Continuum Mechanics and the Finite Element Method
Assignment 2

- Due on March 2nd @ midnight
Suppose you want to simulate this...
The familiar mass-spring system

\[ f_{\text{int}}(x) = -k \left( \left( \frac{l}{l_0} - 1 \right) \frac{x - y_i}{|x - y_i|} \right) \]

Spring length before/after
\[ l = |x - y_i| \]
\[ l_0 = |X - y_i| \]

Deformation Measure
\[ e = \left( \frac{l}{l_0} - 1 \right) \]

Elastic Energy
\[ W = \frac{1}{2} ke^2 \]

Forces
\[ f_{\text{int}} = -\frac{\partial W}{\partial x} \]
Mass Spring Systems

Can be used to model arbitrary elastic/plastic objects, but...

- Behavior depends on tessellation
  - Find good spring layout
  - Find good spring constants
  - Different types of springs interfere
  - No direct map to measurable material properties
Alternative…

- Start from continuum mechanics
- Discretize with Finite Elements
  - Decompose model into simple elements
  - Setup & solve system of algebraic equations
- Advantages
  - Accurate and controllable material behavior
  - Largely independent of mesh structure
Mass Spring vs Continuum Mechanics

- Mass spring systems require:
  1. Measure of Deformation \( \left( \frac{l}{l_0} - 1 \right) \)
  2. Material Model \( k \)
  3. Deformation Energy \( W = \frac{1}{2} k e^2 \)
  4. Internal Forces \( f_{\text{int}} = -\frac{\partial W}{\partial x} \)

- We need to derive the same types of concepts using continuum mechanics principles
Continuum Mechanics: 3D Deformations

• For a deformable body, identify:
  – undeformed state $\Omega \subset \mathbb{R}^3$ described by positions $X$
  – deformed state $\Omega' \subset \mathbb{R}^3$ described by positions $x$

• Displacement field $\mathbf{u}$ describes $\Omega'$ in terms of $\Omega$

$$\mathbf{u} : \Omega \rightarrow \Omega' \quad x = X + \mathbf{u}(X)$$

$$\mathbf{u}(X) = \begin{pmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{pmatrix}$$
Continuum Mechanics: 3D Deformations

- Consider material points $X_1$ and $X_2$ such that $|d|$ is infinitesimal, where $d = X_2 - X_1$
- Now consider deformed vector $d'$

Deformation gradient $F = \frac{\partial x}{\partial X}$

$$d' = x_2 - x_1 \approx (I + \nabla u)d$$

$\nabla u = \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial v} \\
\frac{\partial x}{\partial w} & \frac{\partial x}{\partial w} & \frac{\partial x}{\partial w}
\end{pmatrix}$
So...

- Displacement field transforms points
  \[ \mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}) \]

- Jacobian of displacement field (deformation gradient) transforms differentials (infinitesimal vectors) from undeformed to deformed
  \[ \mathbf{d}' = (\mathbf{I} + \nabla \mathbf{u}) \mathbf{d} = \mathbf{F} \mathbf{d} \quad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \]
In general, displacement field is not explicitly described. Nevertheless, toy examples:

\[ x = X + 5 \]
\[ y = Y + 2 \]
\[ F = I \]
In general, displacement field is not explicitly described. Nevertheless, toy examples:

\[ x = X \cos \theta - Y \sin \theta \]
\[ y = X \sin \theta + Y \cos \theta \]

\[ F = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]
In general, displacement field is not explicitly described. Nevertheless, toy examples:

\[
x = 2.0X + 0.0Y \\
y = 0.0X + 1.5Y \\
F = \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 1.5 \end{bmatrix}
\]
In general, displacement field is not explicitly described. Nevertheless, toy examples:

$$x = 1.0X + 0.0Y$$
$$y = 0.5X + 1.0Y$$

$$F = \begin{bmatrix} 1.0 & 0.0 \\ 0.5 & 1.0 \end{bmatrix}$$
In general, displacement field is not explicitly described. Nevertheless, toy examples:

\[ x = X + \frac{1}{4} XY + 5 \]
\[ y = Y + 4 \]

\[ F = \begin{bmatrix} 1 + \frac{1}{4} Y & \frac{1}{4} X \\ 0 & 1 \end{bmatrix} \]
Measure of deformations

