances, like with real power

  – Who pays for transmission? How do we handle pricing when transmission lines become congested?