Announcements

- Homework is due today 4pm!
- HW3 is out already and HW1 solutions are available.
Combinations

So far we have been talking only about ordered samples. For many applications we only need an unordered sample. In particular we want to figure out how many ways one can form groups of size $r$ from a set of size $n$.

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- **Q:** How many groups of size \( r \) are there of a given population of size \( n \)?
- **Q:** In other words: how many ways can I choose \( r \) unordered elements *without replacement* from a population of size \( n \)?

\[ \binom{n}{r} \]
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▶ How many ways can I choose two digits without replacement from \( \{1, 2, 3\} \)?

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- 12, 23, 13. So $\binom{3}{2} = 3$. 

Combinations

- Earlier we learned about \( n \) permute \( r \). This is how we choose \( r \) elements without replacement, but the **order matters**.
- Let consider all ordered samples of size 2 picked without replacement from 3 numbers.
- Now we want the same. But order does not matter. So \( \{1, 2\} \) and \( \{2, 1\} \) are the same.
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- (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)

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- Now we want the same. But order does not matter. So $\{1, 2\}$ and $\{2, 1\}$ are the same.
- $(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)$
  - $2$ times
- Say we are picking 3 out of 4 numbers. Consider all ordered arrangements.
  - $123, 132, 231, 213, 312, 321, 143, 134, 431, 413, 314, 341, \ldots$
  - $(1, 2, 3)$ appears 6 times
  - $(1, 3, 4)$ appears 6 times
Let us start with the collection of ordered \( r \)-tuplets chosen without replacement from \( n \) nodes. This collection is of size \((n)_r\).

This also sometimes written as \( \binom{n}{r} \).

These are also known as the binomial coefficients.
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Hence the total number of unordered samples/groups of size $r$ is given by

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\binom{n}{r} := \frac{(n)_r}{r!} = \frac{n!}{r!(n-r)!}
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These are also known as the binomial coefficients.
Binomial coefficients: fun stuff

\[ \binom{n}{k} = \binom{n}{n-k} \]

- Number of ways to choose \( k \) out of \( n \) things is the same as choosing \((n - k)\) things out \( n \) and removing them.

\[ \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \]

- How do you choose \( k+1 \) elements out of \( n+1 \) elements?
  - Divide \( n+1 \) into a set \( S \) of \( n \) elements plus one extra element \( x \).
  - The sample you pick either has \( x \) or not.
  - How many ways can you choose \( k+1 \) out of \( n+1 \) so that \( x \) is never included?
    - Just choose \( k+1 \) elements from \( S \)!
    - So in \( \binom{n}{k} \) ways.
  - How many ways can you choose \( k+1 \) elements so that \( x \) is included?
    - You can choose \( k \) elements from \( S \) elements in \( \binom{n}{k} \) ways and add \( x \) to that pile.
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Example 1

- How many binary sequences of length $n$ are there with $k$ ones?
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- How many binary sequences of length $n$ are there with $k$ ones?
- Same as choosing $k$ out of $n$ positions in the sequence. We put a one in these positions and zeros in the rest. So the answer is $\binom{n}{k}$. 
Example II

What is $\sum_{k=0}^{n} \binom{n}{k}$?
Example II

- What is \( \sum_{k=0}^{n} \binom{n}{k} \)?

- Well this is just the total number of possible binary strings, so \( 2^n \).
Practice Problems

You are walking on a grid. You can either go right or up by one step. You start from (0,0). How many paths are there to (5, 10)?

How many of the above paths go via (4, 4)?
Practice Problem: Path counting

Think of each right as a 1 and each up as a 0. Now you have a bijection with every binary sequence of length 15 with 5 “1”s. A bijection is essentially saying that each path you construct can be written as a length 15 binary sequence with 5 “1”s and for any such binary sequence you have a path that takes you to (5, 10).

How many such sequences are there? \( \binom{15}{5} \).