• Displacement field transforms points

\[ x = X + u(X) \]

• Jacobian of displacement field (deformation gradient) transforms vectors

\[ d' = (I + \nabla u) d = Fd \]

\[ F = \frac{\partial x}{\partial X} \]

How can we describe deformations?
Back to spring deformation

- Deformation measure (strain): \[
\left( \frac{l}{l_0} - 1 \right)
\]

- Undeformed spring: \[
\frac{l}{l_0} = 1
\]

- Undeformed (infinitesimal) continuum volume: \[
F = I
\]
Strain (description of deformation in terms of relative displacement)

- Deformation measure (strain): \( \left( \frac{l}{l_0} - 1 \right) \)

- Desirable property: if spring is undeformed, strain is 0 (no change in shape)

- Can we find a similar measure that would work for infinitesimal volumes?
3D Nonlinear Strain

Idea: to quantify change in shape, measure change in squared length for any arbitrary vector

\[ | \mathbf{d}' |^2 - | \mathbf{d} |^2 = \mathbf{d}'^T \mathbf{d}' - \mathbf{d}^T \mathbf{d} = \mathbf{d}^T (\mathbf{F}^T \mathbf{F} - \mathbf{I}) \mathbf{d} \]

\[ \rightarrow \text{Green strain} \quad \mathbf{E} = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - \mathbf{I}) \]
Strain (description of deformation in terms of relative displacement)

- Deformation measure (strain): \( \left( \frac{l}{l_0} - 1 \right) \)

- Desirable property: if spring is undeformed, strain is 0 (no change in shape)

- Can we find a similar measure that would work for infinitesimal volumes?

Green strain: \( E = \frac{1}{2} (F^T F - I) \)
3D Linear Strain

- Green strain is quadratic in displacements
  \[
  E = \frac{1}{2} (F^T F - I) = \frac{1}{2} (\nabla u + \nabla u^T + \nabla u^T \nabla u)
  \]

- Neglecting quadratic term (small deformation assumption) leads to the linear Cauchy strain (small strain)
  \[
  \varepsilon = \frac{1}{2} (\nabla u + \nabla u^t) = \frac{1}{2} (F + F^t) - I
  \]

- Written out:
  \[
  \varepsilon = \frac{1}{2} \begin{pmatrix}
  2\partial_x u & \partial_y u + \partial_x v & \partial_z u + \partial_x w \\
  \partial_x v + \partial_y u & 2\partial_y v & \partial_z v + \partial_y w \\
  \partial_x w + \partial_z u & \partial_y w + \partial_z v & 2\partial_z w
  \end{pmatrix}
  \]

Notation
\[
  \mathbf{u}(x) = \begin{pmatrix}
  u(x, y, z) \\
  v(x, y, z) \\
  w(x, y, z)
  \end{pmatrix}
\]
3D Linear Strain

- **Linear Cauchy strain**

\[
\varepsilon = \frac{1}{2} \begin{pmatrix}
2\partial_x u & \partial_y u + \partial_x v & \partial_z u + \partial_x w \\
\partial_x v + \partial_y u & 2\partial_y v & \partial_z v + \partial_y w \\
\partial_x w + \partial_z u & \partial_y w + \partial_z v & 2\partial_z w
\end{pmatrix} =: \begin{pmatrix}
\varepsilon_x & \gamma_{xy} & \gamma_{xz} \\
\gamma_{xy} & \varepsilon_y & \gamma_{yz} \\
\gamma_{xz} & \gamma_{yz} & \varepsilon_z
\end{pmatrix}
\]

