SDS 321: Introduction to Probability and Statistics
Lecture 13: Expectation and Variance and joint distributions

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Multiple random variables

So far we have been talking about single random variables and associated PMF’s. However, often we are interested in multiple random variables.

- Consider two discrete random variables $X$, and $Y$ associated with the same experiment.
- The joint PMF of $X$ and $Y$ are defined as $p_{X,Y}(x,y) = P(X = x, Y = y)$ for all pairs of values $x, y$ $X$ and $Y$ can take.
- This is none other than $P(\{X = x\} \cap \{Y = y\})$.
- Of course the order does not matter.
Properties of the joint PMF

- Recall that if $A_1, A_2, \ldots, A_K$ is a partition of $\Omega$,
  \[ P(B) = P \left( \bigcup_k (B \cap A_k) \right) = \sum_k P(B \cap A_k). \]

- $\{X = x\}$ is the disjoint union of $\{X = x\} \cap \{Y = y\}$ for all $y$ values $Y$ can take.

- $\{X = x\} \cap \{Y = y\}$ is none other than $\{X = x, Y = y\}$

- We can extend this to PMFs:
  \[ \sum_y P(X = x, Y = y) = \]
  \[ \sum_x P(X = x, Y = y) = 1. \]
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$$\sum_y P(X = x, Y = y) = P(X = x).$$

$$\sum_x P(X = x, Y = y) =$$
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  \[ \sum_{y} P(X = x, Y = y) = P(X = x). \]
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- These are also called the marginal PMF’s.
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- So $\sum_x \sum_y P(X = x, Y = y) = \ldots$.
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- So \[ \sum_x \sum_y P(X = x, Y = y) = \]
  
  - $\{X = x, Y = y\}$ for all pairs of numerical values taken by $X$ and $Y$ form a partition of the sample space.
  - And now the normalization rule gives us the result!
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Contingency tables

Alice says that there are more left handed women than left handed men. Bob gives her some numbers to count probabilities.

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- $P(X = 1, L = 1) = \frac{3}{100}$
- $P(X = 1) = \frac{1}{2} \leftarrow$ Marginal probability!
- $P(L = 1) = \frac{10}{100} \leftarrow$ Marginal probability!
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- \( P(X = 1) = \frac{1}{2} \) ← Marginal probability!
- \( P(L = 1) = \frac{10}{100} \) ← Marginal probability!
- Remember! These really are estimated numbers, and hence approximations. I am estimating the fraction of left handed men in a population via my sample!
Functions of multiple random variables

\[ E(g(X, Y)) = \sum_{x,y} g(x, y)P(X = x, Y = y). \]

Let \( g(X, Y) = aX + bY. \)

\[ E(g(X, Y)) = \sum_{x,y} (ax + by)P(X = x, Y = y) = aE[X] + bE[Y]. \]

What if \( g(X, Y) = aX^2 + bY^2 + c? \)

\[ E[g(X, Y)] = aE[X^2] + bE[Y^2] + c \]

**Common Mistake:** \( E[g(X, Y)] \neq g(E[X], E[Y])! \) unless \( g \) is linear in \( X \) and \( Y! \)
Multiple random variables

How about three random variables?

▶ We will write $p_{X,Y,Z}(x, y, z) = P(X = x, Y = y, Z = z)$

▶ The rules are the same:

$\begin{align*}
P(X = x, Y = y) &= \sum_z P(X = x, Y = y, Z = z). \\
P(X = x) &= \sum_{y,z} P(X = x, Y = y, Z = z). \\
P(Y = y) &= \sum_{x,z} P(X = x, Y = y, Z = z). \\
P(Z = z) &= \sum_{x,y} P(X = x, Y = y, Z = z). \\
\sum_{x,y,z} P(X = x, Y = y, Z = z) &= 1.
\end{align*}$

▶ Generalizes easily to more than 3 random variables.
Linearity of expectation

Perhaps one of the most useful and powerful results!


- More generally,

  $$E[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n] = a_1 E[X_1] + a_2 E[X_2] + \cdots + a_n E[X_n]$$

- This is extremely general! $X_1, \ldots, X_n$ do not have to be mutually independent for this to hold!

