NP complete problems

\[ P = \{ L \mid \text{there is a DTM } M \text{ and a polynomial } p(n) \text{ such that } M \text{ is of time complexity } p(n) \text{ and } L(M) = L \} \]

\[ \text{NP} = \{ L \mid \text{NDTM } \} \]

A language \( L_0 \in \text{NP} \) is \textbf{NP-complete} if the following condition is satisfied:

If we are given a deterministic algorithm of time complexity \( T(n) \geq n \) to recognize \( L_0 \), then for every language \( L \in \text{NP} \) we can effectively find a deterministic algorithm of time complexity \( T(p_L(n)) \) where \( p_L \) is a polynomial
that depends on \( L \). We say \( L \) is reducible to \( L_0 \).

A language \( L \) is polynomially transformable to \( L_0 \) iff there is a deterministic polynomial-time bounded Turing machine \( M \) which will convert each string \( w \) in the alphabet of \( L \) into a string \( w_0 \) in the alphabet of \( L_0 \) such that \( w \in L \) iff \( w_0 \) is in \( L_0 \).

If \( L \) is polynomially transformable to \( L_0 \) and there is a polynomial algorithm for \( L_0 \), then there is also one for \( L \).
Theorem: the problem of determining whether a boolean formula in CNF (conjunctive normal form) is satisfiable is NP-complete.

CNF: \((x_1 + x_2)(x_2 + \overline{x}_1 + \overline{x}_3)\)

\[
\uparrow
\]

"product of sums"

Proof:

Let \(L \in \text{NP}\) and let \(M\) be a non-deterministic polynomially time bounded Turing machine that accepts \(L\). Let \(w\) be some input tape of \(M\). We will show how to polynomially transform the pair \((M, w)\) into a boolean
formula \( f \) such that \( w \in T(M) \) iff \( f \)
is satisfiable.

States of \( M \): \( q_1, q_2, \ldots, q_s \)

tape symbols: \( X_1, X_2, \ldots, X_m \)

\( p(n) \) - time complexity of \( M \)

\(|w| = n \)

If \( M \) accepts \( w \), then there must be a sequence of ID's \( I_0, I_1, \ldots, I_q \) such that

1. \( I_0 \) is the initial ID.
2. \( I_q \) is the final accepting ID.
3. \( I_j \Rightarrow I_{j+1} \) corresponds to a valid move
4. \( q \leq p(n) \)
5. No ID has more than \( p(n) \) tape symbols.

Propositions used in boolean formula
1. \( c_{ijt} \quad 1 \leq i \leq p(n), \quad 1 \leq j \leq M, \quad 0 \leq t \leq p(n) \)
   \( c_{ijt} \) will be true iff the \( i \)th cell on \( M \)'s input tape contains \( X_j \) at time \( t \).
2. \( s_{kt} \quad 1 \leq k \leq s, \quad 0 \leq t \leq p(n) \)
   \( s_{kt} \) will be true iff \( M \) is in state \( q_k \) at time \( t \).
3. Hit \( 1 \leq i \leq p(n), \quad 0 \leq t \leq p(n) \)

Hit will be true if the tape head of \( M \) is scanning cell \( i \) at time \( t \).

Total number of propositions is \( O(p^2(n)) \).

Exactly one of \( x_1, \ldots, x_K \)

\[
U(x_1, \ldots, x_K) = (x_1 + x_2 + \ldots + x_K) \prod_{i \neq j} (x_i + x_j)
\]

length of \( U(x_1, \ldots, x_K) = O(K^2) \).

a. The tape head is scanning exactly one symbol in each ID:

\[
A = A_0 A_1 A_2 \ldots A_{p(n)} \quad \text{where}
\]

\[
A_t = U(H_{1t}, H_{2t}, \ldots, H_{p(n)t})
\]
Note that $|A_1| = O\left(p^3(n)\right)$.

b. Each tape cell contains exactly one symbol at each time:

$$B_i^t = \prod_{i,t} B_{i,t} \text{ where}$$

$$B_{i,t} = \mathbb{U}(C_{i1t}, C_{i2t}, \ldots, C_{imi})$$

Note that $|B_1| = O\left(p^2(n)\right)$.

c. $M$ is in exactly one state at time $t$:

$$C = C_0 C_1 \ldots C_{p(n)} \text{ where}$$

$$C_t = \mathbb{U}(S_{1t}, S_{2t}, \ldots, S_{st})$$

Note that $|C| = O\left(p(n)\right)$.
d. At most one tape cell can change at each time:

\[ D = \prod_{ijt} \left( \left( C_{ijt} = C_{ij(t+1)} \right) + Hit \right) \]

\[(x \equiv y) + z\] is not in CNF.

\[
\left( (x \rightarrow y) \land (y \rightarrow x) + z \right) \\
\left( (x \rightarrow y) + z \right) \left( (y \rightarrow x) + z \right) \\
(x + y + z) (\bar{y} + x + z)
\]

Hence, \(101 = O(p^2(n))\)

e. Transitions satisfy next move function \(S\) of \(M\):

\[ E = \prod_{ijkt} E_{ijkt} \]
\[
E_{ijkt} = \neg c_{ijt} + \neg h_{it} + \neg s_{kt} + \sum_k c_{ijjk(t+1)} s_{kkj(t+1)} h_{jk(t+1)}
\]

and \(i_{k+1} = i + d_k\) where \(d_k = \begin{cases} 1, \text{ stay} \\ -1, \text{ move left} \\ +1, \text{ move right} \end{cases}\)

Can show that \(|E| = O(p^3(n))\)

f. The initial conditions are satisfied:

\[
F = s_{10} h_{10} \prod_{1 \leq i \leq n} c_{ij} \prod_{n < i \leq p(n)} c_{i10}
\]

\(X_1\) is the blank symbol.

\(w_i = x_{ji}\) for \(1 \leq i \leq n\)
Note that $|F| = O(p(n))$

g. $M$ eventually enters its accepting state $G = S \subseteq \rho(n)$
   (we assume that $M$ has been modified so that once it enters
   its accepting state, it stays there)

$I = ABCDEFG$

Clearly, $|I| = O(p^4(n))$

Note also that $I$ is in CNF so we obtain the result that
satisfiability is NP complete for boolean expressions in CNF. □