\(\varepsilon_i\) : normal strains

\(\gamma_i\) : shear strains

- **Geometric interpretation**
Cauchy vs. Green strain

◆ Nonlinear Green strain is rotation-invariant
  • Apply incremental rotation $\mathbf{R}$ to given deformation $\mathbf{F}$ to obtain $\mathbf{F}' = \mathbf{RF}$

  • Then $\mathbf{E}' = \frac{1}{2}(\mathbf{F}'^T \mathbf{F}' - \mathbf{I}) = \mathbf{E}$

◆ Linear Cauchy strain is not rotation-invariant
  $\varepsilon' = \frac{1}{2}(\mathbf{F}' + \mathbf{F}'^t) \neq \varepsilon$
  → artifacts for larger rotations
Mass Spring vs Continuum Mechanics

- **Mass spring systems:**
  1. Measure of Deformation: \( \frac{l}{l_0} - 1 \)
  2. Material Model: \( k \cdot \frac{\partial W}{\partial x} \)
  3. Deformation Energy: \( W = \frac{1}{2} k e^2 \)
  4. Internal Forces: \( f_{int} = -\frac{\partial W}{\partial x} \)

- **Continuum Mechanics:**
  1. Measure of Deformation: Green or Cauchy strain
  2. Material Model
Material Model: linear isotropic material

• Material model links strain to energy (and stress)
• Linear isotropic material (*generalized Hooke’s law*)
  – Energy density \( \Psi = \frac{1}{2} \lambda \text{tr}(\varepsilon)^2 + \mu \text{tr}(\varepsilon^2) \)
  – Lame parameters \( \lambda \) and \( \mu \) are material constants related to Poisson Ratio and Young’s modulus

• Interpretation
  – \( \text{tr}(\varepsilon^2) = \|\varepsilon\|^2_F \) penalizes all strain components equally
  – \( \text{tr}(\varepsilon)^2 \) penalizes dilatations, i.e., volume changes
Volumetric Strain (*dilatation, hydrostatic strain*)

- Consider a cube with side length \(a\)
- For a given deformation \(\varepsilon\), the volumetric strain is

\[
\frac{\Delta V}{V_0} = \frac{(a(1 + \varepsilon_{11}) \cdot a(1 + \varepsilon_{22}) \cdot a(1 + \varepsilon_{33}) - a^3)}{a^3} \\
= (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + O(\varepsilon^2) \approx \text{tr}(\varepsilon)
\]

[Diagram of a cube with side length \(a\) showing volume changes]

http://en.wikipedia.org/wiki/Infinitesimal_strain_theory
Linear isotropic material

Energy density: \( \Psi = \frac{1}{2} \lambda \text{tr}(\varepsilon)^2 + \mu \text{tr}(\varepsilon^2) \)

- Problem: Cauchy strain is not invariant under rotations
  \( \rightarrow \) artifacts for rotations
- Solutions:
  - Corotational elasticity
  - Nonlinear elasticity
Material Model: non-linear isotropic model

- Replace Cauchy strain with Green strain $\rightarrow$ St. Venant-Kirchhoff material (StVK)
- Energy density: $\Psi_{StVK} = \frac{1}{2} \lambda \text{tr}(E)^2 + \mu \text{tr}(E^2)$
- Rotation invariant!
Problems with StVK

- StVK softens under compression

\[ \Psi_{StVK} = \frac{1}{2} \lambda \text{tr}(E)^2 + \mu \text{tr}(E^2) \]
Green Strain $E = \frac{1}{2} (F^t F - I) = \frac{1}{2} (\mathbf{C} - I)$

Split into deviatoric (i.e. shape changing/distortion) and volumetric (dilation, volume changing) deformations

Volumetric: $J = \text{det}(F)$  Deviatoric: $\hat{\mathbf{C}} = \text{det}(F)^{-2/3} \mathbf{C}$

Neo-Hookean material:

$$\Psi_{NH} = \frac{\mu}{2} \text{tr}(\hat{\mathbf{C}} - I) - \mu \ln(J) + \frac{\lambda}{2} \ln(J)^2$$
Different Models