- This generalizes to $E \left[ \sum_i a_i f(X_i) \right] = \sum_i a_i E[f(X_i)]$, as long as the expectations are well defined.
Expectation of $Y \sim \text{Binomial}(n, p)$

Remember that a Binomial($n, p$) random variable is nothing other than the sum of $n$ independent Bernoulli’s!

- $Y = \sum_{i=1}^{n} X_i$, where $X_i \sim \text{Bernoulli}(n, p)$.

- We know that $E[X_i] = p$.

- Using our newfound tool, we have:

  $$E[Y] = E[\sum_{i} X_i] = \sum_{i} E[X_i] = np.$$ 

- We do not need the mutual independence of the Bernoullis to get this result!
I am throwing $m$ distinguishable balls into $n$ distinguishable bins. What is the expected number of empty bins (call this $Y$)?

- Let $X_i = \begin{cases} 1 & \text{The } i^{th} \text{ bin is empty} \\ 0 & \text{Otherwise} \end{cases}$

- We want $E[Y]$.

- $E[Y] = 

- $E[X_i] = 

- When $m = n$, for large $n$, $E[Y] \approx \frac{n}{e}$. 

9
Balls and bins

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- We want $E[Y]$.

- $E[Y] = E[\sum_i X_i] = \sum_i E[X_i]$

- $E[X_i] = \frac{1}{n}$

- When $m = n$, for large $n$, $E[Y] = n \left(1 - \frac{1}{n}\right) \approx \frac{n}{e}$. 9
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- We want $E[Y]$.
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- $E[X_i] = P(\text{No ball falls in bin } i) = \frac{1}{n}$

When $m = n$, for large $n$, $E[Y] \approx \frac{n}{e}$. 
Balls and bins

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- We want $E[Y]$.

- $E[Y] = E[\sum_i X_i] = \sum_i E[X_i]$

- $E[X_i] = P(\text{No ball falls in bin } i) = (1 - 1/n)^m$

- $E[Y] = n(1 - 1/n)^m$
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When $m = n$, for large $n$, $E[Y] = n(1 - 1/n)^n \approx n/e$. 
Conditional PMF

So we have started thinking about how *knowing about one random variable* alters our belief about another random variable. This brings us to conditional PMFs!

- The **conditional PMF** of a random variable $X$, conditioned on a particular event $A$ with $P(A) > 0$, is defined by:

  \[ p_{X|A}(x) = P(X = x|A) = \frac{P(\{X = x\} \cap A)}{P(A)} \]

- So we have
  \[
  \sum_x P(X = x|A) = \sum_x \frac{P(\{X = x\} \cap A)}{P(A)} = \sum_x \frac{P(\{X = x\} \cap A)}{P(A)}
  \]

- But $A$ can be written as a disjoint union of the events $\{X = x\} \cap A$ for all numerical values $X$ takes.

- Total probability rule gives: $P(A) = \sum_x P(\{X = x\} \cap A)$, and so
  \[
  \sum_x P(X = x|A) = 1.
  \]
Conditioning one random variable on another

Let $X$ and $Y$ be two random variables associated with the same experiment. Now the knowledge of $Y = y$ for some particular value $y$ provides us with partial knowledge about what value $X$ may take.

- The **conditional PMF** of $X$ given $Y$ is given by
  \[ p_{X|Y}(x, y) = P(X = x|\{Y = y\}). \]

- Using the same set of rules as before we can write:
  \[ P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p_{X,Y}(x, y)}{p_Y(y)} \]

- For any fixed $y$ such that $P(Y = y) > 0$, we also have:
  \[ \sum_x P(X = x|Y = y) = 1. \]

- So, a conditional PMF satisfies the properties of a PMF.
Conditional PMF

Bob and Alice are interested in finding out the conditional probability of being left handed given a person is a man. Bob finds his data again.

\[ P(L = 1 | X = 0) \] is just the fraction of all men who are left handed.

\[ P(L = 0 | X = 0) + P(L = 1 | X = 0) = 1 \]

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- $P(L = 1|X = 0)$ is just the fraction of all men who are left handed
- $P(L = 1|X = 0) = 7/50$. 
Bob and Alice are interested in finding out the conditional probability of being left handed given a person is a man. Bob finds his data again.

\[ P(L = 1|X = 0) \] is just the fraction of all men who are left handed

\[ P(L = 1|X = 0) = \frac{7}{50}. \]

Let us plug in the formula. \( P(L = 1, X = 0) = \).
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Bob finds his data again. Let us plug in the formula. $P(L = 1, X = 0) = \frac{7}{100}$.

