Nevode's Theorem

A binary relation \( R \subseteq S \times S \) is an equivalence relation iff

1. \( \forall s \in S \ [s \mathrel{Rs}] \)
2. \( \forall s, t \in S \ [sRt \Rightarrow t \mathrel{Rs}] \)
3. \( \forall r, s, t \in S \ [(r \mathrel{Rs} \& s \mathrel{Rt}) \Rightarrow r \mathrel{RT}] \)

An equivalence relation \( R \subseteq S \times S \) partitions \( S \) into disjoint subsets called equivalence classes. Two elements elements \( a, b \in S \) are in the same equivalence class iff \( a \mathrel{R} b \). The equivalence class of \( a \) will be denoted by \( [a] \), i.e. \( [a] = \{ b \mid b \mathrel{Ra} \} \). For any \( a \) and \( b \) in \( S \), either

\( [a] = [b] \) or \( [a] \) and \( [b] \) are disjoint. The index of an equivalence relation is the number of equivalence classes.
Example: \( R = \{(x, y) \in \mathbb{N}^2 \mid x \text{ and } y \text{ have the same remainder when divided by 3}\}. \) Is \( R \) an equivalence relation? What are the equivalence classes? What is the index of \( R \)?

Let \( E \) be an equivalence relation on \( \Sigma^* \), then \( E \) is right invariant iff \( \forall u, v, w [ u E v \Rightarrow u w E v w ] \).

**Theorem** The following statements are equivalent:

1. \( L \) is accepted by a DFA;
2. \( L \) is the union of some equivalence classes of a right invariant equivalence relation of finite index.
3. Let the equivalence relation \( R \) be defined by \( x R y \text{ iff for all } z \in \Sigma^* \)
$x \in L$ iff $y \in L$. Then $R$ is of finite index.

**Proof**

$(1) \Rightarrow (2)$ Let $L = T(M)$ where $M = (S, \Sigma, \delta, s_0, F)$. Define an equivalence relation $E$ on $\Sigma^*$ by $u \ E v$ iff $\delta(s_0, u) = \delta(s_0, v)$.

Now show that $E$ is right invariant.

\[ \delta(s_0, v) = \delta(s_0, v) \Rightarrow \delta(s_0, uw) = \delta(s_0, vw). \]

Hence $u \ E v \Rightarrow uw \ E vw$.

What are the equivalence classes of $E$? Why does $E$ have finite index?
Show that \( L = \bigcup_{t \in F} \{ w \mid \delta(s_0, w) = t \} \)

(2) \(\Rightarrow\) (3) Any \( E \) satisfying (2) must be a refinement of \( R \), i.e., each equivalence class of \( E \) is contained entirely within some equivalence class of \( R \). Consequently, if \( E \) has finite index over \( R \).

(3) \(\Rightarrow\) (1) Define a DFA \( M' \) as follows:
\[
M' = ( \{ [w] \mid w \in \Sigma^* \}, \Sigma, \lambda, \delta, \{ [w] \mid w \in L3 \} )
\]
Let \( \delta([w], x) = [wx] \). Show \( T(M') = L \).

Reduced Automata

Theorem: \( M' \) constructed in (3) \(\Rightarrow\) (1) above is the DFA with the minimum number of states such that \( T(M') = L \).
proof Given M such that \( T(M) = L \).
construct E as in the proof of
(1) \( \Rightarrow \) (2) above. From the proof of
(2) \( \Rightarrow \) (3) we know that E is a
refinement of R.

**Theorem** If M and M' are two
minimal finite automata accepting
the same language L, then \( |S| = |S'| \) and M' can be obtained
from M by simply relabelling the
states of M (M & M' are isomorphic).

**Proof**

How can d determine which state
of M' corresponds to a given
state of M?