- St. Venant-Kirchhoff
- Neo-Hookean
- Linear
Mass Spring vs Continuum Mechanics

Mass spring systems:
1. Measure of Deformation
2. Material Model
3. Deformation Energy
4. Internal Forces

\[
\left( \frac{l}{l_0} - 1 \right)
\]

\[
W = \frac{1}{2} ke^2
\]

Continuum Mechanics:
1. Measure of Deformation: Green or Cauchy strain
2. Material Model: linear, StVK, Neo-Hookean, etc
3. From Energy Density to Deformation Energy:
   Finite Element Discretization
Finite Element Discretization

- Divide domain into discrete elements, e.g., \textit{tetrahedra}

- Explicitly store displacement values at nodes \((x_i)\).

- Displacement field everywhere else obtained through interpolation: \(x(X) = \sum N_i(X)x_i\)

- Deformation Gradient: \(F = \frac{\partial x(X)}{\partial X} = \sum_i x_i \left(\frac{\partial N_i}{\partial X}\right)^t\)
Basis Functions

- Basis functions (aka shape functions) $N_i(X_j): \mathbb{R}^3 \rightarrow \mathbb{R}$
- Satisfy delta-property: $N_i(X_j) = \delta_{ij}$
- Simplest choice: linear basis functions

$$N_i(\bar{x}, \bar{y}, \bar{z}) = a_i \bar{x} + b_i \bar{y} + c_i \bar{z} + d_i$$

- Compute $N_i$ (and $\frac{\partial N_i}{\partial X}$) through

$$\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} = \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ \delta_{4i} \end{pmatrix}$$
Constant Strain Elements (Linear Basis Functions)

- Displacement field is continuous in space
- Deformation Gradient, strain, stress are not
  - Constant Strain per element
- Deformation Gradient can be computed as
  - $\mathbf{F} = \mathbf{eE}^{-1}$ where $\mathbf{e}$ and $\mathbf{E}$ are matrices whose columns are edge vectors in undeformed and deformed configurations
Constant Strain Elements: From energy density to deformation energy

- Integrate energy density over the entire element:
  \[ W^e = \int_{\Omega_e} \Psi(F) \]

- If basis functions are linear:
  - \( F \) is linear in \( x_i \)
  - \( F \) is constant throughout element:
    \[ W^e = \int_{\Omega_e} \Psi(F) = \Psi(F) \cdot \bar{V}^e \]
Mass Spring vs Finite Element Method

Mass spring systems:

1. Measure of Deformation
   \( \left( \frac{l}{l_0} - 1 \right) \)
2. Material Model
   \( k \)
3. Deformation Energy
   \( W = \frac{1}{2} k e^2 \)
4. Internal Forces
   \( f_{\text{int}} = -\frac{\partial W}{\partial x} \)

Continuum Mechanics:

1. Measure of Deformation: Green or Cauchy strain
2. Material Model: linear, StVK, Neo-Hookean, etc
3. Deformation Energy: integrate over elements
4. Internal Forces:
   \( f_{\text{int}} = -\frac{\partial W}{\partial x} \)
**FEM recipe**

- Discretize into elements (triangles/tetraderons, etc)
- For each element
  - Compute deformation gradient $\mathbf{F} = \mathbf{eE}^{-1}$
  - Use material model to define energy density $\Psi(\mathbf{F})$
  - Integrate over elements to compute energy: $W$
  - Compute nodal forces as: $f_{\text{int}} = -\frac{\partial W}{\partial x}$
FEM recipe

St. Venant-Kirchhoff material

\[ E = \frac{1}{2} (F^T F - I) \]

\[ \Psi = \mu \|E\|_F + \frac{\lambda}{2} \text{tr}^2(E) \]

Neohookean elasticity

\[ I_1 = \|F\|^2_F, \quad J = \det F \]

\[ \Psi = \frac{\mu}{2} (I_1 - 3) - \mu \log(J) + \frac{\lambda}{2} \log^2(J) \]

Area/volume of element

\[ f = -\frac{\partial W}{\partial x} = -V \frac{\partial \Psi}{\partial F} \frac{\partial F}{\partial x} \]