Now you need to go via (4, 4). So first count paths from (0, 0) to (4, 4). There are \( \binom{8}{4} \) such paths. How many paths go from (4, 4) to (5, 10)? Change you origin! This is the same as counting paths from (0, 0) to (1, 6). There are \( \binom{7}{1} = 7 \) such paths. So the total number of paths is \( \binom{8}{4} \times 7 \) using the multiplication rule.
Example II

- How many configurations of length $n$ binary strings are there with $k$ 1’s?
  - Think Binomial coefficient!
- How many configurations of length 10 strings are there with three 0’s, four 1’s and three 2’s?
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  - First choose positions of the 0’s in $\binom{10}{3}$ ways.
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  - First choose positions of the 0’s in \( \binom{10}{3} \) ways.
  - Now choose positions of the 1’s from the remaining 7 in \( \binom{7}{4} \) ways.
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  ▶ Now choose positions of the 1’s from the remaining 7 in $\binom{7}{4}$ ways.
  ▶ The remaining three positions are given to the 2’s.
  ▶ So a total of \( \frac{10!}{3!7!} \times \frac{7!}{4!3!} = \frac{10!}{3!4!3!} \) ways.
Multinomial coefficients

- $\binom{n}{k} := \#\text{ways to divide } n \text{ elements into two disjoint groups, where the first group has size } k \text{ and the second size } n - k$.

- $\binom{n}{n_1, n_2, n_3} := \#\text{ways to divide } n \text{ elements into 3 disjoint groups of sizes } n_1, n_2 \text{ and } n_3 = n - n_1 - n_2$ respectively.
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- First we choose \( n_1 \) out of \( n \). Next from the remaining \( n - n_1 \) we choose \( n_2 \). Rest are assigned to the third group.
Multinomial coefficients

- \((\begin{array}{c} n \\ k \end{array})\) := \#ways to divide \(n\) elements into two disjoint groups, where the first group has size \(k\) and the second size \(n-k\).

- \(\left(\begin{array}{c} n \\ n_1, n_2, n_3 \end{array}\right)\) := \#ways to divide \(n\) elements into 3 disjoint groups of sizes \(n_1\), \(n_2\) and \(n_3 = n - n_1 - n_2\) respectively.

- First we choose \(n_1\) out of \(n\). Next from the remaining \(n-n_1\) we choose \(n_2\). Rest are assigned to the third group.

- \(\left(\begin{array}{c} n \\ n_1, n_2, n_3 \end{array}\right) = \left(\begin{array}{c} n \\ n_1 \end{array}\right) \times \left(\begin{array}{c} n-n_1 \\ n_2 \end{array}\right) = \frac{n!}{n_1!n_2!n_3!}\)
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- \( \binom{n}{n_1, n_2, n_3} = \binom{n}{n_1} \times \binom{n - n_1}{n_2} = \frac{n!}{n_1! n_2! n_3!} \)

- Generalizing to \( r \) groups of sizes \( n_1, \ldots, n_r \) with \( n_1 + \cdots + n_r = n \) we have:

\[
\binom{n}{n_1, \ldots, n_r} := \frac{n!}{n_1! \cdots n_r!}
\]
Sum up

- Here are all the things we have learned so far.
- There are $n!$ ways to permute $n$ distinguishable objects.
- $(n)_r$ is the number of ways one can pick $r$ ordered objects from $n$ distinguishable objects. $(n)_r = n!/(n-r)!$
- $\binom{n}{r}$ is the number of ways one can pick $r$ unordered objects from $n$ distinguishable objects.
- $\frac{n!}{n_1!n_2!n_3!}$ is the number of ways one can label $n$ distinguishable objects with $n_1$ labels of one type, $n_2$ labels of a second type, and $n_3 = n - n_1 - n_2$ elements of the third type.
- Today we will learn more about occupancy problems.
So far we have been talking about distinguishable objects. In all these problems, which object was assigned to which group matters.

