1. Prove that the functions in (a)–(b) below are convex, without resorting to second derivatives.
   (a) \( f(x, y) = x^2/y \) for \( y > 0 \) on \( \mathbb{R} \times \mathbb{R}_{++} \).
   (b) \( f(x) = \log(1 + e^{\sum a_i x_i}) \) on \( \mathbb{R}^n \) (\( a_i \in \mathbb{R} \) for \( 1 \leq i \leq n \)).
   (c) Using (b) show that \( \det(X + Y)^{1/n} \geq \det(X)^{1/n} + \det(Y)^{1/n} \) for \( X, Y \in S_{++}^n \).

2. Let \( f : \mathbb{R} \to \mathbb{R}_{++} \). Prove that \( f \) is log-convex if and only if \( e^{cf(x)} \) is convex for every \( c \in \mathbb{R} \) (we assume \( f \) is continuous but not that it is differentiable).

3. Fenchel conjugates:
   (a) Derive the Fenchel conjugate for \( x^T Ax + b^T x \) where \( A \succeq 0 \) may be rank-deficient.
   (b) Consider the quasi-norm \( f(x) := \|x\|_{1/2} := \left[ \sum_{i=1}^n |x_i|^{1/2} \right]^2 \). What is its bi-conjugate \( f^{**} \)?

4. Let a vector \( x \) be split into nonoverlapping subvectors \( x_1, \ldots, x_G \), then we define its \( l_{p,q} \)-mixed norm as
   \[
   \|x\|_{p,q} := \left( \sum_i^G \|x_i\|_q^p \right)^{1/p}, \quad p, q \geq 1.
   \]
   Derive the dual norm to this norm (Hint: it is another mixed-norm).
   (Remark: The norms \( \ell_{1,2}, \ell_{1,\infty} \) and \( \ell_{2,1} \) are perhaps the most interesting examples; they come up in multitask lasso and group lasso problems.)

5. Consider the normed metric space: \( \mathbb{R}^n \). Define the function
   \[
   d(x, y) := \frac{2\|x - y\|}{\|x\| + \|y\| + \|x - y\|}, \quad \forall x, y \in \mathbb{R}^n.
   \]
   Prove that \( d \) is a metric on \( \mathbb{R}^n \setminus \{0\} \).

6. Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is a symmetric function, (i.e., if \( x = [x_1, x_2, \ldots, x_n] \) and \( x_\sigma = [x_\sigma(1), \ldots, x_\sigma(n)] \) for any permutation \( \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \), then \( f(x_\sigma) = f(x) \)). Let \( S_{nn}^n \) be the set of \( n \times n \) symmetric matrices, and \( \lambda : S_{nn}^n \to \mathbb{R}^n \) the eigenvalue map, that maps a symmetric matrix to the sorted (\( \downarrow \)) vector of its eigenvalues. Show that the Fenchel conjugate of the composite function
   \[
   (f \circ \lambda)^* = f^* \circ \lambda.
   \]
   [Hint: This question is simpler than it appears. Use the fact that for any two matrices \( X, Y \in S_{nn}^n \) we have the inequality
   \[
   \text{tr}(XY) \leq \lambda(X)^T \lambda(Y).
   \]
   Also useful is to remember that \( \lambda(\cdot) \) and \( \text{tr} \) enjoy the following invariance: \( \lambda(QAQ^T) = \lambda(A) \) for orthogonal \( Q \), and \( \text{tr}(QAQ^T) = \text{tr}(A) \). To prove the claim, try showing \( (f \circ \lambda)^* \leq f^* \circ \lambda \) and \( (f \circ \lambda)^* \geq f^* \circ \lambda \). It’ll be helpful to consider \( Y = U \text{Diag}(\lambda(Y)) U^T \).]

7. [Bonus] Let \( x \) and \( y \) be vectors whose coordinates are in sorted order, so that
   \[
   x_1 \geq x_2 \geq \ldots \geq x_n, \quad y_1 \geq y_2 \geq \ldots \geq y_n.
   \]
   Suppose now that \( x \) and \( y \) satisfy the following
   \[
   \sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i, \text{ for } 1 \leq k < n
   \]
   \[
   \sum_{i=1}^n x_i = \sum_{i=1}^n y_i,
   \]
   Prove that for convex function \( f : \mathbb{R} \to \mathbb{R} \), it must hold that
   \[
   \sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(y_i).
   \]