First Piola-Kirchhoff stress tensor \( P \)
FEM recipe

**St. Venant-Kirchhoff material**

\[
E = \frac{1}{2} (F^T F - I)
\]

\[
\Psi = \mu \|E\|_F + \frac{\lambda}{2} \text{tr}^2(E)
\]

\[
P = F \left[ 2\mu E + \lambda \text{tr}(E)I \right]
\]

**Neohookean elasticity**

\[
I_1 = \|F\|_F^2, \quad J = \det F
\]

\[
\Psi = \frac{\mu}{2} (I_1 - 3) - \mu \log(J) + \frac{\lambda}{2} \log^2(J)
\]

\[
P = \mu (F - F^{-T}) + \lambda \log(J)F^{-T}
\]

Area/volume of element

\[
f = -\frac{\partial W}{\partial x} = -V \frac{\partial \Psi}{\partial F} \frac{\partial F}{\partial x}
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First Piola-Kirchhoff stress tensor \( P \)
FEM recipe

St. Venant-Kirchhoff material

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E = \frac{1}{2}(F^T F - I)
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\Psi = \mu \|E\|_F + \frac{\lambda}{2} \text{tr}^2(E)
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P = F [2\mu E + \lambda \text{tr}(E)I]
\]

Neohookean elasticity

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I_1 = \|F\|^2_F, \quad J = \det F
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\[
\Psi = \frac{\mu}{2} (I_1 - 3) - \mu \log(J) + \frac{\lambda}{2} \log^2(J)
\]

\[
P = \mu (F - F^{-T}) + \lambda \log(J)F^{-T}
\]

For a tetrahedron, this works out to:

\[
[f_1 \ f_2 \ f_3] = -VPE^{-T}; \ f_4 = -f_1 - f_2 - f_3
\]

Additional reading: http://www.femdefo.org/
Material Parameters

Lame parameters $\lambda$ and $\mu$ are material constants related to the fundamental physical parameters: Poisson’s Ratio and Young’s modulus (http://en.wikipedia.org/wiki/Lamé_parameters)

**St. Venant-Kirchhoff material**

$$E = \frac{1}{2} (F^T F - I)$$

$$\Psi = \mu \|E\|_F + \frac{\lambda}{2} \text{tr}^2(E)$$

**Neo-hookean elasticity**

$$I_1 = \|F\|_F^2, \quad J = \det F$$

$$\Psi = \frac{\mu}{2} (I_1 - 3) - \mu \log(J) + \frac{\lambda}{2} \log^2(J)$$
Young’s Modulus and Poisson Ratio

Lame parameters $\lambda$ and $\mu$ are material constants related to the fundamental physical parameters: Poisson’s Ratio and Young’s modulus (http://en.wikipedia.org/wiki/Lamé_parameters)

Young’s modulus (E), measure of stiffness

Poisson’s ratio ($\nu$), relative transverse to axial deformation
What Do These Parameters Mean

- Stiffness is pretty intuitive

**Pascals = force/area**

OBJECT
E=10000000
GPa

OBJECT
E=60
KPa
Poisson’s Ratio controls volume preservation.

\[
\mu = -\frac{\Delta y}{\Delta x}
\]
What Do These Parameters Mean

Poisson’s Ratio controls volume preservation

\[ \mu = -\frac{\Delta y}{\Delta x} \]
What Do These Parameters Mean

Poisson’s Ratio controls volume preservation

\[ \mu = - \frac{\Delta y}{\Delta x} \]

Poisson’s Ratio \((\nu) = \) ?
What Do These Parameters Mean

Poisson’s Ratio controls volume preservation

Poisson’s Ratio ($\nu$) = 0.5
What Do These Parameters Mean

Poisson’s ratio is between -1 and 0.5

$\text{OBJECT PR} = 0.5$

$\nu = -0.5$
Negative Poisson’s Ratio
Where do material parameters come from?
Simple Measurement: Stiffness

OBJECT

\[ l_0 \]
Simple Measurement

What’s the Force (Stress)?