Sometimes however we are interested in counting frequencies.

How many ways can you divide 10 fruits among 4 children so that everyone gets at least one?

Here, we are not interested in which fruit went to which child. We only care about how many of the fruits went to a child.

So we think of the fruits as indistinguishable objects and the children are distinguishable bins.
Say \( x_i \) is the number of fruits going to child \( i \). So we are looking for ways to write \( x_1 + x_2 + x_3 + x_4 = 10 \), where \( x_i > 0 \).

\( 1 + 1 + 4 + 4 \) is not the same as \( 1 + 4 + 1 + 4 \).

How will you represent \( x_1 = 1, x_2 = 3, x_3 = 2, \) and \( x_4 = 4 \). Write 10 stars for 10 fruits. Put bars to represent bins/children.

In order to divide 10 stars into 4 parts you will need 3 bars.

\[
\begin{array}{c|c|c|c}
* & * * * & * * & * * * \\
1 & 3 & 2 & 4
\end{array}
\]
Say $x_i$ is the number of fruits going to child $i$. So we are looking for ways to write $x_1 + x_2 + x_3 + x_4 = 10$, where $x_i > 0$.

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How will you represent $x_1 = 1, x_2 = 3, x_3 = 2$, and $x_4 = 4$. Write 10 stars for 10 fruits. Put bars to represent bins/children.

In order to divide 10 stars into 4 parts you will need 3 bars.

There are 9 spaces between 10 stars. We want to put 3 bars in these places. There are total $\binom{9}{3}$ ways of doing this.
Say you want to distribute \( n \) fruits among \( k \) children so that everyone gets at least 1.

Say \( x_i \) is the number of fruits going to child \( i \). So we are looking for ways to write \( x_1 + x_2 + \cdots + x_k = n \), where \( x_i > 0 \).

Writing in stars and bars, you want to place \((k - 1)\) bars in \((n - 1)\) spaces between the stars.

So the answer is \( \binom{n-1}{k-1} \).
Now I want to divide 10 fruits among 4 children. A child may or may not get any fruit. How many ways to do this?

▶ We want to use our former idea, but how?
Now I want to divide 10 fruits among 4 children. A child may or may not get any fruit. How many ways to do this?

- We want to use our former idea, but how?

Lightbulb! Why not divide 14 fruits to 4 children so that everyone has at least one, and then remove one fruit from each? The fruits are indistinguishable and so it doesn’t matter which one you take out.
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▶ This way everyone will have zero or more fruits, and total number of fruits is 10.
Now I want to divide 10 fruits among 4 children. A child may or may not get any fruit. How many ways to do this?

- We want to use our former idea, but how?

**Lightbulb!** Why not divide 14 fruits to 4 children so that everyone has at least one, and then remove one fruit from each? The fruits are indistinguishable and so it doesn’t matter which one you take out.

- This way everyone will have zero or more fruits, and total number of fruits is 10.

- So the answer is \( \binom{13}{3} \).
How many ways can I write $n$ as an sum of $k$ non-negative integers? $1 + 2 + 4$ is different from $1 + 4 + 2$.

We want to enumerate the ordered set of $k$-tuplets $\{x_1, \ldots, x_k\}$, such that $x_1 + \ldots + x_k = n$ and $x_i \geq 0$, for $i = 1, \ldots, k$.
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We want to enumerate the ordered set of \( k \)-tuplets \( \{x_1, \ldots, x_k\} \), such that \( x_1 + \ldots x_k = n \) and \( x_i \geq 0 \), for \( i = 1, \ldots, k \).

Define \( y_i = x_i + 1 \). Now for each \( r \)-tuplet of \( x_i \)'s we have an \( r \)-tuplet of \( y_i \)'s such that \( y_1 + \cdots + y_k = n + k \), and \( y_i > 0 \) for \( i = 1, \ldots, k \).
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Define $y_i = x_i + 1$. Now for each $r$-tuplet of $x_i$’s we have an $r$-tuplet of $y_i$’s such that $y_1 + \cdots + y_k = n + k$, and $y_i > 0$ for $i = 1, \ldots, k$.