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$P(L = 1|X = 0)$ is just the fraction of all men who are left handed.

$P(L = 1|X = 0) = \frac{7}{50}$.

Let us plug in the formula. $P(L = 1, X = 0) = \frac{7}{100}$.

$P(X = 0) = \frac{50}{100}$. So $\frac{P(L = 1, X = 0)}{P(X = 0)} = \frac{7}{50}$. 
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- $P(L = 1|X = 0) = \frac{7}{50}$.
- Let us plug in the formula. $P(L = 1, X = 0) = \frac{7}{100}$.
- $P(X = 0) = \frac{50}{100}$. So $\frac{P(L = 1, X = 0)}{P(X = 0)} = \frac{7}{50}$.
- What is $P(L = 0|X = 0)$? Its just the fraction of all men who are right handed! So $\frac{43}{50}$. 
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P(X = 0) = \frac{50}{100}. \text{ So } \frac{P(L = 1, X = 0)}{P(X = 0)} = \frac{7}{50}.
\]

What is \( P(L = 0 | X = 0) \)? Its just the fraction of all men who are right handed! So \( \frac{43}{50} \).

\[
P(L = 0 | X = 0) + P(L = 1 | X = 0) = 1!
\]
Conditional PMF

- Remember that a conditional PMF is a valid PMF.
- Since \( P(X = x | Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} \), we also have the multiplication rule:
  - \( P(X = x, Y = y) = P(X = x | Y = y)P(Y = y) \)
  - But \( P(X = x, Y = y) = P(Y = y, X = x) \), and so we also have: \( P(X = x, Y = y) = P(Y = y | X = x)P(X = x) \).

- Same as multiplication rule from before!
- We can also draw trees to get conditional probabilities!
Independence of random variables

Let's first consider two events \( \{X = x\} \) and \( A \). We know that these two events are independent if

\[
P(\{X = x\}, A) = P(\{X = x\})P(A)
\]

In other words if \( P(A) > 0 \), then \( P(X = x|A) = P(X = x) \), i.e. knowing the occurrence of \( A \) does not change our belief about \( \{X = x\} \).

We will call the random variable \( X \) and event \( A \) to be independent if

\[
P(X = x, A) = P(X = x)P(A) \quad \text{For all } x
\]

Two random variables are said to be independent if

\[
P(X = x, Y = y) = P(X = x)P(Y = y) \quad \text{For all } x \text{ and } y
\]

To put it a bit differently,

\[
P(X = x|Y = y) = P(X = x) \quad \text{For all } x \text{ and } y \text{ such that } P(Y = y) > 0
\]
A super important implication

We saw that $E[X + Y] = E[X] + E[Y]$ no matter whether $X$ and $Y$ are independent or not.

- If $X$ and $Y$ are independent, $E[XY] = E[X]E[Y]$

\[
E[XY] = \sum_{x,y} xyP(X = x, Y = y) = \sum_{x,y} xyP(X = x)P(Y = y)
\]

\[
= \left( \sum_{x} xP(X = x) \right) \left( \sum_{y} yP(Y = y) \right) = E[X]E[Y]
\]

- In fact, $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$
Variance of sum of independent random variables

Let $X$ and $Y$ be two independent random variables. What is $\text{var}(X + Y)$?

- **Remember!** $\text{var}(X + Y) = E[(X + Y)^2] - (E[X + Y])^2$

  
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- $= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 = \text{var}(X) + \text{var}(Y)$

Variance of sum of independent random variables equals the sum of the variances!
Variance of sum of independent random variables

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- $\text{var}(X + Y) = E[(X + Y)^2] - (E[X + Y])^2$

  $= \underbrace{E[X^2] - E[X]^2} + \underbrace{E[Y^2] - E[Y]^2} = \text{var}(X) + \text{var}(Y)$

- **Variance of sum of independent random variables equals the sum of the variances!**
Independence of several random variables

- Three random variables $X$, $Y$ and $Z$ are said to be independent if

$$P(X = x, Y = y, Z = z) = P(X = x)P(Y = y)P(Z = z) \quad \text{For all } x, y, z$$

- If $X$, $Y$, $Z$ are independent, then so are $f(X)$, $g(Y)$ and $h(Z)$.
- Also, any random variable $f(X, Y)$ and $g(Z)$ are independent.
- Are $f(X, Y)$ and $g(Y, Z)$ independent?
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  - Not necessarily, both have $Y$ in common.