What’s the Deformation (Strain)?
Simple Measurement

OBJECT

1kg

\[ l \]

\[ l_0 \]

\[ \sigma \]

\[ E \]
Simple Measurement

2kg

OBJECT

\[ l \]

\[ l_0 \]

\( \sigma \)

\( E \)
Simple Measurement

OBJECT

3kg

$\sigma$

$E$
Simple Measurement

OBJECT

7kg
Simple Measurement

How do we get the stiffness?
Simple Measurement

How do we get the stiffness?
Simple Measurement

How do we get the stiffness?

\[ \text{OBJECT} \quad 7\text{kg} \]

\[ l \quad l_0 \]

\[ \sigma \quad \text{Stiffness} \]
Simple Measurement: Poisson’s Ratio
Simple Measurement: Poisson’s Ratio

$\text{OBJECT}$

$l_0$

$l_0$
Simple Measurement: Poisson’s Ratio

Compute changes in width and height

\[ \Delta l^w \]

\[ \Delta l^h \]
Simple Measurement: Poisson’s Ratio

\[ \nu = \frac{\Delta l_h}{\Delta l_w} \]

Poisson’s Ratio

\[ \Delta l^w \]

\[ \frac{\Delta l^w}{l^w} \]

\[ \frac{l^w}{l_0} \]

OBJECT

\[ \Delta l^h \]

\[ \frac{\Delta l^h}{l^h} \]

\[ \frac{l^h}{l_0} \]
Measurement Devices

Bi-Axial Extensometer with 5 kN Pneumatic Side-Action Grips
Simulating Elastic Materials with CM+FEM

- You now have all the mathematical tools you need
Suppose you want to simulate this...
Plastic and Elastic Materials

◆ Elastic Materials
  • Objects return to their original shape in the absence of other forces

◆ Plastic Deformations:
  • Object does not always return to its original shape
Example: Crushing a Coke Can

Old Reference State

New Reference State
Example: Crushing a van
A Simple Model For Plasticity

- Recall our model for strain: \( \frac{1}{2} (F^T F - I) \)
- Let’s consider how to encode a change of reference shape into this metric
  - Changing undeformed mesh is not easy!
- We want to exchange \( F \) with \( \omega_p F \), a deformation gradient that takes into account the new shape of our object
Continuum Mechanics: Deformation

- deformation gradient maps undeformed vectors (local) to deformed (world) vectors

\[ dx \approx F dX \]
Continuum Mechanics: Deformation

- **deformation gradient** maps undeformed vectors (reference) to deformed (world) vectors

\[ \mathbf{dx} \approx \mathbf{F} \mathbf{dX} \]
Continuum Mechanics: Deformation

- $\mathbf{F}$ transforms a vector from Reference space to World Space
Continuum Mechanics: Deformation

- Introduce a new space

Reference Space → Plastic Space → World Space
Our goal is to use $w \mathbf{F}$ but we only have access to $r \mathbf{F}$. 

References:

- Reference Space
- Plastic Space
- World Space
Our goal is to use $wF$ but we only have $rF$. We have $wF = rF^{-1} p F$.
Continuum Mechanics: Deformation

- Our goal is to use $w_F$ but we only have access to $p_{F^r}$.  
  \[
  w_F = w_{F^r F^{-1}}
  \]

- Keep an estimate of $p_{F^{-1}}$ per element, built incrementally.
How to Compute the Plastic Deformation Gradient

- We compute the strain/stress for each element during simulation.
- When it gets above a certain threshold store $F$ as $\frac{p}{r} F$.
How to Compute the Plastic Deformation Gradient

- We compute the strain/stress for each element during simulation.
- When it gets above a certain threshold store $\mathbf{F}$ as $\frac{p}{r} \mathbf{F}$.
How to Compute the Plastic Deformation Gradient

- Each subsequent simulation step uses

\[ w_F = w_{F^p} F^{F^{-1}} \]
So now you too can simulate this...
Questions?