But this is the same as the former problem, with $n + k$ stars and $k$ bars! So the answer is $\binom{n + k - 1}{k - 1}$.
Our population consists of ten digits \( \{0, 1, \ldots, 9\} \).

I pick a 5 digit number uniformly at random. Such a number can start with zero, and may have repetitions.

What is the probability \( p \) that the five digits are all different?
Our population consists of ten digits \{0, 1, \ldots, 9\}.

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\[ \# \text{all 5 digit numbers is } 10^5. \]
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\[ p = (10)_5 / 10^5. \]
A bus with 5 passengers makes 10 stops. All configurations of discharging the passengers are equally likely.

What is the probability $p$ that no two passengers get down at the same stop?
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- All possible configurations is $10^5$.
- Configurations with 5 passengers each with a different stop is $(10)_5$. 
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- All possible configurations is \( 10^5 \).
- Configurations with 5 passengers each with a different stop is \( (10)_5 \).
- \( p = (10)_5/10^5 \).
The birthdays of \( r \leq 365 \) people form a sample of size \( r \) from the population of all birthdays (365 days in the year). We assume that a person is equally likely to be born on any of the 365 days and no one was born on Feb 29th.

What is the probability that no two people will have the same birthday?

\[
p = \frac{(365)_r}{365^r} = \frac{365!}{(365 - r)!365^r} \quad (1)
\]

Then at least two people must have the same birthday. This is also called the Pigeonhole Principle. So \( p = 0 \).

How will you calculate this for large \( r \), say \( r = 30 \)?
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Probability and counting: example 1c

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- What is the probability if \( r = 366 \)?

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What is the probability if $r = 366$?

Then at least two people must have the same birthday. This is also called the Pigeonhole Principle. So $p = 0$. 
The birthdays of \( r \leq 365 \) people form a sample of size \( r \) from the population of all birthdays (365 days in the year). We assume that a person is equally likely to be born on any of the 365 days and no one was born on Feb 29\(^{th}\).

What is the probability that no two people will have the same birthday?

\[
p = \frac{(365)_r}{365^r} = \frac{365!}{(365 - r)!365^r} \tag{1}
\]

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How will you calculate this for large \( r \), say \( r = 30 \)?
Birthdays

- Problem... my calculator can’t handle $365!$ or $365^{30}$.
- Take logarithms! $365^{30} = 10^{76.8688} = 7.392 \times 10^{76}$.
- We can approximate factorials using Stirling’s approximation:

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$$

- The $\sim$ symbol means the ratio of the two sides tend to 1 as $n \to \infty$.

$$\ln(365!) \approx \frac{1}{2} \ln(2\pi) + \left(365 + \frac{1}{2}\right) \ln(365) - 365 = 1792.3$$

$$\ln(335!) \approx \frac{1}{2} \ln(2\pi) + \left(335 + \frac{1}{2}\right) \ln(335) - 335 = 1616.6$$

$$\ln \left(\frac{365!}{335!}\right) = \ln(365!) - \ln(335!) \approx 1792.3 - 1616.6 = 175.55$$

$$\frac{365!}{335!} \approx e^{175.55} = 2.1711 \times 10^{76}$$

- The actual value is $2.1710 \times 10^{76}$ – not bad!
Birthdays

- So, we with 30 people, we have $7.392 \times 10^{76}$ possible combinations of birthdays.
- $2.171 \times 10^{76}$ of these possible combinations of birthdays have no repeats.
- So, the probability of no one having the same birthday is:
  $$\frac{2.171 \times 10^{76}}{7.392 \times 10^{76}} \approx 0.296$$
- Odds are, there’s a shared birthday!