- For $n$ independent random variables, $X_1, X_2, \ldots, X_n$, we also have:

$$\text{var}(X_1 + X_2 + X_3 + \cdots + X_n) = \text{var}(X_1) + \text{var}(X_2) + \cdots + \text{var}(X_n)$$
Consider $n$ independent Bernoulli variables $X_1, X_2, \ldots, X_n$, each with probability $p$ of having value “1”. The sum $Y = \sum_i X_i$ is a $Binomial(n, p)$ random variable.

- We saw last time that $E[Y] = \sum_i E[X_i] = np$. What about the variance?
- Recall that $\text{var}(X_i) = p(1 - p)$ for $i \in \{1, 2, \ldots, n\}$.
- $\text{var}(Y) = \text{var}(X_1 + X_2 + \cdots + X_n) = \sum_{i=1}^{n} \text{var}(X_i) = np(1 - p)$. 

Variance of a Binomial
Conditional Independence

- Very similar to conditional independence of events!
- \( X \) and \( Y \) are conditionally independent, given a positive probability event \( A \) if

\[
P(X = x, Y = y | A) = P(X = x | A)P(Y = y | A)
\]

For all \( x \) and \( y \)

- Same as saying \( P(X = x | Y = y, A) = P(X = x | A) \), i.e.

- Once you know that \( A \) has occurred, knowing \( \{Y = y\} \) has occurred does not give you any more information!

- Like we learned before, conditional independence does not imply unconditional independence.
Example-conditionally independent but not marginally independent

- I separately phone two students (Alice and Bob) and tell them the midterm grade.
- To each, I report the same grade, \( G \in \{A+, A..., C\} \).
- The signal is bad and, Alice and Bob each independently make an educated guess of what I said.
- Let the grades guessed by Alice and Bob be \( X \) and \( Y \).
- Are \( X \) and \( Y \) marginally independent?
Example—conditionally independent but not marginally independent

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- Let the grades guessed by Alice and Bob be $X$ and $Y$.
- Are $X$ and $Y$ marginally independent?
  - **NO.** you would think, $P(X = A|Y = A) > P(X = A)$.

What if I tell you that $G = A$—?
- Are $X$ and $Y$ conditionally independent given $\{G = A\}$.
- **YES!** Because if we know the grade I actually said, the two variables are no longer dependent.
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- Let the grades guessed by Alice and Bob be $X$ and $Y$.
- Are $X$ and $Y$ marginally independent?
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To each, I report the same grade, $G \in \{A+, A, ..., C\}$.

The signal is bad and, Alice and Bob each independently make an educated guess of what I said.

Let the grades guessed by Alice and Bob be $X$ and $Y$.

Are $X$ and $Y$ marginally independent?

- **NO.** you would think, $P(X = A|Y = A) > P(X = A)$.

What if I tell you that $G = A+$?

- Are $X$ and $Y$ conditionally independent given $\{G = A+\}$.
Example—conditionally independent but not marginally

- I separately phone two students (Alice and Bob) and tell them the midterm grade.
- To each, I report the same grade, $G \in \{A+, A, ..., C\}$.
- The signal is bad and, Alice and Bob each independently make an educated guess of what I said.
- Let the grades guessed by Alice and Bob be $X$ and $Y$.
- Are $X$ and $Y$ marginally independent?
  - **NO.** you would think, $P(X = A \mid Y = A) > P(X = A)$.
- What if I tell you that $G = A-$?
  - Are $X$ and $Y$ conditionally independent given $\{G = A-\}$?
  - **YES!** Because if we know the grade I actually said, the two variables are no longer dependent.
Example—marginally independent but not conditionally

- I toss two dice independently and $X$ and $Y$ are the readings on them.
- Are $X$ and $Y$ independent?
- Now I tell you that $X + Y = 12$. Are they still independent?